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On Idempotency of Linear Combinations of a Quadratic or a Cubic Matrix and an Arbitrary Matrix

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Abstract. Let $\mathbf{A}$ be a quadratic or a cubic $n \times n$ nonzero matrix and $\mathbf{B}$ be an arbitrary $n \times n$ nonzero matrix. In this study, we have established necessary and sufficient conditions for the idempotency of the linear combinations of the form $a\mathbf{A} + b\mathbf{B}$, under the some certain conditions imposed on $\mathbf{A}$ and $\mathbf{B}$, where a, b are nonzero complex numbers.

1. Introduction and Preliminary Results

Let $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^{m\times n}$, and $\mathbb{C}^n$ denote the sets of complex numbers, nonzero complex numbers, all $m \times n$ complex matrices, and all $n \times n$ complex matrices, respectively. $\mathbf{0}$, $\mathbf{0}_n$, and $\mathbf{I}_n$ stand for a zero matrix of appropriate size, a zero matrix of order n, and an identity matrix of order n, respectively. The symbol $\oplus$ will denote the direct sum of matrices. Moreover, a matrix $\mathbf{A} \in \mathbb{C}^n$ is called an idempotent, an involutive, and an $\{\alpha, \beta\}-$ quadratic matrix if $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{A}^2 = \mathbf{I}_n$, and $(\mathbf{A} - \alpha\mathbf{I}_n)(\mathbf{A} - \beta\mathbf{I}_n) = \mathbf{0}$ with $\alpha, \beta \in \mathbb{C}$, respectively [1]. It is noteworthy that an idempotent and an involutive matrix are a $\{1, 0\}-$ quadratic matrix and a $\{1, -1\}-$ quadratic matrix, respectively. As in above, we will call a matrix $\mathbf{A} \in \mathbb{C}^n$ as an $\{\alpha, \beta, \gamma\}-$ cubic matrix if $(\mathbf{A} - \alpha\mathbf{I}_n)(\mathbf{A} - \beta\mathbf{I}_n)(\mathbf{A} - \gamma\mathbf{I}_n) = \mathbf{0}$ with $\alpha, \beta, \gamma \in \mathbb{C}$. Involutive, idempotent, tripotent, and quadratic matrices (that is, some special cases of cubic matrices) have been comprehensively studied in the literature (for example [1–6, 8–11]). Moreover, they have applications to digital image encryption [12].

Consider a linear combination of the form

$$\mathbf{K} = a\mathbf{A} + b\mathbf{B}, \;\; \mathbf{A}, \mathbf{B} \in \mathbb{C}^n, \;\; a, b \in \mathbb{C}^*. \tag{1}$$

Recently, under some conditions, it has been studied some problems of characterizing all situations where a linear combination of the form (1) is a special type of matrix when $\mathbf{A}$ and $\mathbf{B}$ are special types of matrices (see, for example, [2–4, 9–11]). Liu et al. characterize the involutiveness of the form (1) when $\mathbf{A}$ is a quadratic or a tripotent matrix and $\mathbf{B}$ is an arbitrary matrix [8].

The aim of this paper is to obtain the necessary and sufficient conditions for $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ to be an idempotent matrix, where $\mathbf{A}$ is a quadratic or a cubic matrix and $\mathbf{B}$ is an arbitrary matrix with some certain conditions. Moreover, some examples are given related to the obtained results.

It was established a useful expression for quadratic matrices in [9]. Now the following lemma, inspired by it, can be given for cubic matrices.

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Lemma 1.1. *Let $\mathbf{A} \in \mathbb{C}^n$. The following statements are equivalent.*

(a) *There exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$ and*

$$(\mathbf{A} - \alpha\mathbf{I}_n)(\mathbf{A} - \beta\mathbf{I}_n)(\mathbf{A} - \gamma\mathbf{I}_n) = \mathbf{0}. \tag{2}$$

(b) *$\mathbf{A}$ is diagonalizable and its spectrum $\sigma(\mathbf{A})$ is a subset of $\{\alpha, \beta, \gamma\}$.*

(c) *There exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$ and three idempotent matrices $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{C}^n$ such that $\mathbf{A} = \alpha\mathbf{X} + \beta\mathbf{Y} + \gamma\mathbf{Z}$, $\mathbf{X} + \mathbf{Y} + \mathbf{Z} = \mathbf{I}_n$, and $\mathbf{XY} = \mathbf{YX} = \mathbf{XZ} = \mathbf{ZX} = \mathbf{YZ} = \mathbf{ZY} = \mathbf{0}$.*

(d) *There exist $a, b, c \in \mathbb{C}$ and two commuting idempotent matrices $\mathbf{X}, \mathbf{Y}$ such that $a \neq 0$, $b \neq 0$ and $\mathbf{A} = a\mathbf{X} + b\mathbf{Y} + c\mathbf{I}_n$.*

Proof.

(a) ⇒ (b): It is clear from the fact that a matrix is diagonalizable if and only if every eigenvalue of it has multiplicity 1 as a zero of its minimal polynomial [7, Corollary 3.3.10].

(b) ⇒ (c): Let $\mathbf{A}$ be a diagonalizable matrix and $\sigma(\mathbf{A}) \subset \{\alpha, \beta, \gamma\}$, then there exists a nonsingular matrix $\mathbf{S} \in \mathbb{C}^n$ such that

$$\mathbf{A} = \mathbf{S}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_q \oplus \gamma\mathbf{I}_r\right)\mathbf{S}^{-1}$$

with $p, q, r \in \{0, \ldots, n\}$ and $p + q + r = n$. Let $\mathbf{X} = \mathbf{S}\left(\mathbf{I}_p \oplus \mathbf{0} \oplus \mathbf{0}\right)\mathbf{S}^{-1}$, $\mathbf{Y} = \mathbf{S}\left(\mathbf{0} \oplus \mathbf{I}_q \oplus \mathbf{0}\right)\mathbf{S}^{-1}$, and $\mathbf{Z} = \mathbf{S}\left(\mathbf{0} \oplus \mathbf{0} \oplus \mathbf{I}_r\right)\mathbf{S}^{-1}$. Observe that $\mathbf{A} = \alpha\mathbf{X} + \beta\mathbf{Y} + \gamma\mathbf{Z}$, $\mathbf{X} + \mathbf{Y} + \mathbf{Z} = \mathbf{I}_n$, and $\mathbf{XY} = \mathbf{YX} = \mathbf{XZ} = \mathbf{ZX} = \mathbf{YZ} = \mathbf{ZY} = \mathbf{0}$ as desired.

(c) ⇒ (d): Since $\mathbf{A} = \alpha\mathbf{X} + \beta\mathbf{Y} + \gamma\mathbf{Z}$ and $\mathbf{Z} = \mathbf{I}_n - \mathbf{X} - \mathbf{Y}$, we can write

$$\mathbf{A} = (\alpha - \gamma)\mathbf{X} + (\beta - \gamma)\mathbf{Y} + \gamma\mathbf{I}_n,$$

and the desired result is obtained by taking $a = \alpha - \gamma$, $b = \beta - \gamma$, and $c = \gamma$.

(d) ⇒ (a): Since $\mathbf{X}$ commutes with $\mathbf{Y}$ and they are idempotent, there exists a nonsingular matrix $\mathbf{S} \in \mathbb{C}^n$ such that $\mathbf{X} = \mathbf{S}\left(\mathbf{I}_p \oplus \mathbf{0} \oplus \mathbf{0}\right)\mathbf{S}^{-1}$ and $\mathbf{Y} = \mathbf{S}\left(\mathbf{0} \oplus \mathbf{I}_q \oplus \mathbf{0}\right)\mathbf{S}^{-1}$ with $rank(\mathbf{X}) = p$ and $rank(\mathbf{Y}) = q$ [7, Theorem 1.3.12]. So, it can be written

$$\begin{aligned}\mathbf{A} &= a\mathbf{S}\left(\mathbf{I}_p \oplus \mathbf{0} \oplus \mathbf{0}\right)\mathbf{S}^{-1} + b\mathbf{S}\left(\mathbf{0} \oplus \mathbf{I}_q \oplus \mathbf{0}\right)\mathbf{S}^{-1} + c\mathbf{S}\left(\mathbf{I}_p \oplus \mathbf{I}_q \oplus \mathbf{I}_{n-p-q}\right)\mathbf{S}^{-1} \\ &= \mathbf{S}\left((a + c)\mathbf{I}_p \oplus (b + c)\mathbf{I}_q \oplus c\mathbf{I}_{n-p-q}\right)\mathbf{S}^{-1}\end{aligned}$$

by the hypothesis. Let $\alpha = a + c$, $\beta = b + c$, and $\gamma = c$. Hence, we have

$$\mathbf{A} - \alpha\mathbf{I}_n = \mathbf{S}\left(\mathbf{0} \oplus (\beta - \alpha)\mathbf{I}_q \oplus (\gamma - \alpha)\mathbf{I}_{n-p-q}\right)\mathbf{S}^{-1},$$

$$\mathbf{A} - \beta\mathbf{I}_n = \mathbf{S}\left((\alpha - \beta)\mathbf{I}_p \oplus \mathbf{0} \oplus (\gamma - \beta)\mathbf{I}_{n-p-q}\right)\mathbf{S}^{-1},$$

and

$$\mathbf{A} - \gamma\mathbf{I}_n = \mathbf{S}\left((\alpha - \gamma)\mathbf{I}_p \oplus (\beta - \gamma)\mathbf{I}_q \oplus \mathbf{0}\right)\mathbf{S}^{-1}.$$

So, the proof is completed. □

Therefore some properties have been given for $\{\alpha, \beta, \gamma\}-$ cubic matrices. In view of the fact that a cubic matrix can be written as in (2), some results previously worked about special type of matrices can be generalized. Now we can give the main results.

2. Main Results

In this section, we will investigate the idempotency of the linear combination of the form (1), under some certain conditions. The following result, concerning with a cubic and an arbitrary matrix, is striking.

Theorem 2.1. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha \neq 0$, $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$. Moreover, let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an $\{\alpha, \beta, \gamma\}-$cubic matrix and an arbitrary matrix, respectively. Furthermore, let $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$ and $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ with $a, b \in \mathbb{C}^*$. Then $\mathbf{K}$ is an idempotent matrix if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma\mathbf{I}_{n-p-q} \end{pmatrix}\mathbf{V}^{-1}$$

and $\mathbf{B}$ satisfies one of the following cases.

