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Existence and Global Attractivity of Periodic Solutions to Some Classes of Difference Equations

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Abstract. Existence and global attractivity of periodic solutions to some subclasses of the following class of difference equations

$$x_{n+1} = q_n x_n + f(n, x_n, x_{n-1}, \dots, x_{n-k}), \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}_0$, $(q_n)_{n\in\mathbb{N}_0}$ is a T-periodic sequence ($T \in \mathbb{N}$), and $f : \mathbb{N}_0 \times \mathbb{R}^{k+1} \to \mathbb{R}$ is a T-periodic function in the first variable, which for each $n \in \{0, 1, \dots, T-1\}$ is continuous in other variables, are studied.

1. Introduction

Throughout the paper, by $\mathbb{Z}$ is denoted the set of all integers, while by $\mathbb{N}_l$, where $l \in \mathbb{Z}$, is denoted the set of all $n \in \mathbb{Z}$ such that $n \ge l$. If $k, l \in \mathbb{Z}$, then the notation $n = \overline{k, l}$ stands for the set of all $n \in \mathbb{Z}$ such that $k \le n \le l$.

Throughout the paper we use the conventions

$$\sum_{i=s}^{t} c_i = 0 \quad \text{and} \quad \prod_{i=s}^{t} c_i = 1,$$

when $t < s$, $t, s \in \mathbb{N}_0$.

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The linear first-order difference equation

$$x_{n+1} = q_n x_n + f_n, \quad n \in \mathbb{N}_0, \tag{1}$$

where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are sequences of numbers is a very important and useful solvable difference equation (many classical solvable difference equations and systems, or their invariants, can be found, for example, in [1, 2, 7, 8, 11, 12, 14–16, 18, 19, 22–29, 34, 36, 61]). For some methods for solving equation (1), see, for example, [8, 12, 14, 25] (book [25] contains a nice presentation of three methods for solving it, essentially corresponding to the three methods for solving the linear differential equation of first order).

The general solution to equation (1) is

$$x_n = x_0 \prod_{j=0}^{n-1} q_j + \sum_{i=0}^{n-1} f_i \prod_{j=i+1}^{n-1} q_j, \quad n \in \mathbb{N}_0. \tag{2}$$

To describe the usefulness of equation (1), we mention that many nonlinear difference equations and systems of difference equations are solved by transforming them into some of special cases of the equation, by using one or several changes of variables along with some algebra (see, for example, the difference equations in [30, 38, 41, 47, 55, 56], systems in [5, 42–45, 51, 52, 54, 55, 57–59], as well as the equations and systems in references therein).

An interesting case is when the sequences q_n and f_n are periodic. Recall that a sequence $(a_n)_{n\in\mathbb{N}_0}$ is eventually periodic if there are $T \in \mathbb{N}$ and $n_0 \in \mathbb{N}_0$, such that

$$a_{n+T} = a_n, \quad \text{for} \quad n \geq n_0.$$

When $T = 1$ for the sequence is said that is eventually constant (see, for example, [13]). If $n_0 = 0$ then it is said that the sequence is T-periodic, although many authors understand that every eventually T-periodic sequence is T-periodic. Some basic facts on periodic solutions to equation (1), can be found in [12], while some more complex ones can be found in [3]. For some other results on periodicity and related topics, e.g., [6, 37, 39, 40, 53] and the references therein.

We may assume that sequences q_n and f_n are periodic with the same period, since if q_n is periodic with period T_1 and f_n is periodic with period T_2, then both sequences are periodic with period $T = \operatorname{lcm}(T_1, T_2)$ (the least common multiplier of natural numbers T_1 and T_2).

Since in this paper q_n will always denote a T-periodic sequence, from now on we will use the following notation

$$\lambda := \prod_{j=0}^{T-1} q_j.$$

Note that the T-periodicity of q_n, implies

$$\prod_{j=n}^{n+T-1} q_j = \lambda, \tag{3}$$

for every $n \in \mathbb{N}_0$.

In [3] is quoted the following result which essentially appears in [12].

Theorem 1. *Assume that $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are two T-periodic sequences. Then the following statements are true.*

(a) *If*

$$0 \neq \lambda \neq 1, \tag{4}$$

then (1) *has a unique T-periodic solution given by the initial condition*

$$x_0 = \frac{\sum_{i=0}^{T-1} f_i \prod_{j=i+1}^{T-1} q_j}{1-\lambda}.$$

(b) *If*

$$\lambda = 1 \quad and \quad \sum_{i=0}^{T-1} f_i \prod_{j=i+1}^{T-1} q_j = 0, \tag{5}$$

then all solutions to equation (1) *are T-periodic.*

(c) *If*

$$\lambda = 1 \quad and \quad \sum_{i=0}^{T-1} f_i \prod_{j=i+1}^{T-1} q_j \neq 0,$$

then equation (1) *has no T-periodic solutions.*

Remark 1. Difference equation (1) is a special case of the general first-order difference equation

$$x_{n+1} = f(n, x_n), \quad n \in \mathbb{N}_0, \tag{6}$$

where $f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$. Note simply that for the case of equation (1) we have

$$f(n, t) = q_n t + f_n,$$

for $n \in \mathbb{N}_0$ and $t \in \mathbb{R}$.

The following condition

$$f(n + T, x) = f(n, x), \tag{7}$$

where $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$, essentially corresponds to the T-periodicity of sequences q_n and f_n.

By a direct calculation from (6) we have

$$x_n = f(n-1, f(n-2, \dots f(1, f(0, x_0)) \dots)), \quad n \in \mathbb{N},$$

from which it follows that if $(\widehat{x_n})_{n\in\mathbb{N}_0}$ is a T-periodic solution to equation (6), then the following condition must hold

$$\widehat{x_T} = f(T-1, f(T-2, \dots f(1, f(0, \widehat{x_0})) \dots)) = \widehat{x_0}, \tag{8}$$

that is, the following nonlinear algebraic equation

$$f(T-1, f(T-2, \dots f(1, f(0, x)) \dots)) = x, \tag{9}$$

must have a solution.

On the other hand, if equation (9) has a solution, say $\widehat{x_0}$, then (8) holds. From this, (6), (7) and (8), it follows that

$$\widehat{x_{T+1}} = f(T, \widehat{x_T}) = f(0, \widehat{x_0}) = \widehat{x_1}.$$

Similarly, by an inductive argument, it follows that

$$\widehat{x_{n+T}} = \widehat{x_n},$$

for every $n \in \mathbb{N}_0$.

Hence, the following proposition holds.