(a) $\alpha = 1$, $\beta = 0$, *and* $a\gamma = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{\gamma-1}{\gamma b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1}{\gamma b}\mathbf{I}_{p-r} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{4_2} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{4_3} & b\mathbf{B}_{6_2}\mathbf{B}_{7_2} & \mathbf{0} & \mathbf{0}_{q-s} & \mathbf{B}_{6_2} \\ \mathbf{0} & \mathbf{B}_{7_2} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{B}_{4_2} \in \mathbb{C}^{s\times(p-r)}$, $\mathbf{B}_{4_3} \in \mathbb{C}^{(q-s)\times r}$, $\mathbf{B}_{6_2} \in \mathbb{C}^{(q-s)\times(n-p-q)}$, *and* $\mathbf{B}_{7_2} \in \mathbb{C}^{(n-p-q)\times(p-r)}$ *arbitrary.*

(b) $\alpha = 1$, $\gamma = 0$, *and* $a\beta = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{\beta-1}{\beta b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1}{\beta b}\mathbf{I}_{p-r} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{4_2} & \mathbf{0}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{7_2} & \mathbf{0} & \frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{B}_{7_3} & b\mathbf{B}_{8_2}\mathbf{B}_{4_2} & \mathbf{B}_{8_2} & \mathbf{0} & \mathbf{0}_{n-p-q-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{B}_{4_2} \in \mathbb{C}^{q\times(p-r)}$, $\mathbf{B}_{7_2} \in \mathbb{C}^{t\times(p-r)}$, $\mathbf{B}_{7_3} \in \mathbb{C}^{(n-p-q-t)\times r}$, *and* $\mathbf{B}_{8_2} \in \mathbb{C}^{(n-p-q-t)\times q}$ *arbitrary.*

(c) $\beta = 1$, $\gamma = 0$, *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{B}_{2_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{q-s} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{8_2} & \frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{B}_{7_2} & \mathbf{B}_{8_3} & b\mathbf{B}_{7_2}\mathbf{B}_{2_2} & \mathbf{0} & \mathbf{0}_{n-p-q-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{B}_{2_2} \in \mathbb{C}^{p\times(q-s)}$, $\mathbf{B}_{7_2} \in \mathbb{C}^{(n-p-q-t)\times p}$, $\mathbf{B}_{8_2} \in \mathbb{C}^{t\times(q-s)}$, *and* $\mathbf{B}_{8_3} \in \mathbb{C}^{(n-p-q-t)\times s}$ *arbitrary.*

(d) $\gamma = 1$, $\beta = 0$, *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{3_2} \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} & \mathbf{B}_{6_2} \\ \mathbf{B}_{4_2} & \mathbf{0} & \mathbf{0}_{q-s} & \mathbf{B}_{6_3} & b\mathbf{B}_{4_2}\mathbf{B}_{3_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-q-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{B}_{3_2} \in \mathbb{C}^{p\times(n-p-q-t)}$, $\mathbf{B}_{4_2} \in \mathbb{C}^{(q-s)\times p}$, $\mathbf{B}_{6_2} \in \mathbb{C}^{s\times(n-p-q-t)}$, *and* $\mathbf{B}_{6_3} \in \mathbb{C}^{(q-s)\times t}$ *arbitrary.*

Proof. From Lemma 1.1, we can write a cubic matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha \mathbf{I}_p \oplus \beta \mathbf{I}_q \oplus \gamma \mathbf{I}_{n-p-q}\right)\mathbf{U}^{-1},$$

where $p, q \in \{0, \ldots, n\}$, $p+q \leq n$ and $\mathbf{U} \in \mathbb{C}^n$ is a nonsingular matrix. Let us write $\mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \mathbf{B}_4 & \mathbf{B}_5 & \mathbf{B}_6 \\ \mathbf{B}_7 & \mathbf{B}_8 & \mathbf{B}_9 \end{pmatrix}\mathbf{U}^{-1}$, where $\mathbf{B}_1 \in \mathbb{C}^p$, $\mathbf{B}_5 \in \mathbb{C}^q$. Observe that under the hypotheses $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$ and $\alpha \neq 0$, one has

$$\begin{array}{lll} \mathbf{B}_1 = \alpha \mathbf{B}_1, & \mathbf{B}_2 = \beta \mathbf{B}_2, & \mathbf{B}_3 = \gamma \mathbf{B}_3, \\ \beta^2 \mathbf{B}_4 = \alpha\beta^2 \mathbf{B}_4, & \beta^2 \mathbf{B}_5 = \beta^3 \mathbf{B}_5, & \beta^2 \mathbf{B}_6 = \beta^2\gamma \mathbf{B}_6, \\ \gamma^2 \mathbf{B}_7 = \alpha\gamma^2 \mathbf{B}_7, & \gamma^2 \mathbf{B}_8 = \beta\gamma^2 \mathbf{B}_8, & \gamma^3 \mathbf{B}_9 = \gamma^2 \mathbf{B}_9. \end{array} \tag{3}$$

Let us assume that $\mathbf{K}$ is an idempotent matrix. Hence,

$$\begin{array}{l} b^2(\mathbf{B}_2\mathbf{B}_4 + \mathbf{B}_3\mathbf{B}_7) + \left(a\alpha \mathbf{I}_p + b\mathbf{B}_1\right)^2 = a\alpha \mathbf{I}_p + b\mathbf{B}_1,\ ab(\alpha+\beta)\mathbf{B}_2 + b^2(\mathbf{B}_2\mathbf{B}_5 + \mathbf{B}_3\mathbf{B}_8 + \mathbf{B}_1\mathbf{B}_2) = b\mathbf{B}_2, \\ ab(\alpha+\gamma)\mathbf{B}_3 + b^2(\mathbf{B}_2\mathbf{B}_6 + \mathbf{B}_3\mathbf{B}_9 + \mathbf{B}_1\mathbf{B}_3) = b\mathbf{B}_3,\ ab(\alpha+\beta)\mathbf{B}_4 + b^2(\mathbf{B}_4\mathbf{B}_1 + \mathbf{B}_6\mathbf{B}_7 + \mathbf{B}_5\mathbf{B}_4) = b\mathbf{B}_4, \\ b^2(\mathbf{B}_4\mathbf{B}_2 + \mathbf{B}_6\mathbf{B}_8) + \left(a\beta \mathbf{I}_q + b\mathbf{B}_5\right)^2 = a\beta \mathbf{I}_q + b\mathbf{B}_5,\ ab(\beta+\gamma)\mathbf{B}_6 + b^2(\mathbf{B}_4\mathbf{B}_3 + \mathbf{B}_6\mathbf{B}_9 + \mathbf{B}_5\mathbf{B}_6) = b\mathbf{B}_6, \\ ab(\alpha+\gamma)\mathbf{B}_7 + b^2(\mathbf{B}_7\mathbf{B}_1 + \mathbf{B}_8\mathbf{B}_4 + \mathbf{B}_9\mathbf{B}_7) = b\mathbf{B}_7,\ ab(\beta+\gamma)\mathbf{B}_8 + b^2(\mathbf{B}_7\mathbf{B}_2 + \mathbf{B}_8\mathbf{B}_5 + \mathbf{B}_9\mathbf{B}_8) = b\mathbf{B}_8, \\ b^2(\mathbf{B}_7\mathbf{B}_3 + \mathbf{B}_8\mathbf{B}_6) + \left(a\gamma \mathbf{I}_{n-p-q} + b\mathbf{B}_9\right)^2 = \gamma a \mathbf{I}_{n-p-q} + b\mathbf{B}_9. \end{array} \tag{4}$$

The proof can be divided into following cases depending on the scalars α, β, γ.

(i) Let $\alpha = 1$.

From (3), it is seen that $\mathbf{B}_2$ and $\mathbf{B}_3$ are zero matrices. Depending on the β, let us split this case into two cases.

(i-1) Let $\beta = 0$.

From (3), it is seen that $\mathbf{B}_8$ and $\mathbf{B}_9$ are zero matrices. Reorganizing the equations of (4) it follows that

$$\begin{array}{l} \left(a\mathbf{I}_p + b\mathbf{B}_1\right)^2 = a\mathbf{I}_p + b\mathbf{B}_1,\ \ (b\mathbf{B}_5)^2 = b\mathbf{B}_5,\ \ \left(a\gamma \mathbf{I}_{n-p-q}\right)^2 = a\gamma \mathbf{I}_{n-p-q}, \\ ab\gamma \mathbf{B}_6 + b^2\mathbf{B}_5\mathbf{B}_6 = b\mathbf{B}_6,\ \ ab(1+\gamma)\mathbf{B}_7 + b^2\mathbf{B}_7\mathbf{B}_1 = b\mathbf{B}_7, \\ ab\mathbf{B}_4 + b^2(\mathbf{B}_4\mathbf{B}_1 + \mathbf{B}_6\mathbf{B}_7 + \mathbf{B}_5\mathbf{B}_4) = b\mathbf{B}_4. \end{array} \tag{5}$$

From the first and second equations in (5), it is clear that $a\mathbf{I}_p + b\mathbf{B}_1$ and $b\mathbf{B}_5$ are idempotent. Since an idempotent matrix is a $\{1, 0\}-$ quadratic matrix, there exist $r \in \{0, \ldots, p\}$, $s \in \{0, \ldots, q\}$ and nonsingular matrices $\mathbf{V}_1 \in \mathbb{C}^p$, $\mathbf{V}_2 \in \mathbb{C}^q$ such that

$$a\mathbf{I}_p + b\mathbf{B}_1 = \mathbf{V}_1\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{p-r} \end{pmatrix}\mathbf{V}_1^{-1},\ \ b\mathbf{B}_5 = \mathbf{V}_2\begin{pmatrix} \mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q-s} \end{pmatrix}\mathbf{V}_2^{-1},$$

respectively. So, we obtain that

$$\mathbf{B}_1 = \mathbf{V}_1\begin{pmatrix} \frac{1-a}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{p-r} \end{pmatrix}\mathbf{V}_1^{-1},\ \ \mathbf{B}_5 = \mathbf{V}_2\begin{pmatrix} \frac{1}{b}\mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q-s} \end{pmatrix}\mathbf{V}_2^{-1}. \tag{6}$$

Let $\mathbf{B}_6$ and $\mathbf{B}_7$ be written as

$$\mathbf{B}_6 = \mathbf{V}_2\begin{pmatrix} \mathbf{B}_{6_1} \\ \mathbf{B}_{6_2} \end{pmatrix} \text{ and } \mathbf{B}_7 = \begin{pmatrix} \mathbf{B}_{7_1} & \mathbf{B}_{7_2} \end{pmatrix}\mathbf{V}_1^{-1}, \tag{7}$$

where $\mathbf{B}_{6_1} \in \mathbb{C}^{s\times(n-p-q)}$ and $\mathbf{B}_{7_1} \in \mathbb{C}^{(n-p-q)\times r}$. Substituting (6), (7) into the forth and fifth equations in (5) yield

$$(ab\gamma - b)\mathbf{V}_2\begin{pmatrix} \mathbf{B}_{6_1} \\ \mathbf{B}_{6_2} \end{pmatrix} + b^2\mathbf{V}_2\begin{pmatrix} \frac{1}{b}\mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q-s} \end{pmatrix}\begin{pmatrix} \mathbf{B}_{6_1} \\ \mathbf{B}_{6_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

and

$$[ab(1+\gamma)-b]\begin{pmatrix} \mathbf{B}_{7_1} & \mathbf{B}_{7_2} \end{pmatrix}\mathbf{V}_1^{-1} + b^2\begin{pmatrix} \mathbf{B}_{7_1} & \mathbf{B}_{7_2} \end{pmatrix}\begin{pmatrix} \frac{1-a}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{p-r} \end{pmatrix}\mathbf{V}_1^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Therefore, it can be written $\begin{pmatrix} a\gamma\mathbf{B}_{6_1} \\ (a\gamma-1)\mathbf{B}_{6_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} a\gamma\mathbf{B}_{7_1} & (a\gamma-1)\mathbf{B}_{7_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix}$. Moreover, from the third equation in (5), it is clear that $a\gamma = 1$. Hence, $\mathbf{B}_6$ and $\mathbf{B}_7$ reduce to

$$\mathbf{B}_6 = \mathbf{V}_2\begin{pmatrix} \mathbf{0} \\ \mathbf{B}_{6_2} \end{pmatrix} \quad \text{and} \quad \mathbf{B}_7 = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{7_2} \end{pmatrix}\mathbf{V}_1^{-1}, \tag{8}$$

where $\mathbf{B}_{6_2} \in \mathbb{C}^{(q-s)\times(n-p-q)}$ and $\mathbf{B}_{7_2} \in \mathbb{C}^{(n-p-q)\times(p-r)}$ are arbitrary matrices.