Proposition 1. *Consider equation* (6)*, where* $f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$*, is a function satisfying condition* (7)*. Then, the equation has a T-periodic solution if and only if the nonlinear equation* (9) *has a solution.*

Since for the case of difference equation (1), by using formula (2) and (3) with $n = 0$, we have

$$f(T-1, f(T-2, \dots f(1, f(0, x_0)) \dots)) = \lambda x_0 + \sum_{i=0}^{T-1} f_i \prod_{j=i+1}^{T-1} q_j, \tag{10}$$

the first and third statements in Theorem 1 follow from Proposition 1, whereas if (5) holds, then from (10) we easily see that for every $x_0 \in \mathbb{R}$ we have

$$x_0 = x_T,$$

from which along with the periodicity of the sequences q_n and f_n the second statement in Theorem 1 follows.

By using (2) and (3), for every solution $(x_n)_{n\in\mathbb{N}_0}$ to difference equation (1), we have

$$\begin{aligned} x_{n+T} &= x_0 \prod_{j=0}^{n+T-1} q_j + \sum_{i=0}^{n+T-1} f_i \prod_{j=i+1}^{n+T-1} q_j \\ &= \lambda\Big(x_0 \prod_{j=0}^{n-1} q_j + \sum_{i=0}^{n-1} f_i \prod_{j=i+1}^{n-1} q_j\Big) + \sum_{i=n}^{n+T-1} f_i \prod_{j=i+1}^{n+T-1} q_j \\ &= \lambda x_n + c_n, \end{aligned}$$

for every $n \in \mathbb{N}_0$, where

$$c_n := \sum_{i=n}^{n+T-1} f_i \prod_{j=i+1}^{n+T-1} q_j. \tag{11}$$

Hence, if $\lambda = 0$, we have

$$x_{n+T} = c_n, \tag{12}$$

for $n \in \mathbb{N}_0$.

Now we quote an interesting lemma, whose special cases have been essentially used in the literature (for example, in [33, 50]).

Lemma 1. *Let $(q_n)_{n\in\mathbb{N}_0}$, $(f_n^{(j)})_{n\in\mathbb{N}_0}$, $j = \overline{1,p}$, be T-periodic sequences, and a sequence $(b_n)_{n\geq k}$ is defined by*

$$b_n := \sum_{j=n-k}^{n+l} \sum_{t=1}^{p} f_j^{(t)} \prod_{i=j+s}^{n+m} q_i, \quad n \geq k, \tag{13}$$

where $k, l, m, p, s \in \mathbb{N}_0$.

Then, the sequence $(b_n)_{n\geq k}$ is also T-periodic.

Proof. We have

$$\begin{aligned} b_{n+T} &= \sum_{j=n+T-k}^{n+T+l} \sum_{t=1}^{p} f_j^{(t)} \prod_{i=j+s}^{n+T+m} q_i = \sum_{j'=n-k}^{n+l} \sum_{t=1}^{p} f_{j'+T}^{(t)} \prod_{i=j'+T+s}^{n+T+m} q_i \\ &= \sum_{j'=n-k}^{n+l} \sum_{t=1}^{p} f_{j'}^{(t)} \prod_{i'=j'+s}^{n+m} q_{i'+T} = \sum_{j'=n-k}^{n+l} \sum_{t=1}^{p} f_{j'}^{(t)} \prod_{i'=j'+s}^{n+m} q_{i'} = b_n, \end{aligned}$$

for all $n \geq k$, from which the lemma follows. □

If in (13) is chosen $k = 0$, $l = T - 1$, $m = T - 1$, $p = 1$, $s = 1$ and $f_n^{(1)} = f_n$, $n \in \mathbb{N}_0$, then from Lemma 1 we obtain the following corollary.

Corollary 1. *Let $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ be two T-periodic sequences, and a sequence $(c_n)_{n\in\mathbb{N}_0}$ is defined by* (11). *Then, the sequence $(c_n)_{n\in\mathbb{N}_0}$ is also T-periodic.*

From Corollary 1 and (12) the following result follows.

Corollary 2. *Let $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ be two T-periodic sequences, and $\lambda = 0$. Then, every solution $(x_n)_{n\in\mathbb{N}_0}$ to equation* (1) *is eventually T-periodic.*

The relationship between the periodic and other solutions to equation (1) was not considered in [3]. The problem, among other ones, has been recently tackled in our paper [50], where we considered the equation not only on the domain $\mathbb{N}_0$, but on the set of all integers $\mathbb{Z}$, which is possible if $q_n \neq 0$, $n \leq -1$ (for the case of equation (6) such a condition is in general case impossible to find). For a class of linear second-order difference equations some results of this type, among other ones, have been recently proved in [49].

Definition. For a sequence $(a_n)_{n\geq l}$, $l \in \mathbb{Z}$, is said that it converges geometrically to a sequence $(\tilde{a}_n)_{n\geq l}$ if there are $L \geq 0$ and $q \in (0, 1)$, such that

$$|a_n - \tilde{a}_n| \leq Lq^n,$$

for $n \geq l$.

The following result, among other ones, was proved in [48].

Theorem 2. *Assume $(q_n)_{n\in\mathbb{Z}}$ and $(f_n)_{n\in\mathbb{Z}}$ are two T-periodic sequences, and that* (4) *holds. Then equation* (1) *has a unique T-periodic solution, and the following statements are true.*

(a) *If $0 < |\lambda| < 1$, then all the solutions to equation* (1) *converge geometrically to the periodic one as $n \to +\infty$, while they are getting away geometrically from the periodic one as $n \to -\infty$.*
(b) *If $|\lambda| > 1$, then all the solutions to equation* (1) *converge geometrically to the periodic one as $n \to -\infty$, while they are getting away geometrically from the periodic one as $n \to +\infty$.*

Remark 2. Note that the condition $\lambda \neq 0$ in Theorem 2 implies

$$q_n \neq 0, \quad \text{for} \quad n \in \mathbb{Z},$$

from which it follows that every solution to equation (1) is defined on the whole $\mathbb{Z}$, in this case.

On the other hand, in [31] were studied positive solutions to the following difference equation

$$x_{n+1} = q_n x_n + f(n, x_{n-k}), \quad n \in \mathbb{N}_0, \tag{14}$$

where $k \in \mathbb{N}_0$, $(q_n)_{n\in\mathbb{N}_0}$ is a positive T-periodic sequence, and $f : \mathbb{N}_0 \times [0, \infty) \to (0, \infty)$ is a T-periodic function in the first variable, which for each $n \in \{0, 1, \dots, T - 1\}$ is continuous in the second variable.