Lastly, let

$$\mathbf{B}_4 = \mathbf{V}_2\begin{pmatrix} \mathbf{B}_{4_1} & \mathbf{B}_{4_2} \\ \mathbf{B}_{4_3} & \mathbf{B}_{4_4} \end{pmatrix}\mathbf{V}_1^{-1}, \tag{9}$$

where $\mathbf{B}_{4_1} \in \mathbb{C}^{s\times r}$. Substituting (6), (8), and (9) into the sixth equation in (5) yields

$$\begin{pmatrix} \mathbf{B}_{4_1} & \mathbf{0} \\ \mathbf{0} & b\mathbf{B}_{6_2}\mathbf{B}_{7_2} - \mathbf{B}_{4_4} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Therefore, $\mathbf{B}_4$ turns into

$$\mathbf{B}_4 = \mathbf{V}_2\begin{pmatrix} \mathbf{0} & \mathbf{B}_{4_2} \\ \mathbf{B}_{4_3} & b\mathbf{B}_{6_2}\mathbf{B}_{7_2} \end{pmatrix}\mathbf{V}_1^{-1}, \tag{10}$$

where $\mathbf{B}_{4_2} \in \mathbb{C}^{s\times(p-r)}$ and $\mathbf{B}_{4_3} \in \mathbb{C}^{(q-s)\times r}$ are arbitrary matrices.

Let us define $\mathbf{V} := \mathbf{U}\left(\mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \mathbf{I}_{n-p-q}\right)$. In view of (6), (8), and (10) we obtain that

$$\begin{aligned}
\mathbf{A} &= \mathbf{U}\begin{pmatrix} \mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma\mathbf{I}_{n-p-q} \end{pmatrix}\mathbf{U}^{-1} \\
&= \mathbf{V}\begin{pmatrix} \mathbf{V}_1^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-p-q} \end{pmatrix}\begin{pmatrix} \mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma\mathbf{I}_{n-p-q} \end{pmatrix}\begin{pmatrix} \mathbf{V}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-p-q} \end{pmatrix}\mathbf{V}^{-1} \\
&= \mathbf{V}\begin{pmatrix} \mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma\mathbf{I}_{n-p-q} \end{pmatrix}\mathbf{V}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{B} &= \mathbf{U}\begin{pmatrix} \mathbf{V}_1\begin{pmatrix} \frac{1-a}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{p-r} \end{pmatrix}\mathbf{V}_1^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{V}_2\begin{pmatrix} \mathbf{0} & \mathbf{B}_{4_2} \\ \mathbf{B}_{4_3} & b\mathbf{B}_{6_2}\mathbf{B}_{7_2} \end{pmatrix}\mathbf{V}_1^{-1} & \mathbf{V}_2\begin{pmatrix} \frac{1}{b}\mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q-s} \end{pmatrix}\mathbf{V}_2^{-1} & \mathbf{V}_2\begin{pmatrix} \mathbf{0} \\ \mathbf{B}_{6_2} \end{pmatrix} \\ \begin{pmatrix} \mathbf{0} & \mathbf{B}_{7_2} \end{pmatrix}\mathbf{V}_1^{-1} & \mathbf{0} & \mathbf{0}_{n-p-q} \end{pmatrix}\mathbf{U}^{-1} \\
&= \mathbf{V}\begin{pmatrix} \frac{\gamma-1}{\gamma b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1}{\gamma b}\mathbf{I}_{p-r} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{4_2} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{4_3} & b\mathbf{B}_{6_2}\mathbf{B}_{7_2} & \mathbf{0} & \mathbf{0}_{q-s} & \mathbf{B}_{6_2} \\ \mathbf{0} & \mathbf{B}_{7_2} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q} \end{pmatrix}\mathbf{V}^{-1}
\end{aligned}$$

which establishes part (a).

(i-2) Let $\beta \neq 0$.

From (3), it is seen that $\mathbf{B}_5$ and $\mathbf{B}_6$ are zero matrices. Reorganizing the equations of (4) it follows that

$$\begin{array}{l}\left(a\mathbf{I}_p + b\mathbf{B}_1\right)^2 = a\mathbf{I}_p + b\mathbf{B}_1, \;\; \left(a\beta\mathbf{I}_q\right)^2 = a\beta\mathbf{I}_q, \;\; \left(a\gamma\mathbf{I}_{n-p-q} + b\mathbf{B}_9\right)^2 = a\gamma\mathbf{I}_{n-p-q} + b\mathbf{B}_9, \\ ab\left(1+\beta\right)\mathbf{B}_4 + b^2\mathbf{B}_4\mathbf{B}_1 = b\mathbf{B}_4, \;\; ab\left(\beta+\gamma\right)\mathbf{B}_8 + b^2\mathbf{B}_9\mathbf{B}_8 = b\mathbf{B}_8, \\ ab\left(1+\gamma\right)\mathbf{B}_7 + b^2\left(\mathbf{B}_7\mathbf{B}_1 + \mathbf{B}_8\mathbf{B}_4 + \mathbf{B}_9\mathbf{B}_7\right) = b\mathbf{B}_7.\end{array} \tag{11}$$

From the first equation in (11), it is obvious that $a\mathbf{I}_p + b\mathbf{B}_1$ is an idempotent matrix. Thus, there exist $r \in \{0, \ldots, p\}$ and a nonsingular matrix $\mathbf{Y}_1 \in \mathbb{C}^p$ such that

$$\mathbf{B}_1 = \mathbf{Y}_1 \begin{pmatrix} \frac{1-a}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{p-r} \end{pmatrix} \mathbf{Y}_1^{-1}. \tag{12}$$

From the second and third equations in (11), it is clear that $a\beta = 1$ and $a\gamma\mathbf{I}_{n-p-q} + b\mathbf{B}_9$ is idempotent. However, from the last equation in (3), $\gamma = 0$ or $\gamma = 1$ or $\mathbf{B}_9 = \mathbf{0}$. It is clear that $\gamma \neq 1$. Moreover, if $\mathbf{B}_9 = \mathbf{0}$ then $a\gamma = 0$ or $a\gamma = 1$. But this latter equality contradicts the hypothesis $\beta \neq \gamma$. Thus γ must be zero. So, there exist $t \in \{0, \ldots, n-p-q\}$ and a nonsingular matrix $\mathbf{Y}_2 \in \mathbb{C}^{(n-p-q)}$ such that

$$\mathbf{B}_9 = \mathbf{Y}_2 \begin{pmatrix} \frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-p-q-t} \end{pmatrix} \mathbf{Y}_2^{-1}. \tag{13}$$

Let $\mathbf{B}_4$ and $\mathbf{B}_8$ be written as

$$\mathbf{B}_4 = \begin{pmatrix} \mathbf{B}_{4_1} & \mathbf{B}_{4_2} \end{pmatrix} \mathbf{Y}_1^{-1} \;\text{ and }\; \mathbf{B}_8 = \mathbf{Y}_2 \begin{pmatrix} \mathbf{B}_{8_1} \\ \mathbf{B}_{8_2} \end{pmatrix}, \tag{14}$$

where $\mathbf{B}_{4_1} \in \mathbb{C}^{q\times r}$, $\mathbf{B}_{8_1} \in \mathbb{C}^{t\times q}$. Substituting (12), (14) and (13), (14) into the forth and fifth equations in (11) yield $\begin{pmatrix} a\beta\mathbf{B}_{4_1} & (a\beta-1)\mathbf{B}_{4_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} a\beta\mathbf{B}_{8_1} \\ (a\beta-1)\mathbf{B}_{8_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$, respectively. Moreover using $a\beta = 1$, $\mathbf{B}_4$ and $\mathbf{B}_8$ reduce to

$$\mathbf{B}_4 = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{4_2} \end{pmatrix} \mathbf{Y}_1^{-1} \;\text{ and }\; \mathbf{B}_8 = \mathbf{Y}_2 \begin{pmatrix} \mathbf{0} \\ \mathbf{B}_{8_2} \end{pmatrix}, \tag{15}$$

where $\mathbf{B}_{4_2} \in \mathbb{C}^{q\times(p-r)}$ and $\mathbf{B}_{8_2} \in \mathbb{C}^{(n-p-q-t)\times q}$ are arbitrary matrices.

Lastly, let

$$\mathbf{B}_7 = \mathbf{Y}_2 \begin{pmatrix} \mathbf{B}_{7_1} & \mathbf{B}_{7_2} \\ \mathbf{B}_{7_3} & \mathbf{B}_{7_4} \end{pmatrix} \mathbf{Y}_1^{-1}, \tag{16}$$

where $\mathbf{B}_{7_1} \in \mathbb{C}^{t\times r}$. Substituting (12), (13), (15), and (16) into the sixth equation in (11) yields

$$\begin{pmatrix} \mathbf{B}_{7_1} & \mathbf{0} \\ \mathbf{0} & b\mathbf{B}_{8_2}\mathbf{B}_{4_2} - \mathbf{B}_{7_4} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

So, $\mathbf{B}_7$ reduces to

$$\mathbf{B}_7 = \mathbf{Y}_2 \begin{pmatrix} \mathbf{0} & \mathbf{B}_{7_2} \\ \mathbf{B}_{7_3} & b\mathbf{B}_{8_2}\mathbf{B}_{4_2} \end{pmatrix} \mathbf{Y}_1^{-1}, \tag{17}$$

where $\mathbf{B}_{7_2} \in \mathbb{C}^{t\times(p-r)}$ and $\mathbf{B}_{7_3} \in \mathbb{C}^{(n-p-q-t)\times r}$ are arbitrary matrices.