The main reason for studying only positive solutions to equation (14) in [31], is the fact that some special cases of the equation are discrete analogues of some models in biology. For example, the equation

$$x_{n+1} = ax_n + \frac{b}{1 + x_{n-k}^{\gamma}}, \quad n \in \mathbb{N}_0,$$

where

$$\min\{a, b, \gamma\} > 0$$

and $k \in \mathbb{N}_0$, is a discrete analogue of a model that has been used in studying of the blood cell production ([17]).

The following result was proved in [31].

Theorem 3. *Assume that $k \in \mathbb{N}_0$, $(q_n)_{n\in\mathbb{N}_0}$ is a positive T-periodic sequence such that $q_n \in (0,1]$, $n \in \mathbb{N}_0$, and $f : \mathbb{N}_0 \times [0,\infty) \to (0,\infty)$ is a T-periodic function in the first variable, which for each $n \in \{0,1,\dots,T-1\}$ is nonincreasing and continuous in the second variable and for each $n \in \{0,1,\dots,T-1\}$ there are nonnegative constants L_n such that*

$$|f(n,x) - f(n,y)| \le L_n|x-y|, \tag{15}$$

for every $x, y \in [0,\infty)$, and

$$\sum_{j=n}^{n+k} L_j \prod_{i=j+1}^{n+k} q_i < 1, \tag{16}$$

for each $n \in \{0,1,\dots,T-1\}$.

Then equation (14) has a unique positive T-periodic solution $(\tilde{x}_n)_{n\ge-k}$, and every positive solution $(x_n)_{n\ge-k}$ to the equation satisfies

$$\lim_{n\to\infty}(\tilde{x}_n - x_n) = 0.$$

Note that the main difference between Theorems 2 and 3, is that the later one considers only positive solutions to equation (14), unlike the former one which considers arbitrary solutions to equation (1). For some related results see [32]. It is a natural problem to get some related results to Theorem 3 for some more general equations. In the recent paper [33] a generalization of Theorem 3 has been given.

Motivated by the problem, as well as some results in [31, 33, 46, 48–50], here we consider the existence and global attractivity of periodic solutions to some subclasses of the following class of difference equations

$$x_{n+1} = q_n x_n + f(n, x_n, x_{n-1}, \dots, x_{n-k}), \quad n \in \mathbb{N}_0, \tag{17}$$

where $k \in \mathbb{N}_0$, $f : \mathbb{N}_0 \times \mathbb{R}^{k+1} \to \mathbb{R}$ is a function such that for each $n \in \mathbb{N}_0$ it is continuous in other $k+1$ variables and

$$f(n+T, t_1, \dots, t_{k+1}) = f(n, t_1, \dots, t_{k+1}),$$

for some $T \in \mathbb{N}$ and every $t_j \in \mathbb{R}$, $j = \overline{1, k+1}$.

Let $l_T^\infty(\mathbb{N}_m)$, where $m \in \mathbb{Z} \setminus \mathbb{N}$, be the space of all T-periodic sequences $x = (x_n)_{n\ge m}$, with the following norm

$$\|x\|_{ps} = \max_{m\le n\le m+T-1} |x_n|. \tag{18}$$

It is known that $l_T^\infty(\mathbb{N}_m)$ with norm (18) is a Banach space.

One of the standard methods for showing the existence of a specific type of solutions to difference equations is application of fixed-point theorems. Beside the contraction mapping theorem, which was formulated and proved in [4] (some interesting applications of the theorem have been recently presented in [46], [50] and [60]), one of the fixed-point theorems which is frequently applied is the Schauder fixed point theorem ([35]). For some applications of the theorem, see, for example [9, 10, 20, 21, 33] and the related references therein.

2. Main results

This section states and proves the main results in this paper. The first result is about the existence of periodic solutions to equation (17) and is motivated by Theorem 1 in [33].

Theorem 4. *Assume $(q_n)_{n\in\mathbb{N}_0}$ is a T-periodic sequence such that $\lambda \neq 0,1$, function $f : \mathbb{N}_0 \times [a,b]^{k+1} \to \mathbb{R}$, $a,b \in \mathbb{R}$, is continuous on $[a,b]^{k+1}$ for each $n \in \mathbb{N}_0$, and is T-periodic in the first variable, and the following conditions*

$$\begin{aligned} a \le & \min_{n=\overline{0,T-1}} \min_{a\le t_j\le b, j=\overline{1,k+1}} \sum_{i=n}^{n+T-1} \frac{f(i,t_1,\dots,t_{k+1})}{1-\lambda} \prod_{j=i+1}^{n+T-1} q_j \\ \le & \max_{n=\overline{0,T-1}} \max_{a\le t_j\le b, j=\overline{1,k+1}} \sum_{i=n}^{n+T-1} \frac{f(i,t_1,\dots,t_{k+1})}{1-\lambda} \prod_{j=i+1}^{n+T-1} q_j \le b, \end{aligned} \tag{19}$$

hold.

Then, equation (17) *has a T-periodic solution.*

Proof. Let

$$A_{a,b} := \{(x_n)_{n\ge -k} \in l_T^\infty(\mathbb{N}_{-k}) : a \le x_n \le b, n \ge -k\}.$$

It is easy to see that $A_{a,b}$ is a convex and compact subset of the linear space $l_T^\infty(\mathbb{N}_{-k})$.

Using (2) and (3), we have that for every solution $(x_n)_{n\ge -k}$ to equation (17) holds

$$\begin{aligned} x_{n+T} =& x_0 \prod_{j=0}^{n+T-1} q_j + \sum_{i=0}^{n+T-1} f(i,x_i,\dots,x_{i-k}) \prod_{j=i+1}^{n+T-1} q_j \\ =& \lambda\Big(x_0 \prod_{j=0}^{n-1} q_j + \sum_{i=0}^{n-1} f(i,x_i,\dots,x_{i-k}) \prod_{j=i+1}^{n-1} q_j\Big) + \sum_{i=n}^{n+T-1} f(i,x_i,\dots,x_{i-k}) \prod_{j=i+1}^{n+T-1} q_j \\ =& \lambda x_n + d_n, \end{aligned} \tag{20}$$

where

$$d_n := \sum_{i=n}^{n+T-1} f(i,x_i,\dots,x_{i-k}) \prod_{j=i+1}^{n+T-1} q_j, \tag{21}$$

for $n \in \mathbb{N}_0$.

If $(x_n)_{n\ge -k}$ is a T-periodic sequence, then the following sequence

$$\widetilde{f_n} := f(n,x_n,\dots,x_{n-k}), \quad n \in \mathbb{N}_0,$$

is also T-periodic. Indeed, by using the T-periodicity of function f in the first variable and T-periodicity of $(x_n)_{n\ge -k}$, we have

$$\widetilde{f_{n+T}} = f(n+T,x_{n+T},\dots,x_{n+T-k}) = f(n,x_n,\dots,x_{n-k}) = \widetilde{f_n},$$

for every $n \in \mathbb{N}_0$.

Since sequences q_n and $\widetilde{f_n}$ are T-periodic, by Lemma 1 it follows that for every T-periodic sequence $(x_n)_{n\ge -k}$ the corresponding sequence d_n defined in (21) is also T-periodic.

Let $\widehat{T}$ be an operator on the set $A_{a,b}$ defined by

$$\widehat{T}x_n = \frac{\sum_{i=n}^{n+T-1} f(i,x_i,\dots,x_{i-k}) \prod_{j=i+1}^{n+T-1} q_j}{1-\lambda}, \tag{22}$$

for $n \in \mathbb{N}_0$.

Then, from (19) and (22), we have

$$a \leq \widehat{T}x_n \leq b, \tag{23}$$

for every $n \in \mathbb{N}_0$ and $(x_n)_{n\geq -k} \in A_{a,b}$. The periodicity of the sequence $(\widehat{T}x_n)_{n\in\mathbb{N}_0}$ follows from the T-periodicity of the sequence defined in (21). Hence, $\widehat{T}(A_{a,b}) \subseteq A_{a,b}$.

Since for each $n \in \{0, 1, \dots, T-1\}$, $f(n, t_1, \dots, t_{k+1})$ is continuous on the set $[a, b]^{k+1}$, it is not difficult to see, by using the standard ε-δ technique, that the operator $\widehat{T}$ is continuous on the set $A_{a,b}$.

By using the Schauder fixed point theorem, operator $\widehat{T}$ has a fixed point in the set $A_{a,b}$, that is, there is a sequence $(\widehat{x}_n)_{n\geq -k} \in A_{a,b}$ such that

$$\widehat{T}\widehat{x}_n = \widehat{x}_n, \quad \text{for} \quad n \in \mathbb{N}_0,$$

which can be written as

$$\widehat{x}_n = \frac{\sum_{i=n}^{n+T-1} f(i, \widehat{x}_i, \dots, \widehat{x}_{i-k}) \prod_{j=i+1}^{n+T-1} q_j}{1-\lambda},$$

for $n \in \mathbb{N}_0$.

Since sequences q_n and $\widehat{x}_n$ are T-periodic, as well as the function f in the first variable, we have

$$\begin{aligned}
\widehat{x}_{n+1} &= \frac{\sum_{i=n+1}^{n+T} f(i, \widehat{x}_i, \dots, \widehat{x}_{i-k}) \prod_{j=i+1}^{n+T} q_j}{1-\lambda} \\
&= \frac{q_{n+T} \sum_{i=n}^{n+T-1} f(i, \widehat{x}_i, \dots, \widehat{x}_{i-k}) \prod_{j=i+1}^{n+T-1} q_j}{1-\lambda} \\
&\quad + \frac{f(n+T, \widehat{x}_{n+T}, \dots, \widehat{x}_{n+T-k}) - f(n, \widehat{x}_n, \dots, \widehat{x}_{n-k}) \prod_{j=n+1}^{n+T} q_j}{1-\lambda} \\
&= q_n\widehat{x}_n + f(n, \widehat{x}_n, \dots, \widehat{x}_{n-k}),
\end{aligned} \tag{24}$$

which shows that the sequence $(\widehat{x}_n)_{n\geq -k}$ is a T-periodic solution to difference equation (17). □

Remark 3. The introduction of the operator defined in (22) is quite natural and is a folklore thing. Namely, if a solution $(x_n)_{n\geq -k}$ to equation (17) is T-periodic, then from (20) it follows that, in the case $\lambda \neq 1$, it must be

$$x_n = \frac{d_n}{1-\lambda}, \quad n \in \mathbb{N}_0,$$

which strikingly suggests the introduction of the operator.

Theorem 5. *Consider the following difference equation*

$$x_{n+1} = q_n x_n + f(n, x_n), \quad n \in \mathbb{N}_0, \tag{25}$$

where $(q_n)_{n\in\mathbb{N}_0}$ is a positive T-periodic sequence such that $\lambda < 1$, $f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ is a T-periodic function in the first variable, which for each $n \in \{0, 1, \dots, T-1\}$ is nonincreasing in the second variable, and that the functions

$$g_l(x) = q_l x + f(l, x), \tag{26}$$

are nondecreasing for each $l \in \{0, 1, \dots, T-1\}$.

If equation (25) *has a T-periodic solution, then every solution to the equation converges to the periodic one geometrically as $n \to +\infty$.*

Proof. First note that if $x_0 \leq \widehat{x}_0$, then from (25) and the monotonicity of function $g_1(x)$, it follows that

$$x_1 = q_1 x_0 + f(1, x_0) \leq q_1\widehat{x}_0 + f(1, \widehat{x}_0) = \widehat{x}_1. \tag{27}$$

Assume that we have proved

$$x_n \le \widehat{x}_n, \tag{28}$$

for some $n \in \mathbb{N}$, and let $n = mT + l$, where $m \in \mathbb{N}_0$ and $l \in \{0, 1, \dots, T-1\}$. Then from the assumption and the monotonicity of function $g_l(x)$, T-periodicity of the sequence q_n and function f in the first variable, it follows that

$$\begin{aligned} x_{n+1} &= q_n x_n + f(n, x_n) = q_l x_n + f(l, x_n) \\ &\le q_l \widehat{x}_n + f(l, \widehat{x}_n) = q_n \widehat{x}_n + f(n, \widehat{x}_n) = \widehat{x}_{n+1}. \end{aligned} \tag{29}$$

From (27), (29) and by the induction, it follows that (28) holds for every $n \in \mathbb{N}_0$.

Now assume that $(\tilde{x}_n)_{n\in\mathbb{N}_0}$ is a T-periodic solution to equation (25), and that $(x_n)_{n\in\mathbb{N}_0}$ is another solution to the equation.

First, assume that $x_0 \le \widetilde{x}_0$. Then, by (28) we have

$$x_n \le \widetilde{x}_n, \tag{30}$$

for every $n \in \mathbb{N}_0$.

If there is $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = \widetilde{x}_{n_0}$, then from (25) is obtained that $x_n = \widetilde{x}_n$ for $n \ge n_0$, from which the result follows in this case.

Otherwise, from (30), we have

$$y_n := \widetilde{x}_n - x_n > 0, \tag{31}$$

for every $n \in \mathbb{N}_0$.