Let us define $\mathbf{V} := \mathbf{U}\left(\mathbf{Y}_1 \oplus \mathbf{I}_q \oplus \mathbf{Y}_2\right)$. In view of (12), (13), (15), and (17) we obtain that

$$\begin{aligned}\mathbf{A} &= \mathbf{U}\begin{pmatrix}\mathbf{I}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \beta\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q}\end{pmatrix}\mathbf{U}^{-1}\\ &= \mathbf{V}\begin{pmatrix}\mathbf{Y}_1{}^{-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{Y}_2{}^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{I}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \beta\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q}\end{pmatrix}\begin{pmatrix}\mathbf{Y}_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{Y}_2\end{pmatrix}\mathbf{V}^{-1}\\ &= \mathbf{V}\begin{pmatrix}\mathbf{I}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \beta\mathbf{I}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q}\end{pmatrix}\mathbf{V}^{-1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{B} &= \mathbf{U}\begin{pmatrix}\mathbf{Y}_1\begin{pmatrix}\frac{1-a}{b}\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{p-r}\end{pmatrix}\mathbf{Y}_1{}^{-1} & \mathbf{0} & \mathbf{0}\\ \begin{pmatrix}\mathbf{0} & \mathbf{B}_{4_2}\end{pmatrix}\mathbf{Y}_1{}^{-1} & \mathbf{0}_q & \mathbf{0}\\ \mathbf{Y}_2\begin{pmatrix}\mathbf{0} & \mathbf{B}_{7_2}\\ \mathbf{B}_{7_3} & b\mathbf{B}_{8_2}\mathbf{B}_{4_2}\end{pmatrix}\mathbf{Y}_1{}^{-1} & \mathbf{Y}_2\begin{pmatrix}\mathbf{0}\\ \mathbf{B}_{8_2}\end{pmatrix} & \mathbf{Y}_2\begin{pmatrix}\frac{1}{b}\mathbf{I}_t & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_{n-p-q-t}\end{pmatrix}\mathbf{Y}_2{}^{-1}\end{pmatrix}\mathbf{U}^{-1}\\ &= \mathbf{V}\begin{pmatrix}\frac{\beta-1}{\beta b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{-1}{\beta b}\mathbf{I}_{p-r} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_{4_2} & \mathbf{0}_q & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_{7_2} & \mathbf{0} & \frac{1}{b}\mathbf{I}_t & \mathbf{0}\\ \mathbf{B}_{7_3} & b\mathbf{B}_{8_2}\mathbf{B}_{4_2} & \mathbf{B}_{8_2} & \mathbf{0} & \mathbf{0}_{n-p-q-t}\end{pmatrix}\mathbf{V}^{-1}\end{aligned}$$

which yields part (b).

(ii) Let $\beta = 1$.

From (3), it is seen that $\mathbf{B}_1, \mathbf{B}_3, \mathbf{B}_4$, and $\mathbf{B}_6$ are zero matrices. Reorganizing the equations of (4) it can be written

$$\begin{aligned}&\left(a\alpha\mathbf{I}_p\right)^2 = a\alpha\mathbf{I}_p,\ \left(a\mathbf{I}_q + b\mathbf{B}_5\right)^2 = a\mathbf{I}_q + b\mathbf{B}_5,\ \left(a\gamma\mathbf{I}_{n-p-q} + b\mathbf{B}_9\right)^2 = a\gamma\mathbf{I}_{n-p-q} + b\mathbf{B}_9\\ &ab\left(\alpha+1\right)\mathbf{B}_2 + b^2\mathbf{B}_2\mathbf{B}_5 = b\mathbf{B}_2,\ \ ab\left(\alpha+\gamma\right)\mathbf{B}_7 + b^2\mathbf{B}_9\mathbf{B}_7 = b\mathbf{B}_7,\\ &ab\left(1+\gamma\right)\mathbf{B}_8 + b^2\left(\mathbf{B}_7\mathbf{B}_2 + \mathbf{B}_8\mathbf{B}_5 + \mathbf{B}_9\mathbf{B}_8\right) = b\mathbf{B}_8.\end{aligned} \tag{18}$$

It is clear that $a\alpha = 1$ and $a\mathbf{I}_q + b\mathbf{B}_5$ is an idempotent matrix from the first and second equations in (18), respectively. There exist $s \in \{0,\ldots,q\}$ and a nonsingular matrix $\mathbf{T}_1 \in \mathbb{C}^q$ such that

$$\mathbf{B}_5 = \mathbf{T}_1\begin{pmatrix}\frac{1-a}{b}\mathbf{I}_s & \mathbf{0}\\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{q-s}\end{pmatrix}\mathbf{T}_1{}^{-1}. \tag{19}$$

From the third equation in (18), it is clear that $a\gamma\mathbf{I}_{n-p-q} + b\mathbf{B}_9$ is idempotent. However, from the last equation in (3), $\gamma = 0$ or $\gamma = 1$ or $\mathbf{B}_9 = \mathbf{0}$. It is obvious that $\gamma \neq 1$. Moreover, if $\mathbf{B}_9 = \mathbf{0}$ then $a\gamma = 0$ or $a\gamma = 1$. But this latter equality contradicts the hypothesis $\alpha \neq \gamma$. Thus γ must be zero. So, there exist $t \in \{0,\ldots,n-p-q\}$ and a nonsingular matrix $\mathbf{T}_2 \in \mathbb{C}^{(n-p-q)}$ such that

$$\mathbf{B}_9 = \mathbf{T}_2\begin{pmatrix}\frac{1}{b}\mathbf{I}_t & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_{n-p-q-t}\end{pmatrix}\mathbf{T}_2{}^{-1}. \tag{20}$$

Let $\mathbf{B}_2$ and $\mathbf{B}_7$ be written as

$$\mathbf{B}_2 = \begin{pmatrix}\mathbf{B}_{2_1} & \mathbf{B}_{2_2}\end{pmatrix}\mathbf{T}_1{}^{-1} \ \text{ and } \ \mathbf{B}_7 = \mathbf{T}_2\begin{pmatrix}\mathbf{B}_{7_1}\\ \mathbf{B}_{7_2}\end{pmatrix}, \tag{21}$$

where $\mathbf{B}_{2_1} \in \mathbb{C}^{p\times s}$ and $\mathbf{B}_{7_1} \in \mathbb{C}^{t\times p}$. Substituting (19), (21) and (20), (21) into the forth and fifth equations in (18) yield $\begin{pmatrix} a\alpha\mathbf{B}_{2_1} & (a\alpha-1)\,\mathbf{B}_{2_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} a\alpha\mathbf{B}_{7_1} \\ (a\alpha-1)\,\mathbf{B}_{7_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$, respectively. Moreover, since $a\alpha = 1$, we obtain that

$$\mathbf{B}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{2_2} \end{pmatrix}\mathbf{T}_1{}^{-1} \quad \text{and} \quad \mathbf{B}_7 = \mathbf{T}_2\begin{pmatrix} \mathbf{0} \\ \mathbf{B}_{7_2} \end{pmatrix}, \tag{22}$$

where $\mathbf{B}_{2_2} \in \mathbb{C}^{p\times(q-s)}$ and $\mathbf{B}_{7_2} \in \mathbb{C}^{(n-p-q-t)\times p}$ are arbitrary matrices.

Let

$$\mathbf{B}_8 = \mathbf{T}_2\begin{pmatrix} \mathbf{B}_{8_1} & \mathbf{B}_{8_2} \\ \mathbf{B}_{8_3} & \mathbf{B}_{8_4} \end{pmatrix}\mathbf{T}_1{}^{-1}, \tag{23}$$

where $\mathbf{B}_{8_1} \in \mathbb{C}^{t\times s}$. Substituting (19), (20), (22), and (23) into the sixth equation in (18) yields

$$\begin{pmatrix} \mathbf{B}_{8_1} & \mathbf{0} \\ \mathbf{0} & b\mathbf{B}_{7_2}\mathbf{B}_{2_2} - \mathbf{B}_{8_4} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

So, $\mathbf{B}_8$ turns into

$$\mathbf{B}_8 = \mathbf{T}_2\begin{pmatrix} \mathbf{0} & \mathbf{B}_{8_2} \\ \mathbf{B}_{8_3} & b\mathbf{B}_{7_2}\mathbf{B}_{2_2} \end{pmatrix}\mathbf{T}_1{}^{-1}, \tag{24}$$

where $\mathbf{B}_{8_2} \in \mathbb{C}^{t\times(q-s)}$ and $\mathbf{B}_{8_3} \in \mathbb{C}^{(n-p-q-t)\times s}$ are arbitrary matrices. Let us define $\mathbf{V} := \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{T}_1 \oplus \mathbf{T}_2\right)$. In view of (19), (20), (22), and (24) we obtain that

$$\begin{aligned}
\mathbf{A} &= \mathbf{U}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q} \end{pmatrix}\mathbf{U}^{-1} \\
&= \mathbf{V}\begin{pmatrix} \mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_1{}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_2{}^{-1} \end{pmatrix}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q} \end{pmatrix}\begin{pmatrix} \mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_2 \end{pmatrix}\mathbf{V}^{-1} \\
&= \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q} \end{pmatrix}\mathbf{V}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{B} &= \mathbf{U}\begin{pmatrix} \mathbf{0}_p & \begin{pmatrix} \mathbf{0} & \mathbf{B}_{2_2} \end{pmatrix}\mathbf{T}_1{}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_1\begin{pmatrix} \frac{1-a}{b}\mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{q-s} \end{pmatrix}\mathbf{T}_1{}^{-1} & \mathbf{0} \\ \mathbf{T}_2\begin{pmatrix} \mathbf{0} \\ \mathbf{B}_{7_2} \end{pmatrix} & \mathbf{T}_2\begin{pmatrix} \mathbf{0} & \mathbf{B}_{8_2} \\ \mathbf{B}_{8_3} & b\mathbf{B}_{7_2}\mathbf{B}_{2_2} \end{pmatrix}\mathbf{T}_1{}^{-1} & \mathbf{T}_2\begin{pmatrix} \frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-p-q-t} \end{pmatrix}\mathbf{T}_2{}^{-1} \end{pmatrix}\mathbf{U} \\
&= \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{B}_{2_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{q-s} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{8_2} & \frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{B}_{7_2} & \mathbf{B}_{8_3} & b\mathbf{B}_{7_2}\mathbf{B}_{2_2} & \mathbf{0} & \mathbf{0}_{n-p-q-t} \end{pmatrix}\mathbf{V}^{-1}
\end{aligned}$$

which establishes part (c).

(iii) Let $\gamma = 1$.