Since $\widetilde{x}_n$ is a solution to equation (25), we have

$$x_{n+1} + y_{n+1} = q_n(x_n + y_n) + f(n, x_n + y_n), \quad n \in \mathbb{N}_0. \tag{32}$$

Using (25) in (32), it follows that

$$y_{n+1} = q_n y_n + f(n, y_n + x_n) - f(n, x_n), \tag{33}$$

for $n \in \mathbb{N}_0$.

From (33) along with the monotonicity of $f(n,t)$ in the second variable and (31), it follows that

$$y_{n+1} \le q_n y_n, \tag{34}$$

for every $n \in \mathbb{N}_0$, from which it follows that

$$0 < y_n \le y_0 \prod_{j=0}^{n-1} q_j, \quad n \in \mathbb{N}_0. \tag{35}$$

Hence, if $n = mT + l$ for some $m \in \mathbb{N}_0$ and $l \in \{0, 1, \dots, T-1\}$, form (35) and by using (3), we get

$$\begin{aligned} 0 < \tilde{x}_n - x_n = y_n = y_{mT+l} &\le y_0 \prod_{j=0}^{mT+l-1} q_j \\ &= y_0 \left(\prod_{j=0}^{l-1} q_j \right) \lambda^m = y_0 \left(\prod_{j=0}^{l-1} \frac{q_j}{\sqrt[T]{\lambda}} \right) (\sqrt[T]{\lambda})^n \\ &\le (\tilde{x}_0 - x_0) \max_{l=\overline{1,T}} \left(\prod_{j=0}^{l-1} \frac{q_j}{\sqrt[T]{\lambda}} \right) (\sqrt[T]{\lambda})^n, \end{aligned} \tag{36}$$

for every $n \in \mathbb{N}_0$.

Since $\lambda \in (0,1)$, by letting $n \to +\infty$ in (36) the result follows in this case.

Now assume that $\widetilde{x}_0 < x_0$. Then, by (28) we have

$$\widetilde{x}_n \le x_n, \tag{37}$$

for every $n \in \mathbb{N}_0$.

If there is $n_1 \in \mathbb{N}$, such that $\widetilde{x}_{n_1} = x_{n_1}$, then as above we have $x_n = \widetilde{x}_n$ for $n \ge n_1$, from which the result follows in this case.

Otherwise, from (37), we have

$$y_n := \widetilde{x}_n - x_n < 0, \tag{38}$$

for every $n \in \mathbb{N}_0$, and that (33) holds, from which along with (38) and the monotonicity of $f(n,t)$ in the second variable, it follows that

$$q_n y_n \le y_{n+1}, \tag{39}$$

for every $n \in \mathbb{N}_0$.

From (38) and (39), we have

$$0 < (-y_n) \le (-y_0) \prod_{j=0}^{n-1} q_j, \quad n \in \mathbb{N}_0. \tag{40}$$

Hence, if $n = mT + l$ for some $m \in \mathbb{N}_0$ and $l \in \{0, 1, \dots, T-1\}$, then form (40), as in (36), we get

$$\begin{aligned} 0 < x_n - \widetilde{x}_n = (-y_{mT+l}) &\le (-y_0) \prod_{j=0}^{mT+l-1} q_j \\ &\le (x_0 - \widetilde{x}_0) \max_{l=\overline{1,T}} \left(\prod_{j=0}^{l-1} \frac{q_j}{\sqrt[T]{\lambda}} \right) (\sqrt[T]{\lambda})^n, \end{aligned} \tag{41}$$

for every $n \in \mathbb{N}_0$.

Since $\lambda \in (0,1)$, by letting $n \to +\infty$ in (41) the result follows in this case, finishing the proof of the theorem. □

Remark 4. Since in Theorem 5 any solution to equation (25) converges to the chosen periodic one, it follows that the periodic solution is unique.

From Theorems 4 and 5 the following corollary follows.

Corollary 3. *Consider equation* (25)*, where* $(q_n)_{n\in\mathbb{N}_0}$ *is a positive* T*-periodic sequence such that* $\lambda < 1$*,* $f : \mathbb{N}_0 \times [a,b] \to \mathbb{R}$*,* $a, b \in \mathbb{R}$*, is a* T*-periodic function in the first variable, which for each* $n \in \{0, 1, \dots, T-1\}$ *is a continuous and nonincreasing in the second variable, the functions in* (26) *are nondecreasing for each* $l \in \{0, 1, \dots, T-1\}$*, and that the following conditions are satisfied*

$$a \le \min_{n=\overline{0,T-1}} \min_{a\le t\le b} \sum_{i=n}^{n+T-1} \frac{f(i,t)}{1-\lambda} \prod_{j=i+1}^{n+T-1} q_j \le \max_{n=\overline{0,T-1}} \max_{a\le t\le b} \sum_{i=n}^{n+T-1} \frac{f(i,t)}{1-\lambda} \prod_{j=i+1}^{n+T-1} q_j \le b. \tag{42}$$

Then, equation (25) *has a unique* T*-periodic solution* $(\widetilde{x}_n)_{n\in\mathbb{N}_0}$*, such that*

$$a \le \widetilde{x}_n \le b, \quad n \in \mathbb{N}_0, \tag{43}$$

and every solution to the equation converges to the periodic one geometrically as $n \to +\infty$*.*

Proof. From Theorem 4 it follows that equation (25) has a T-periodic solution. By Theorem 5 it follows that every solution to the equation converges to the periodic one geometrically as $n \to +\infty$, from which it follows that it is unique (Remark 4). □

Example 1. Here we present an example of equation (25) which satisfies the conditions in Theorem 5. Let $(q_n)_{n\in\mathbb{N}_0}$ be a positive T-periodic sequence such that $\lambda < 1$, and

$$f(n,x) = -\frac{\widetilde{q_l}x}{1+|x|}, \tag{44}$$

where $n = Tm + l$, for some $m \in \mathbb{N}_0$, $l \in \{0,\dots,T-1\}$, $0 < \widetilde{q_l} < q_l$, for every $l \in \{0,\dots,T-1\}$. It is clear from (44) that f is decreasing for each $l \in \{0,\dots,T-1\}$, and consequently for each $n \in \mathbb{N}_0$, and that by definition it is T-periodic in the first variable. On the other hand, since

$$g(n,x) = q_n x + f(n,x) = \frac{q_n x|x| + (q_n - \widetilde{q_n})x}{1+|x|},$$

it is easy to see that g is increasing in the second variable for each $n \in \mathbb{N}_0$.