From (3), it is easily seen that $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_7$, and $\mathbf{B}_8$ are zero matrices. Reorganizing the equations of (4) it follows that

$$\begin{aligned} &\left(a\alpha\mathbf{I}_p\right)^2 = a\alpha\mathbf{I}_p, \quad \left(a\beta\mathbf{I}_q + b\mathbf{B}_5\right)^2 = a\beta\mathbf{I}_q + b\mathbf{B}_5, \quad \left(a\mathbf{I}_{n-p-q} + b\mathbf{B}_9\right)^2 = a\mathbf{I}_{n-p-q} + b\mathbf{B}_9, \\ &ab\left(\alpha+1\right)\mathbf{B}_3 + b^2\mathbf{B}_3\mathbf{B}_9 = b\mathbf{B}_3, \quad ab\left(\alpha+\beta\right)\mathbf{B}_4 + b^2\mathbf{B}_5\mathbf{B}_4 = b\mathbf{B}_4, \\ &ab\left(\beta+1\right)\mathbf{B}_6 + b^2\left(\mathbf{B}_4\mathbf{B}_3 + \mathbf{B}_6\mathbf{B}_9 + \mathbf{B}_5\mathbf{B}_6\right) = b\mathbf{B}_6. \end{aligned} \tag{25}$$

It is clear that $a\alpha = 1$ and $a\beta\mathbf{I}_q + b\mathbf{B}_5$ is an idempotent matrix from the first and second equations in (25), respectively. However from the fifth equation in (3), $\beta = 0$ or $\beta = 1$ or $\mathbf{B}_5 = \mathbf{0}$. It is obvious that $\beta \neq 1$. Moreover, if $\mathbf{B}_5 = \mathbf{0}$ then $a\beta = 0$ or $a\beta = 1$. But this latter equality contradicts the hypothesis $\alpha \neq \beta$. Thus β must be zero. So, there exist $s \in \{0, \ldots, q\}$ and a nonsingular matrix $\mathbf{Z}_1 \in \mathbb{C}^q$ such that

$$\mathbf{B}_5 = \mathbf{Z}_1 \begin{pmatrix} \frac{1}{b}\mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{q-s} \end{pmatrix} \mathbf{Z}_1^{-1}. \tag{26}$$

Moreover, from the third equation in (25), it is clear that $a\mathbf{I}_{n-p-q} + b\mathbf{B}_9$ is idempotent. Then there exist $t \in \{0, \ldots, n-p-q\}$ and a nonsingular matrix $\mathbf{Z}_2 \in \mathbb{C}^{(n-p-q)}$ such that

$$\mathbf{B}_9 = \mathbf{Z}_2 \begin{pmatrix} \frac{1-a}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{n-p-q-t} \end{pmatrix} \mathbf{Z}_2^{-1}. \tag{27}$$

Let $\mathbf{B}_3$ and $\mathbf{B}_4$ be written as

$$\mathbf{B}_3 = \begin{pmatrix} \mathbf{B}_{3_1} & \mathbf{B}_{3_2} \end{pmatrix} \mathbf{Z}_2^{-1} \quad \text{and} \quad \mathbf{B}_4 = \mathbf{Z}_1 \begin{pmatrix} \mathbf{B}_{4_1} \\ \mathbf{B}_{4_2} \end{pmatrix}, \tag{28}$$

where $\mathbf{B}_{3_1} \in \mathbb{C}^{p\times t}$ and $\mathbf{B}_{4_1} \in \mathbb{C}^{s\times p}$. Substituting (27), (28) and (26), (28) into the forth and fifth equations in (25) it is obtained that $\begin{pmatrix} a\alpha\mathbf{B}_{3_1} & (a\alpha-1)\mathbf{B}_{3_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} a\alpha\mathbf{B}_{4_1} \\ (a\alpha-1)\mathbf{B}_{4_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$, respectively. Moreover using $a\alpha = 1$, $\mathbf{B}_3$ and $\mathbf{B}_4$ turn to

$$\mathbf{B}_3 = \begin{pmatrix} \mathbf{0} & \mathbf{B}_{3_2} \end{pmatrix} \mathbf{Z}_2^{-1} \quad \text{and} \quad \mathbf{B}_4 = \mathbf{Z}_1 \begin{pmatrix} \mathbf{0} \\ \mathbf{B}_{4_2} \end{pmatrix}, \tag{29}$$

where $\mathbf{B}_{3_2} \in \mathbb{C}^{p\times(n-p-q-t)}$ and $\mathbf{B}_{4_2} \in \mathbb{C}^{(q-s)\times p}$ are arbitrary matrices.

Lastly, let

$$\mathbf{B}_6 = \mathbf{Z}_1 \begin{pmatrix} \mathbf{B}_{6_1} & \mathbf{B}_{6_2} \\ \mathbf{B}_{6_3} & \mathbf{B}_{6_4} \end{pmatrix} \mathbf{Z}_2^{-1}, \tag{30}$$

where $\mathbf{B}_{6_1} \in \mathbb{C}^{s\times t}$. Substituting (26), (27), (29), and (30) into the sixth equation in (25) it is obtained that

$$\begin{pmatrix} \mathbf{B}_{6_1} & \mathbf{0} \\ \mathbf{0} & b\mathbf{B}_{4_2}\mathbf{B}_{3_2} - \mathbf{B}_{6_4} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

So, $\mathbf{B}_6$ reduces to

$$\mathbf{B}_6 = \mathbf{Z}_1 \begin{pmatrix} \mathbf{0} & \mathbf{B}_{6_2} \\ \mathbf{B}_{6_3} & b\mathbf{B}_{4_2}\mathbf{B}_{3_2} \end{pmatrix} \mathbf{Z}_2^{-1}, \tag{31}$$

where $\mathbf{B}_{6_2} \in \mathbb{C}^{s\times(n-p-q-t)}$ and $\mathbf{B}_{6_3} \in \mathbb{C}^{(q-s)\times t}$ are arbitrary matrices.

Let us define $\mathbf{V} := \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{Z}_1 \oplus \mathbf{Z}_2\right)$. In view of (26), (27), (29), and (31) we obtain that

$$\begin{aligned}\mathbf{A} &= \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \mathbf{0}_q \oplus \mathbf{I}_{n-p-q}\right)\mathbf{U}^{-1}\\ &= \mathbf{V}\begin{pmatrix}\mathbf{I}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & {\mathbf{Z}_1}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & {\mathbf{Z}_2}^{-1}\end{pmatrix}\begin{pmatrix}\alpha\mathbf{I}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-p-q}\end{pmatrix}\begin{pmatrix}\mathbf{I}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Z}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{Z}_2\end{pmatrix}\mathbf{V}^{-1}\end{aligned}$$

and

$$\mathbf{B} = \mathbf{U}\begin{pmatrix}\mathbf{0}_p & \mathbf{0} & \begin{pmatrix}\mathbf{0} & \mathbf{B}_{3_2}\end{pmatrix}{\mathbf{Z}_2}^{-1}\\ \mathbf{Z}_1\begin{pmatrix}\mathbf{0}\\ \mathbf{B}_{4_2}\end{pmatrix} & \mathbf{Z}_1\begin{pmatrix}\frac{1}{b}\mathbf{I}_s & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_{q-s}\end{pmatrix}{\mathbf{Z}_1}^{-1} & \mathbf{Z}_1\begin{pmatrix}\mathbf{0} & \mathbf{B}_{6_2}\\ \mathbf{B}_{6_3} & b\mathbf{B}_{4_2}\mathbf{B}_{3_2}\end{pmatrix}{\mathbf{Z}_2}^{-1}\\ \mathbf{0} & \mathbf{0} & \mathbf{Z}_2\begin{pmatrix}\frac{1-a}{b}\mathbf{I}_t & \mathbf{0}\\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{n-p-q-t}\end{pmatrix}{\mathbf{Z}_2}^{-1}\end{pmatrix}\mathbf{U}^{-1}$$

$$= \mathbf{V}\begin{pmatrix}\mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{3_2}\\ \mathbf{0} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} & \mathbf{B}_{6_2}\\ \mathbf{B}_{4_2} & \mathbf{0} & \mathbf{0}_{q-s} & \mathbf{B}_{6_3} & b\mathbf{B}_{4_2}\mathbf{B}_{3_2}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_t & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-q-t}\end{pmatrix}\mathbf{V}^{-1}$$

which establishes part (d).

(iv) Let $\beta \neq 1$, $\alpha \neq 1$, and $\gamma \neq 1$.

From (3), it is easily seen that $\mathbf{B}_1, \mathbf{B}_2$, and $\mathbf{B}_3$ are zero matrices. Depending on the β, let us split this case into two cases.

(iv-1) Let $\beta = 0$.

From (3), it is seen that $\mathbf{B}_7, \mathbf{B}_8$, and $\mathbf{B}_9$ are zero matrices. Reorganizing the equations of (4) it can be written

$$\begin{aligned}&\left(a\alpha\mathbf{I}_p\right)^2 = a\alpha\mathbf{I}_p, \quad (b\mathbf{B}_5)^2 = b\mathbf{B}_5, \quad \left(a\gamma\mathbf{I}_{n-p-q}\right)^2 = a\gamma\mathbf{I}_{n-p-q},\\ &ab\alpha\mathbf{B}_4 + b^2\mathbf{B}_5\mathbf{B}_4 = b\mathbf{B}_4, \quad ab\gamma\mathbf{B}_6 + b^2\mathbf{B}_5\mathbf{B}_6 = b\mathbf{B}_6.\end{aligned} \tag{32}$$

It is clear that $a\alpha = 1$ and $a\gamma = 1$ from the first and third equations in (32), respectively. However these equalities contradict the hypothesis of $\alpha \neq \gamma$. So, in this case, there is no matrix form of $\mathbf{B}$ such that the linear combination matrix $\mathbf{K}$ is idempotent.

(iv-2) Let $\beta \neq 0$.

From (3), it is seen that $\mathbf{B}_4, \mathbf{B}_5$, and $\mathbf{B}_6$ are zero matrices. Reorganizing the equations of (4) it follows that

$$\begin{aligned}&\left(a\alpha\mathbf{I}_p\right)^2 = a\alpha\mathbf{I}_p, \quad \left(a\beta\mathbf{I}_q\right)^2 = a\beta\mathbf{I}_q, \quad \left(a\gamma\mathbf{I}_{n-p-q} + b\mathbf{B}_9\right)^2 = a\gamma\mathbf{I}_{n-p-q} + b\mathbf{B}_9,\\ &ab\left(\alpha + \gamma\right)\mathbf{B}_7 + b^2\mathbf{B}_9\mathbf{B}_7 = b\mathbf{B}_7, \quad ab\left(\beta + \gamma\right)\mathbf{B}_8 + b^2\mathbf{B}_9\mathbf{B}_8 = b\mathbf{B}_8.\end{aligned} \tag{33}$$

It is obvious that $a\alpha = 1$ and $a\beta = 1$ from the first and second equations in (33), respectively. However these equalities contradict the hypothesis of $\alpha \neq \beta$. So, in this case, there is no matrix form of $\mathbf{B}$ such that the linear combination matrix $\mathbf{K}$ is idempotent.

Thus, the necessity has been proved. The sufficiency is obvious. □

In the following example it is sought scalars such that the linear combination of a cubic matrix and an arbitrary matrix is an idempotent matrix.

Example 2.2. *Let*

$$\mathbf{A} = \begin{pmatrix}-1 & 0 & -1 & 0\\ -2 & 1 & -1 & 0\\ 2 & 0 & 2 & 0\\ 4 & -2 & 3 & -1\end{pmatrix}$$

and

$$\mathbf{B}_1 = \begin{pmatrix} -1 & 0 & 5 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 2 & 0 & 0 & -2 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} -1 & 0 & -1 & 0 \\ -4 & 2 & -2 & 0 \\ 2 & 0 & 2 & 0 \\ 5 & -2 & 3 & 0 \end{pmatrix}.$$

Let us find all ordered pair (a, b) such that $\mathbf{K}_i$ is idempotent, $i = 1, 2$, where $a, b \in \mathbb{C}^*$ and $\mathbf{K}_i = a_i\mathbf{A} + b_i\mathbf{B}_i$. It is clear that $\mathbf{A}$ is a $\{-1, 1, 0\}-$ cubic matrix (note that it is not tripotent) and $\mathbf{A}^2\mathbf{B}_i\mathbf{A} = \mathbf{A}^2\mathbf{B}_i$, $i = 1, 2$. Moreover,

$$\mathbf{K}_1 = \begin{pmatrix} -a-b & 0 & 5b-a & b \\ -2a & a & -a-b & 0 \\ 2a+b & 0 & 2a+b & -b \\ 4a+2b & -2a & 3a & -a-2b \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -a-b & 0 & -a-b & 0 \\ -2a-4b & a+2b & -a-2b & 0 \\ 2a+2b & 0 & 2a+2b & 0 \\ 4a+5b & -2a-2b & 3a+3b & -a \end{pmatrix},$$

$$\mathbf{K}_1{}^2 = \begin{pmatrix} -a^2+15ab+8b^2 & -2ab & -a^2+8ab & -ab-8b^2 \\ -2a^2-ab-b^2 & a^2 & -a^2-14ab-b^2 & b^2-ab \\ 2a^2-3ab-2b^2 & 2ab & (2a+b)(a+6b)-3ab & ab+2b^2 \\ 2a^2-13ab-6b^2 & 4ab & a^2+17ab+10b^2 & a^2+5ab+6b^2 \end{pmatrix},$$

$$\mathbf{K}_2{}^2 = \begin{pmatrix} -(a+b)^2 & 0 & -a^2-2ab-b^2 & 0 \\ -2a^2-8ab-8b^2 & (a+2b)^2 & -(a+2b)^2 & 0 \\ 2a^2+4ab+2b^2 & 0 & 2(a+b)^2 & 0 \\ 2a^2+10ab+9b^2 & -4ab-4b^2 & a^2+6ab+5b^2 & a^2 \end{pmatrix}.$$

The solution sets of nonlinear equations $\mathbf{K}_1{}^2 = \mathbf{K}_1$ and $\mathbf{K}_2{}^2 = \mathbf{K}_2$ are $\{(0,0)\}$ and $\{(0,0), (-1,1)\}$, respectively, $i = 1, 2$. While the pair $(-1, 1)$ implies the idempotency of $\mathbf{K}_2$, there is no appropriate pair (a_1, b_1) to imply that $\mathbf{K}_1$ is idempotent. Because the matrix $\mathbf{B}_1$ should have been in the form of the matrix $\mathbf{B}$, in the part (c) of Theorem 2.1, but $\mathbf{B}_1$ does not match the desired form. However, $\mathbf{B}_2$ satisfies aforementioned form of the matrix $\mathbf{B}$.

Example 2.3. *Let us define $\mathbf{A}$ as in the previous example and let us find $a \in \mathbb{C}^*$ and all matrices $\mathbf{B} \in \mathbb{C}^4$ such that $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$ and $a\mathbf{A} + \mathbf{B}$ is idempotent. A diagonalized form of $\mathbf{A}$ and the matrix $\mathbf{V}$ that diagonalize it are*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{V}^{-1} \text{ and } \mathbf{V} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -2 & -2 & -1 \\ 1 & -1 & -3 & -1 \end{pmatrix}.$$

By using the notations of Theorem 2.1, let us assume $\alpha = -1$, $\beta = 1$, and $\gamma = 0$. Then only part (c) of Theorem 2.1 can be implied, so we get $a = 1/\alpha = -1$, $p = 1$, $q = 2$, $s \in \{0, 1, 2\}$, and $t \in \{0, 1\}$. Therefore, depending on the appearing and disappearing blocks of $\mathbf{V}^{-1}\mathbf{B}\mathbf{V}$, it can be written the following possible cases:

POSSIBILITIES FOR $\mathbf{V}^{-1}\mathbf{B}\mathbf{V}$			
t/s	$s = 0$	$s = 1$	$s = 2$
$t = 0$	$\begin{pmatrix} 0 & c & d & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e & e.c & e.d & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & j & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ k & l & k.j & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ w & x & y & 0 \end{pmatrix}$
$t = 1$	$\begin{pmatrix} 0 & f & g & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & h & i & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & u & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & v & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

where c, d, e, f, g, h, i, j, k, l, u, v, w, x, and y are arbitrary complex numbers.

Involutiveness of the linear combination of the form (1) of a quadratic matrix and an arbitrary matrix under the condition $\mathbf{ABA} = \mathbf{BA}$ was considered in [8]. It is noteworthy that the above theorem gives the solution to the problem of the idempotency of linear combination of the form (1) of a cubic matrix and an arbitrary matrix under the condition $\mathbf{A}^2\mathbf{BA} = \mathbf{A}^2\mathbf{B}$. Then it may be interesting to reconsider of the same problem under the condition $\mathbf{ABA} = \mathbf{BA}$.

Theorem 2.4. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha \neq 0$, $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$. Moreover, let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an $\{\alpha, \beta, \gamma\}-$cubic matrix and an arbitrary matrix, respectively. Furthermore, let $\mathbf{ABA} = \mathbf{BA}$ and $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ with $a, b \in \mathbb{C}^*$. Then $\mathbf{K}$ is an idempotent matrix if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \gamma\mathbf{I}_{n-p-q} \end{pmatrix}\mathbf{V}^{-1}$$

and $\mathbf{B}$ satisfies one of the following cases.

(a) $\alpha = 1, \beta = 0$, *and* $a\gamma = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{\gamma-1}{\gamma b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} & \mathbf{B}_{2_2} & \mathbf{0} \\ \mathbf{0} & \frac{-1}{\gamma b}\mathbf{I}_{p-r} & \mathbf{B}_{2_3} & b\mathbf{B}_{3_2}\mathbf{B}_{8_2} & \mathbf{B}_{3_2} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{q-s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{8_2} & \mathbf{0}_{n-p-q} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{B}_{2_2} \in \mathbb{C}^{r\times(q-s)}$, $\mathbf{B}_{2_3} \in \mathbb{C}^{(p-r)\times s}$, $\mathbf{B}_{3_2} \in \mathbb{C}^{(p-r)\times(n-p-q)}$, *and* $\mathbf{B}_{8_2} \in \mathbb{C}^{(n-p-q)\times(q-s)}$ *arbitrary.*

(b) $\alpha = 1, \gamma = 0$, *and* $a\beta = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{\beta-1}{\beta b}\mathbf{I}_r & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{3_2} \\ \mathbf{0} & \frac{-1}{\beta b}\mathbf{I}_{p-r} & \mathbf{B}_{2_2} & \mathbf{B}_{3_3} & b\mathbf{B}_{2_2}\mathbf{B}_{6_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_q & \mathbf{0} & \mathbf{B}_{6_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{B}_{2_2} \in \mathbb{C}^{(p-r)\times q}$, $\mathbf{B}_{3_2} \in \mathbb{C}^{r\times(n-p-q-t)}$, $\mathbf{B}_{3_3} \in \mathbb{C}^{(p-r)\times t}$, *and* $\mathbf{B}_{6_2} \in \mathbb{C}^{q\times(n-p-q-t)}$ *arbitrary.*

(c) $\beta = 1, \gamma = 0$, *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{3_2} \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} & \mathbf{B}_{6_2} \\ \mathbf{B}_{4_2} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{q-s} & \mathbf{B}_{6_3} & b\mathbf{B}_{4_2}\mathbf{B}_{3_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-q-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{B}_{3_2} \in \mathbb{C}^{p\times(n-p-q-t)}$, $\mathbf{B}_{4_2} \in \mathbb{C}^{(q-s)\times p}$, $\mathbf{B}_{6_2} \in \mathbb{C}^{s\times(n-p-q-t)}$, *and* $\mathbf{B}_{6_3} \in \mathbb{C}^{(q-s)\times t}$ *arbitrary.*

(d) $\gamma = 1, \beta = 0$, *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{B}_{2_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{b}\mathbf{I}_s & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{q-s} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{8_2} & \frac{\alpha-1}{\alpha b}\mathbf{I}_t & \mathbf{0} \\ \mathbf{B}_{7_2} & \mathbf{B}_{8_3} & b\mathbf{B}_{7_2}\mathbf{B}_{2_2} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-q-t} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{B}_{2_2} \in \mathbb{C}^{p\times(q-s)}$, $\mathbf{B}_{7_2} \in \mathbb{C}^{(n-p-q-t)\times p}$, $\mathbf{B}_{8_2} \in \mathbb{C}^{t\times(q-s)}$, *and* $\mathbf{B}_{8_3} \in \mathbb{C}^{(n-p-q-t)\times s}$ *arbitrary.*

The proof of Theorem 2.4 is similar to the proof of Theorem 2.1, so it is omitted.

Remark 2.5. *Note that a must be different from 1 in Theorems 2.1 and 2.4 since the hypotheses of $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$. There is same situation in some parts of some results in the sequel.*

$\mathbf{A}$ is considered as a cubic matrix in Theorem 2.1. It may be interesting to reconsider $\mathbf{A}$ as a quadratic matrix under the same condition $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$.

Theorem 2.6. *Let $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, $\alpha \neq \beta$. Moreover, let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an $\{\alpha,\beta\}-$ quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$ and $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ with $a, b \in \mathbb{C}^*$. Then $\mathbf{K}$ is an idempotent matrix if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1}$$

and $\mathbf{B}$ satisfies one of the following cases.

(a) $\beta \neq 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Z}_2 \in \mathbb{C}^{r\times(p-q)}$ and $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ arbitrary.

(b) $\beta = 1$ *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Y}_2 \in \mathbb{C}^{p\times(n-p-r)}$ arbitrary.