Now note that the corresponding difference equation

$$x_{n+1} = \frac{q_n x_n|x_n| + (q_n - \widetilde{q_n})x_n}{1+|x_n|}, \quad n \in \mathbb{N}_0, \tag{45}$$

has the trivial solution

$$x_n = 0, \quad n \in \mathbb{N}_0, \tag{46}$$

which is periodic of all periods, hence, it is also T-periodic. Since, all the conditions of Theorem 5 are satisfied, it follows that all the solutions to equation (45) converge geometrically to solution (46) as $n \to +\infty$.

Theorem 6. *Consider equation* (25), *where* $(q_n)_{n\in\mathbb{Z}}$ *is a positive T-periodic sequence such that* $\lambda > 1$, *and* $f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ *is a T-periodic function in the first variable, which for each* $n \in \{0,1,\dots,T-1\}$ *is nondecreasing in the second variable.*

If equation (25) *has a T-periodic solution, then each other solution to the equation is getting away geometrically from the periodic one as* $n \to +\infty$.

Proof. Since $q_l > 0$ for every $l \in \{0,1,\dots,T-1\}$, this fact along with the monotonicity of the function $f(l,x)$ in the second variable implies that the functions defined in (26) are strictly increasing on $\mathbb{R}$. Hence, by an argument as in the proof of Theorem 5 we see that if $x_0 \le \widehat{x_0}$, then

$$x_n \le \widehat{x_n} \tag{47}$$

for every $n \in \mathbb{N}_0$, and that in (47) strict inequality holds if and only if

$$x_0 < \widehat{x_0}.$$

Let $(\widetilde{x_n})_{n\in\mathbb{N}_0}$ be the T-periodic solution to (25) and $(x_n)_{n\in\mathbb{N}_0}$ be another solution to the equation.

First, assume that $x_0 < \widetilde{x_0}$. Then, if $y_n := \widetilde{x_n} - x_n$, we have $y_n > 0$, for every $n \in \mathbb{N}_0$. From (25) and (32), we get (33), from which along with the monotonicity of $f(n,x)$ in the second variable, it follows that

$$y_{n+1} \ge q_n y_n, \tag{48}$$

for every $n \in \mathbb{N}_0$, and consequently

$$y_n \ge y_0 \prod_{j=0}^{n-1} q_j, \tag{49}$$

for $n \in \mathbb{N}_0$.

Hence, if $n = mT + l$ for some $m \in \mathbb{N}_0$ and $l \in \{0, 1, \dots, T-1\}$, form inequality (49) we obtain

$$\begin{aligned}\tilde{x}_n - x_n = y_n &\geq y_0 \left(\prod_{j=0}^{l-1} q_j\right) \lambda^m \\ &= y_0 \left(\prod_{j=0}^{l-1} \frac{q_j}{\sqrt[T]{\lambda}}\right) (\sqrt[T]{\lambda})^n \\ &\geq (\widetilde{x}_0 - x_0) \min_{l=\overline{1,T}} \left(\prod_{j=0}^{l-1} \frac{q_j}{\sqrt[T]{\lambda}}\right) (\sqrt[T]{\lambda})^n, \end{aligned} \tag{50}$$

for every $n \in \mathbb{N}_0$.

Since $\lambda > 1$, and

$$\min_{l=\overline{1,T}} \left(\prod_{j=0}^{l-1} \frac{q_j}{\sqrt[T]{\lambda}}\right) > 0, \tag{51}$$

by letting $n \to +\infty$ in (50) the result follows in this case.

Now, assume that $\widetilde{x}_0 < x_0$. Then, we have $y_n < 0$, for every $n \in \mathbb{N}_0$. From this along with (33), and the monotonicity of $f(n, x)$ in the second variable, it follows that

$$y_{n+1} \leq q_n y_n, \tag{52}$$

for every $n \in \mathbb{N}_0$, and consequently

$$(-y_n) \geq (-y_0) \prod_{j=0}^{n-1} q_j, \tag{53}$$

for $n \in \mathbb{N}_0$.

Hence, if $n = mT + l$ for some $m \in \mathbb{N}_0$ and $l \in \{0, 1, \dots, T-1\}$, form (53), as above, we have

$$\begin{aligned}x_n - \widetilde{x}_n = (-y_n) &\geq (-y_0) \left(\prod_{j=0}^{l-1} q_j\right) \lambda^m = (-y_0) \left(\prod_{j=0}^{l-1} \frac{q_j}{\sqrt[T]{\lambda}}\right) (\sqrt[T]{\lambda})^n \\ &\geq (x_0 - \widetilde{x}_0) \min_{l=\overline{1,T}} \left(\prod_{j=0}^{l-1} \frac{q_j}{\sqrt[T]{\lambda}}\right) (\sqrt[T]{\lambda})^n, \end{aligned} \tag{54}$$

for every $n \in \mathbb{N}_0$.

Since $\lambda > 1$ and (51) holds, by letting $n \to +\infty$ in (54) the result follows in this case, completing the proof of the theorem. □

Remark 5. Since in Theorem 6 any other solution to difference equation (25) is getting away geometrically from the periodic one, it follows that the periodic solution is unique.

Combining Theorem 4 and Theorem 6 it is easy to see that the following corollary holds.

Corollary 4. *Consider difference equation* (25)*, where* $(q_n)_{n \in \mathbb{N}_0}$ *is a positive T-periodic sequence such that* $\lambda > 1$*,* $f : \mathbb{N}_0 \times [a, b] \to \mathbb{R}$*,* $a, b \in \mathbb{R}$*, is a T-periodic function in the first variable, which for each* $n \in \{0, 1, \dots, T-1\}$ *is continuous and nondecreasing in the second variable, and that the conditions in* (42) *are satisfied. Then, the equation has a unique T-periodic solution* $(\widetilde{x}_n)_{n \in \mathbb{N}_0}$ *satisfying* (43)*, and each other solution to the equation is getting away geometrically from the periodic one as* $n \to +\infty$*.*

The following theorem is proved similarly to Theorem 6, thus the proof is omitted.

Theorem 7. *Consider equation* (17)*, where* $(q_n)_{n\in\mathbb{N}_0}$ *is a positive T-periodic sequence such that* $\lambda > 1$, $f : \mathbb{N}_0\times\mathbb{R}^{k+1} \to \mathbb{R}$ *is a T-periodic function in the first variable, which for each* $n \in \{0, 1, \dots, T-1\}$ *is nondecreasing in the other* $k+1$ *variables.*

If equation (17) *has a T-periodic solution* $(\tilde{x}_n)_{n\in\mathbb{N}_0}$*, then each other solution to the equation which does not oscillate about* $(\tilde{x}_n)_{n\in\mathbb{N}_0}$ *is getting away geometrically from the periodic one as* $n \to +\infty$*.*

Remark 6. It is interesting to note that if we consider equation (17), where $(q_n)_{n\in\mathbb{N}_0}$ is a positive T-periodic sequence such that $\lambda < 1$, $f : \mathbb{N}_0 \times \mathbb{R}^{k+1} \to \mathbb{R}$ is a T-periodic function in the first variable, which for each $n \in \{0, 1, \dots, T-1\}$ is nonincreasing in the other $k+1$ variables, and if we assume that the functions

$$g_l(t_1, \dots, t_{k+1}) = q_l t_1 + f(l, t_1, \dots, t_{k+1}), \tag{55}$$

are nondecreasing in variables t_j, $j = \overline{1, k+1}$, for each $l \in \{0, 1, \dots, T-1\}$, then if equation (17) has a T-periodic solution $(\tilde{x}_n)_{n\in\mathbb{N}_0}$, each other solution to the equation which does not oscillate about $(\tilde{x}_n)_{n\in\mathbb{N}_0}$ will converge to the periodic one geometrically as $n \to +\infty$. The result looks like a generalization of Theorem 5 for such solutions. However, the posed conditions on the monotonicity of functions f and g_l, $l = \overline{1, k+1}$, imply that the function f is constant in variables $t_2, \dots, t_{k+1}$, so that the result does not improve essentially Theorem 5 for such solution.

From the same reason, it is not possible to extend Theorem 3 in [31], in this way. For example, the following result, which is motivated also by Theorem 2 in [33] holds, but it is not an essential improvement of the theorems.

Theorem 8. *Assume* $k \in \mathbb{N}_0$, $(q_n)_{n\in\mathbb{N}_0}$ *is a positive T-periodic sequence such that* $\lambda < 1$, $f : \mathbb{N}_0 \times \mathbb{R}^{k+1} \to \mathbb{R}$ *is a T-periodic function in the first variable, which for each* $n \in \{0, 1, \dots, T-1\}$ *is nonincreasing in the other* $k+1$ *variables, the functions in* (55) *are nondecreasing in variables* t_j, $j = \overline{1, k+1}$*, for each* $l \in \{0, 1, \dots, T-1\}$*, there are nonnegative T-periodic sequences* $(L_n^{(l)})_{n\in\mathbb{N}_0}$, $l = \overline{1, k+1}$*, such that the following condition holds*

$$|f(n, t_1, \dots, t_{k+1}) - f(n, \widetilde{t_1}, \dots, \widetilde{t}_{k+1})| \le \sum_{l=1}^{k+1} L_n^{(l)} |t_l - \widetilde{t_l}|, \tag{56}$$

for every $t_j, \widetilde{t_j} \in [0, \infty)$, $j = \overline{1, k+1}$*, and*

$$\max_{j=\overline{0,T-1}} q_j \le 1 \quad \text{and} \quad \max_{n=\overline{0,T-1}} \sum_{j=n}^{n+k} \sum_{l=1}^{k+1} L_j^{(l)} \prod_{i=j+1}^{n+k} q_i < 1, \tag{57}$$

or

$$\max_{n=\overline{0,T-1}} \sum_{j=n}^{n+k+T-1} \sum_{l=1}^{k+1} L_j^{(l)} \prod_{i=j+1}^{n+k+T-1} q_i < 1. \tag{58}$$

If equation (17) *has a positive T-periodic solution* $(\widetilde{x}_n)_{n\ge -k}$*, then for every solution* $(x_n)_{n\ge -k}$ *to the equation the following relation holds*