Proof. From Theorem 2.1 in [9], we can write a quadratic matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1},$$

where $p \in \{0,\ldots,n\}$ and $\mathbf{U} \in \mathbb{C}^n$ is a nonsingular matrix. We can represent $\mathbf{B}$ as $\mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{T} \end{pmatrix}\mathbf{U}^{-1}$, where $\mathbf{X} \in \mathbb{C}^p$. In view of the hypotheses $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$ and $\alpha \neq 0$ we can write

$$\alpha\mathbf{X} = \mathbf{X}, \quad \beta\mathbf{Y} = \mathbf{Y}, \quad \alpha\beta^2\mathbf{Z} = \beta^2\mathbf{Z}, \quad \beta^3\mathbf{T} = \beta^2\mathbf{T}. \tag{34}$$

Now let us assume that $\mathbf{K}$ is an idempotent matrix then we can write

$$\begin{aligned} &\left(a\alpha\mathbf{I}_p + b\mathbf{X}\right)^2 + b^2\mathbf{Y}\mathbf{Z} = a\alpha\mathbf{I}_p + b\mathbf{X}, \ ab\left(\alpha+\beta\right)\mathbf{Y} + b^2\left(\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{T}\right) = b\mathbf{Y}, \\ &ab\left(\alpha+\beta\right)\mathbf{Z} + b^2\left(\mathbf{Z}\mathbf{X} + \mathbf{T}\mathbf{Z}\right) = b\mathbf{Z}, \ b^2\mathbf{Z}\mathbf{Y} + \left(a\beta\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = a\beta\mathbf{I}_{n-p} + b\mathbf{T}. \end{aligned} \tag{35}$$

Depending on the scalars α and β we have the following cases.

(i) Let $\beta \neq 1$.
From (34), it is seen that $\mathbf{Y} = \mathbf{0}$. Reorganizing the equations of (35), it can be written

$$\begin{aligned} &\left(a\alpha\mathbf{I}_p + b\mathbf{X}\right)^2 = a\alpha\mathbf{I}_p + b\mathbf{X}, \ \left(a\beta\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = a\beta\mathbf{I}_{n-p} + b\mathbf{T}, \\ &ab\left(\alpha+\beta\right)\mathbf{Z} + b^2\left(\mathbf{Z}\mathbf{X} + \mathbf{T}\mathbf{Z}\right) = b\mathbf{Z}. \end{aligned} \tag{36}$$

It is clear that $a\alpha\mathbf{I}_p + b\mathbf{X}$ and $a\beta\mathbf{I}_{n-p} + b\mathbf{T}$ are idempotent matrices from the first and second equations in (36), respectively. Since an idempotent matrix is an $\{1,0\}-$ quadratic matrix, there exist $q \in \{0,\dots,p\}$, $r \in \{0,\dots,n-p\}$ and nonsingular matrices $\mathbf{S}_1 \in \mathbb{C}^p$, $\mathbf{S}_2 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{X} = \mathbf{S}_1 \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} \end{pmatrix} \mathbf{S}_1^{-1} \text{ and } \mathbf{T} = \mathbf{S}_2 \begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{S}_2^{-1}. \tag{37}$$

Let us write $\mathbf{Z}$ as

$$\mathbf{Z} = \mathbf{S}_2 \begin{pmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{pmatrix} \mathbf{S}_1^{-1}, \tag{38}$$

where $\mathbf{Z}_1 \in \mathbb{C}^{r\times q}$. Substituting (37) and (38) into the third equation in (36) it is obtained that

$$\begin{pmatrix} b\mathbf{Z}_1 & \mathbf{0} \\ \mathbf{0} & -b\mathbf{Z}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Then $\mathbf{Z}$ reduces to

$$\mathbf{Z} = \mathbf{S}_2 \begin{pmatrix} \mathbf{0} & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_1^{-1}, \tag{39}$$

where $\mathbf{Z}_2 \in \mathbb{C}^{r\times(p-q)}$ and $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ are arbitrary matrices.

Let us define $\mathbf{V} := \mathbf{U}(\mathbf{S}_1 \oplus \mathbf{S}_2)$. In view of (37) and (39), we obtain that

$$\begin{aligned} \mathbf{A} &= \mathbf{U} \begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix} \mathbf{U}^{-1} \\ &= \mathbf{V} \begin{pmatrix} \mathbf{S}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix} \begin{pmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{pmatrix} \mathbf{V}^{-1} \\ &= \mathbf{V} \begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix} \mathbf{V}^{-1} \end{aligned}$$

and

$$\begin{aligned} \mathbf{B} &= \mathbf{U} \begin{pmatrix} \mathbf{S}_1 \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} \end{pmatrix} \mathbf{S}_1^{-1} & \mathbf{0} \\ \mathbf{S}_2 \begin{pmatrix} \mathbf{0} & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_1^{-1} & \mathbf{S}_2 \begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{S}_2^{-1} \end{pmatrix} \mathbf{U}^{-1} \\ &= \mathbf{V} \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix} \mathbf{V}^{-1} \end{aligned}$$

which establishes part (a).

(ii) Let $\beta = 1$.

From the first and third equations in (34), we obtain $\mathbf{X} = \mathbf{0}$ and $\mathbf{Z} = \mathbf{0}$, respectively. Reorganizing the equations of (35), it is obtained that

$$(a\alpha)^2\mathbf{I}_p = a\alpha\mathbf{I}_p,\ \left(a\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = a\mathbf{I}_{n-p} + b\mathbf{T},\ ab(\alpha+1)\mathbf{Y} + b^2\mathbf{Y}\mathbf{T} = b\mathbf{Y}. \tag{40}$$

It is obvious that $a\alpha = 1$ and $a\mathbf{I}_{n-p} + b\mathbf{T}$ is an idempotent matrix from the first and second equations in (40), respectively. Hence, there exist $r \in \{0, \ldots, n-p\}$ and a nonsingular matrix $\mathbf{S} \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{T} = \mathbf{S}\begin{pmatrix} \frac{1-a}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{S}^{-1}. \tag{41}$$

Let us write $\mathbf{Y}$ as

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \end{pmatrix}\mathbf{S}^{-1}, \tag{42}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{p\times r}$. Substituting (41) and (42) into the third equation in (40) yields

$$\begin{pmatrix} b\,(a\alpha)\,\mathbf{Y}_1 & b\,(a\alpha - 1)\,\mathbf{Y}_2 \end{pmatrix}\mathbf{S}^{-1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Since $a\alpha = 1$, $\mathbf{Y}$ reduces to

$$\mathbf{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \end{pmatrix}\mathbf{S}^{-1},$$

where $\mathbf{Y}_2 \in \mathbb{C}^{p\times(n-p-r)}$ is an arbitrary matrix.

Hence, we can easily write

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \mathbf{I}_{n-p}\right)\mathbf{U}^{-1} = \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)\left(\alpha\mathbf{I}_p \oplus \mathbf{I}_{n-p}\right)\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}$$

and

$$\begin{aligned}\mathbf{B} &= \mathbf{U}\begin{pmatrix} \mathbf{0}_p & \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \end{pmatrix}\mathbf{S}^{-1} \\ \mathbf{0} & \mathbf{S}\begin{pmatrix} \frac{1-a}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{S}^{-1} \end{pmatrix}\mathbf{U}^{-1} \\ &= \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-r} \end{pmatrix}\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}.\end{aligned}$$

The necessity part of the proof is completed by defining $\mathbf{V}$ as $\mathbf{V} := \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)$. The sufficiency is obvious. □

Example 2.7. *Let*

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 & -1 \\ 0 & 3 & 2 & -1 \\ 0 & -2 & -1 & 1 \\ 0 & -2 & -2 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 4 & 0 \end{pmatrix}.$$

Let us find all ordered pair (a, b) such that $\mathbf{K}$ is idempotent, where $a, b \in \mathbb{C}^*$ and $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$. It is clear that $\mathbf{A}$ is a $\{1,2\}$ − quadratic matrix and $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$. Moreover,

$$\mathbf{K} = \begin{pmatrix} 2a & 2a-b & 3a & -a-b \\ 0 & 3a+b & 2a+2b & -a-b \\ 0 & b-2a & -a & a+b \\ 0 & 4b-2a & 4b-2a & 2a \end{pmatrix}$$

and

$$\mathbf{K}^2 = \begin{pmatrix} 4a^2 & 6a^2-2ab-5b^2 & 9a^2-6b^2 & -3a^2-4ab-b^2 \\ 0 & 7a^2+2ab-b^2 & 6a^2-4ab-2b^2 & -3a^2-2ab+b^2 \\ 0 & -6a^2+2ab+5b^2 & -5a^2+6b^2 & 3a^2+2ab-b^2 \\ 0 & -6a^2+8ab+8b^2 & -6a^2+8ab+8b^2 & 4a^2 \end{pmatrix}.$$

From the idempotency of $\mathbf{K}$*, it is obtained that* $(a, b) \in \{(0, 0)\}$*. Therefore, it is seen that there is no appropriate pair* (a, b) *to imply that* $\mathbf{K}$ *is idempotent. Note that,* $\mathbf{V} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{pmatrix}$ *and it diagonalize* $\mathbf{A}$*. Moreover,* $\mathbf{B}$ *is not compatible with the part (a) of Theorem 2.6.*

Example 2.8. *Let* $\mathbf{A} = \begin{pmatrix} 5 & -4 & 0 \\ 8 & -7 & 0 \\ -4 & 4 & 1 \end{pmatrix}$*. Let us find* $a \in \mathbb{C}^*$ *and all matrices* $\mathbf{B} \in \mathbb{C}^3$ *such that* $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{A}^2\mathbf{B}$ *and* $a\mathbf{A} + \mathbf{B}$ *is idempotent. A diagonalized form of* $\mathbf{A}$ *and the matrix* $\mathbf{V}$ *that diagonalize it are*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{V}^{-1} \text{ and } \mathbf{V} = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Using the notations of Theorem 2.6, let us assume $\alpha = -3$ *and* $\beta = 1$*. Then only part (b) of Theorem 2.6 can be implied, so we get* $a = -1/3$*,* $p = 1$*, and* $r \in \{0, 1, 2\}$*. Therefore, depending on the appearing and disappearing blocks of* $\mathbf{V}^{-1}\mathbf{B}\mathbf{V}$*, the following possible cases of* $\mathbf{V}^{-1}\mathbf{B}\mathbf{V}$ *are obtained for the values of* $r = 0, 1, 2$*, respectively.*

$$\begin{pmatrix} 0 & c & d \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & e \\ 0 & 4/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{pmatrix},$$

where c*,* d*, and* e *are arbitrary complex numbers.*

It can be interesting to consider the above theorem under the condition in Theorem 2.4.

Theorem 2.9. *Let* $\alpha, \beta \in \mathbb{C}$ *with* $\alpha \neq 0$*,* $\alpha \neq \beta$*. Moreover, let* $\mathbf{A}$ *and* $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *be an* $\{\alpha, \beta\}-$ *quadratic matrix and an arbitrary matrix, respectively. Furthermore, let* $\mathbf{ABA} = \mathbf{BA}$ *and* $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ *with* $a, b \in \mathbb{C}^*$*. Then* $\mathbf{K}$ *is an idempotent matrix if and only if there exists a nonsingular matrix* $\mathbf{V} \in \mathbb{C}^n$ *such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1}$$

and $\mathbf{B}$ *satisfies one of the following cases.*

(a) $\beta \neq 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$ *and* $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ *arbitrary.*

(b) $\beta = 1$ *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\alpha-1}{\alpha b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-1}{\alpha b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Z}_2 \in \mathbb{C}^{(n-p-r)\times p}$ *arbitrary.*

The proof of Theorem 2.9 is similar to the proof of Theorem 2.6, so it is omitted. Under the different conditions from the previous results, the problem of idempotency of the linear combination of the form (1) of a quadratic matrix and an arbitrary matrix is reconsidered in the following three results.