$$\lim_{n\to\infty} (\widetilde{x}_n - x_n) = 0. \tag{59}$$

References

[1] D. Adamović, Problem 194, *Mat. Vesnik*, **22** (2) (1970), 270.
[2] D. Adamović, Solution to problem 194, *Mat. Vesnik*, **23** (1971), 236-242.
[3] R. P. Agarwal and J. Popenda, Periodic solutions of first order linear difference equations, *Math. Comput. Modelling* **22** (1) (1995), 11-19.
[4] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* **3** (1922), 133-181.
[5] L. Berg and S. Stević, On some systems of difference equations, *Appl. Math. Comput.* **218** (2011), 1713-1718.
[6] G. Boole, *A Treatsie on the Calculus of Finite Differences*, Third Edition, Macmillan and Co., London, 1880.
[7] L. Brand, A sequence defined by a difference equation, *Amer. Math. Monthly* **62** (7) (1955), 489-492.
[8] L. Brand, *Differential and Difference Equations*, John Wiley & Sons, Inc. New York, 1966.
[9] J. Diblik and E. Schmeidel, On the existence of solutions of linear Volterra difference equations asymptotically equivalent to a given sequence, *Appl. Math. Comput.* **218** (2012) 9310-9320.
[10] A. Drozdowicz and J. Popenda, Asymptotic behavior of the solutions of the second order difference equation, *Proc. Amer. Math. Soc.* **99** (1) (1987), 135140.
[11] L. Eulero, *Introductio in Analysin Infinitorum, Tomus Primus*, Lausannae, 1748. (in Latin)
[12] T. Fort, *Finite Differences and Difference Equations in the Real Domain*, Oxford Univ. Press, London, 1948.
[13] B. Iričanin and S. Stević, Eventually constant solutions of a rational difference equation, *Appl. Math. Comput.* **215** (2009), 854-856.
[14] C. Jordan, *Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1956.
[15] V. A. Krechmar, *A Problem Book in Algebra*, Mir Publishers, Moscow, 1974.
[16] H. Levy and F. Lessman, *Finite Difference Equations*, Dover Publications, Inc., New York, 1992.
[17] M. C. Mackey and L. Glass, Oscillation and chaos in physiological control systems, *Science* **197** (1977) 287-289.
[18] A. A. Markoff, *Differenzenrechnung*, Leipzig, B.G. Teubner 1896 (in German).
[19] A. A. Markov, *Ischislenie Konechnykh Raznostey*, 2 izdanie, Odessa, 1910. (in Russian)
[20] J. Migda, Approximative solutions of difference equations, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2014, Article No. 13, (2014), 26 pages.
[21] M. Migda, E. Schmeidel and M. Zdanowicz, Bounded solutions of k-dimensional system of nonlinear difference equations of neutral type, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2015, Article No. 80, (2015), 17 pages.
[22] L. M. Milne-Thomson, *The Calculus of Finite Differences*, MacMillan and Co., London, 1933.
[23] D. S. Mitrinović, *Mathematical Induction, Binomial Formula, Combinatorics*, Gradjevinska Knjiga, Beograd, Serbia, 1980. (in Serbian)
[24] D. S. Mitrinović and D. D. Adamović, *Sequences and Series*, Naučna Knjiga, Beograd, 1980 (in Serbian).
[25] D. S. Mitrinović and J. D. Kečkić, *Methods for Calculating Finite Sums*, Naučna Knjiga, Beograd, 1984 (in Serbian).
[26] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, Springer, 1924. (in German)
[27] G. Papaschinopoulos and C. J. Schinas, Invariants for systems of two nonlinear difference equations. *Differ. Equ. Dyn. Syst.* **7** (1999), 181–196.
[28] G. Papaschinopoulos and C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations. *Nonlinear Anal. Theory Methods Appl.* **46** (2001), 967–978.
[29] G. Papaschinopoulos, C. J. Schinas, and G. Stefanidou, On a k-order system of Lyness-type difference equations, *Adv. Difference Equ.* Vol. 2007, Article ID 31272, (2007), 13 pages.
[30] G. Papaschinopoulos and G. Stefanidou, Asymptotic behavior of the solutions of a class of rational difference equations, *Inter. J. Difference Equations* **5** (2) (2010), 233-249.
[31] C. Qian, Global attractivity of periodic solutions in a higher order difference equation, *Appl. Math. Lett.* **26** (2013), 578-583.
[32] C. Qian, Global attractivity in a nonlinear difference equation and applications to a biological model, *Int. J. Difference Equ.* **9** (2014), 233-242.
[33] C. Qian and J. Smith, Existence and global attractivity of periodic solutions in a higher order difference equation, *Arch. Math. (Brno)* **54** (2) (2018), 91-110.
[34] C. H. Richardson, *An Introduction to the Calculus of Finite Differences*, D. Van Nostrand Company Inc. Toronto, New York, London, 1954.
[35] J. Schauder, Der fixpunktsatz in funktionalräumen, *Studia Math.* **2** (1930), 171-180.
[36] C. Schinas, Invariants for some difference equations, *J. Math. Anal. Appl.* **212** (1997) 281-291.
[37] S. Stević, On a generalized max-type difference equation from automatic control theory, *Nonlinear Anal. TMA* **72** (2010), 1841-1849.
[38] S. Stević, On the difference equation $x_n = x_{n-2}/(b_n + c_n x_{n-1} x_{n-2})$, *Appl. Math. Comput.* **218** (2011), 4507-4513.
[39] S. Stević, Periodicity of a class of nonautonomous max-type difference equations, *Appl. Math. Comput.* **217** (2011), 9562-9566.
[40] S. Stević, On some periodic systems of max-type difference equations, *Appl. Math. Comput.* **218** (2012), 11483-11487.
[41] S. Stević, On the difference equation $x_n = x_{n-k}/(b + c x_{n-1} \cdots x_{n-k})$, *Appl. Math. Comput.* **218** (2012), 6291-6296.
[42] S. Stević, On the system of difference equations $x_n = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$, *Appl. Math. Comput.* **219** (2013), 4755-4764.
[43] S. Stević, On the system $x_{n+1} = y_n x_{n-k}/(y_{n-k+1}(a_n + b_n y_n x_{n-k}))$, $y_{n+1} = x_n y_{n-k}/(x_{n-k+1}(c_n + d_n x_n y_{n-k}))$, *Appl. Math. Comput.* **219** (2013), 4526-4534.
[44] S. Stević, First-order product-type systems of difference equations solvable in closed form, *Electron. J. Differential Equations* Vol. 2015, Article No. 308, (2015), 14 pages.
[45] S. Stević, Product-type system of difference equations of second-order solvable in closed form, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2015, Article No. 56, (2015), 16 pages.

[46] S. Stević, On periodic solutions of a class of k-dimensional systems of max-type difference equations, *Adv. Difference Equ.* Vol. 2016, Article No. 251, (2016), 10 pages.

[47] S. Stević, Solvable subclasses of a class of nonlinear second-order difference equations, *Adv. Nonlinear Anal.* **5** (2) (2016), 147-165.

[48] S. Stević, Bounded and periodic solutions to the linear first-order difference equation on the integer domain, *Adv. Difference Equ.* Vol. 2017, Article No. 283, (2017), 17 pages.

[49] S. Stević, Bounded solutions to nonhomogeneous linear second-order difference equations, *Symmetry* **9**, Article No. 227, (2017), 31 pages.

[50] S. Stević, Existence of a unique bounded solution to a linear second order difference equation and the linear first order difference equation, *Adv. Difference Equ.* Vol. 2017, Article No. 169, (2017), 13 pages.

[51] S. Stević, Solvable product-type system of difference equations whose associated polynomial is of the fourth order, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2017, Article No. 13, (2017), 29 pages.

[52] S. Stević, M. A. Alghamdi, A. Alotaibi and E. M. Elsayed, Solvable product-type system of difference equations of second order, *Electron. J. Differential Equations* Vol. 2015, Article No. 169, (2015), 20 pages.

[53] S. Stević, M. A. Alghamdi, A. Alotaibi and N. Shahzad, Boundedness character of a max-type system of difference equations of second order, *Electron. J. Qual. Theory Differ. Equ.* Vol. 2014, Article No. 45, (2014), 12 pages.

[54] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, On a third-order system of difference equations with variable coefficients, *Abstr. Appl. Anal.* Vol. 2012, Article ID 508523, (2012), 22 pages.

[55] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, On some solvable difference equations and systems of difference equations, *Abstr. Appl. Anal.* Vol. 2012, Article ID 541761, (2012), 11 pages.

[56] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, Solvability of nonlinear difference equations of fourth order, *Electron. J. Differential Equations* Vol. 2014, Article No. 264, (2014), 14 pages.

[57] S. Stević, B. Iričanin and Z. Šmarda, On a product-type system of difference equations of second order solvable in closed form, *J. Inequal. Appl.* Vol. 2015, Article No. 327, (2015), 15 pages.

[58] S. Stević, B. Iričanin and Z. Šmarda, Solvability of a close to symmetric system of difference equations, *Electron. J. Differential Equations* Vol. 2016, Article No. 159, (2016), 13 pages.

[59] S. Stević, B. Iričanin and Z. Šmarda, Two-dimensional product-type system of difference equations solvable in closed form, *Adv. Difference Equ.* Vol. 2016, Article No. 253, (2016), 20 pages.

[60] S. Stević, B. Iričanin and Z. Šmarda, Note on bounded solutions to a class of nonhomogenous linear difference equations, *Electron. J. Differential Equations* Vol. 2017, Article No. 286, (2017), 22 pages.

[61] N. N. Vorobiev, *Fibonacci Numbers*, Birkhäuser, Basel, 2002. (Russian first edition 1950)