Theorem 2.10. *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, $\alpha \neq \beta$. Moreover, let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an $\{\alpha, \beta\}$ − quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $\mathbf{BAB} = \mathbf{AB}^2$ and $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ with $a, b \in \mathbb{C}^*$. Then $\mathbf{K}$ is an idempotent matrix if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1} \tag{43}$$

and $\mathbf{B}$ satisfies one of the following cases.

(a) $\beta = 0$ *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-1}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{44}$$

being $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$, $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ *arbitrary and* $\mathbf{Y}_2\mathbf{Z}_3 = \mathbf{0}$, $\mathbf{Z}_3\mathbf{Y}_2 = \mathbf{0}$.

(b) $\beta = 0$ *and* $a\alpha \neq 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{45}$$

being $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$ *arbitrary.*

(c) $\beta \neq 0$ *and* $a\beta = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{46}$$

being $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ *arbitrary.*

(d) $\beta \neq 0$ *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{47}$$

being $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ *arbitrary.*

(e) $\beta \neq 0$, $a\beta \neq 1$, *and* $a\alpha \neq 1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}. \tag{48}$$

Proof. We can write a quadratic matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1},$$

where $p \in \{0, \ldots, n\}$ and $\mathbf{U} \in \mathbb{C}^n$ is a nonsingular matrix. We can represent $\mathbf{B}$ as $\mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{T} \end{pmatrix}\mathbf{U}^{-1}$, where $\mathbf{X} \in \mathbb{C}^p$. Observe that under the hypotheses $\mathbf{BAB} = \mathbf{AB}^2$, $\alpha \neq 0$, and $\alpha \neq \beta$, one has

$$\mathbf{YZ} = \mathbf{0},\ \mathbf{YT} = \mathbf{0},\ \mathbf{ZX} = \mathbf{0},\ \mathbf{ZY} = \mathbf{0}. \tag{49}$$

Let us assume that $\mathbf{K}$ is an idempotent matrix then one can deduce that

$$\begin{aligned} &\left(a\alpha\mathbf{I}_p + b\mathbf{X}\right)^2 + b^2\mathbf{YZ} = a\alpha\mathbf{I}_p + b\mathbf{X},\ ab\left(\alpha+\beta\right)\mathbf{Y} + b^2\left(\mathbf{XY} + \mathbf{YT}\right) = b\mathbf{Y}, \\ &ab\left(\alpha+\beta\right)\mathbf{Z} + b^2\left(\mathbf{ZX} + \mathbf{TZ}\right) = b\mathbf{Z},\ b^2\mathbf{ZY} + \left(a\beta\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = a\beta\mathbf{I}_{n-p} + b\mathbf{T}. \end{aligned} \tag{50}$$

Considering (49) and (50), we get the following equalities

$$\begin{aligned} &\left(a\alpha\mathbf{I}_p + b\mathbf{X}\right)^2 = a\alpha\mathbf{I}_p + b\mathbf{X},\ \left(a\beta\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = a\beta\mathbf{I}_{n-p} + b\mathbf{T}, \\ &ab\left(\alpha+\beta\right)\mathbf{Y} + b^2\mathbf{XY} = b\mathbf{Y},\ ab\left(\alpha+\beta\right)\mathbf{Z} + b^2\mathbf{TZ} = b\mathbf{Z}. \end{aligned} \tag{51}$$

It is clear that $a\alpha\mathbf{I}_p + b\mathbf{X}$ and $a\beta\mathbf{I}_{n-p} + b\mathbf{T}$ are idempotent matrices from the first and second equations in (51), respectively. So, there exist $q \in \{0, \ldots, p\}$, $r \in \{0, \ldots, n-p\}$ and nonsingular matrices $\mathbf{S}_1 \in \mathbb{C}^p$ and $\mathbf{S}_2 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{X} = \mathbf{S}_1\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} \end{pmatrix}\mathbf{S}_1^{-1}, \tag{52}$$

$$\mathbf{T} = \mathbf{S}_2\begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{S}_2^{-1}. \tag{53}$$

Let $\mathbf{Y}$ and $\mathbf{Z}$ be written as

$$\mathbf{Y} = \mathbf{S}_1\begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{Y}_4 \end{pmatrix}\mathbf{S}_2^{-1} \text{ and } \mathbf{Z} = \mathbf{S}_2\begin{pmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{pmatrix}\mathbf{S}_1^{-1}, \tag{54}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{q\times r}$ and $\mathbf{Z}_1 \in \mathbb{C}^{r\times q}$. Besides, defining $\mathbf{V} := \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)$, it follows that

$$\begin{aligned} \mathbf{A} &= \mathbf{U}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{U}^{-1} \\ &= \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)\begin{pmatrix} \mathbf{S}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2^{-1} \end{pmatrix}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\begin{pmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{pmatrix}\left(\mathbf{S}_1^{-1} \oplus \mathbf{S}_2^{-1}\right)\mathbf{U}^{-1} \\ &= \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1}. \end{aligned}$$

Substituting (52), (54) and (53), (54) into the third and forth equations in (51) it is obtained that

$$\begin{pmatrix} a\beta\mathbf{Y}_1 & a\beta\mathbf{Y}_2 \\ \left(a\beta-1\right)\mathbf{Y}_3 & \left(a\beta-1\right)\mathbf{Y}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} a\alpha\mathbf{Z}_1 & a\alpha\mathbf{Z}_2 \\ \left(a\alpha-1\right)\mathbf{Z}_3 & \left(a\alpha-1\right)\mathbf{Z}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{55}$$

Depending on the $a\beta$ and $a\alpha$, we have the following cases for $\mathbf{Y}$ and $\mathbf{Z}$.

(i) Let $a\beta = 0$ and $a\alpha = 1$.
It is clear that $\mathbf{Y}_3, \mathbf{Y}_4$ and $\mathbf{Z}_1, \mathbf{Z}_2$ are zero matrices from the equations in (55). So, $\mathbf{Y}$ and $\mathbf{Z}$ reduce to

$$\mathbf{Y} = \mathbf{S}_1\begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{S}_2^{-1} \text{ and } \mathbf{Z} = \mathbf{S}_2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{pmatrix}\mathbf{S}_1^{-1}.$$

Substituting **X**, **Y**, **Z**, and **T** into (49), **Y** and **Z** are obtained as

$$\mathbf{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ and } \mathbf{Z} = \mathbf{S}_2 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_1{}^{-1}, \tag{56}$$

where $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$, $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ are arbitrary matrices and $\mathbf{Y}_2\mathbf{Z}_3 = \mathbf{0}$, $\mathbf{Z}_3\mathbf{Y}_2 = \mathbf{0}$. Therefore, **B** we get as

$$\begin{aligned} \mathbf{B} &= \mathbf{U} \begin{pmatrix} \mathbf{S}_1 \begin{pmatrix} \mathbf{0}_q & \mathbf{0} \\ \mathbf{0} & \frac{-1}{b}\mathbf{I}_{p-q} \end{pmatrix} \mathbf{S}_1{}^{-1} & \mathbf{S}_1 \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{S}_2{}^{-1} \\ \mathbf{S}_2 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_1{}^{-1} & \mathbf{S}_2 \begin{pmatrix} \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix} \mathbf{S}_2{}^{-1} \end{pmatrix} \mathbf{U}^{-1} \\ &= \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right) \begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-1}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix} \left(\mathbf{S}_1{}^{-1} \oplus \mathbf{S}_2{}^{-1}\right) \mathbf{U}^{-1}. \end{aligned}$$

which establishes part (a).

(ii) Let $a\beta = 0$ and $a\alpha \neq 1$.
From the equations in (55), it is clear that $\mathbf{Y}_3, \mathbf{Y}_4$, and **Z** are zero matrices. Thus, **Y** is as in (56) and then

$$\begin{aligned} \mathbf{B} &= \mathbf{U} \begin{pmatrix} \mathbf{S}_1 \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} \end{pmatrix} \mathbf{S}_1{}^{-1} & \mathbf{S}_1 \begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{S}_2{}^{-1} \\ \mathbf{0} & \mathbf{S}_2 \begin{pmatrix} \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix} \mathbf{S}_2{}^{-1} \end{pmatrix} \mathbf{U}^{-1} \\ &= \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right) \begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix} \left(\mathbf{S}_1{}^{-1} \oplus \mathbf{S}_2{}^{-1}\right) \mathbf{U}^{-1}, \end{aligned}$$

where $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$ is an arbitrary matrix. So, it is completed part (b).

(iii) Let $a\beta = 1$ and $a\alpha \neq 1$.
It is obvious that $\mathbf{Y}_1, \mathbf{Y}_2$, and **Z** are zero matrices from the equations in (55). So, **Y** reduces to

$$\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Y}_3 & \mathbf{Y}_4 \end{pmatrix} \mathbf{S}_2{}^{-1}. \tag{57}$$

Using (53) and (57) in the second equation of (49), we get the equality $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1}{b}\mathbf{Y}_4 \end{pmatrix} = \mathbf{0}_{p\times(n-p)}$. Therefore

$$\mathbf{Y} = \mathbf{S}_1 \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Y}_3 & \mathbf{0} \end{pmatrix} \mathbf{S}_2{}^{-1},$$

where $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ is an arbitrary matrix and

$$\begin{aligned}\mathbf{B} &= \mathbf{U}\begin{pmatrix} \mathbf{S}_1\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q}\end{pmatrix}\mathbf{S}_1^{-1} & \mathbf{S}_1\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Y}_3 & \mathbf{0}\end{pmatrix}\mathbf{S}_2^{-1} \\ \mathbf{0} & \mathbf{S}_2\begin{pmatrix} \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & -\frac{1}{b}\mathbf{I}_{n-p-r}\end{pmatrix}\mathbf{S}_2^{-1}\end{pmatrix}\mathbf{U}^{-1} \\ &= \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{b}\mathbf{I}_{n-p-r}\end{pmatrix}\left(\mathbf{S}_1^{-1} \oplus \mathbf{S}_2^{-1}\right)\mathbf{U}^{-1}.\end{aligned}$$

which completes part (c).

(iv) Let $a\beta \neq 0$, $a\beta \neq 1$, and $a\alpha = 1$.
From the equations in (55), it is clear that $\mathbf{Y} = \mathbf{0}$ and the form of $\mathbf{Z}$ is as in (56). Hence,

$$\begin{aligned}\mathbf{B} &= \mathbf{U}\begin{pmatrix} \mathbf{S}_1\begin{pmatrix} \mathbf{0}_q & \mathbf{0} \\ \mathbf{0} & -\frac{1}{b}\mathbf{I}_{p-q}\end{pmatrix}\mathbf{S}_1^{-1} & \mathbf{0} \\ \mathbf{S}_2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0}\end{pmatrix}\mathbf{S}_1^{-1} & \mathbf{S}_2\begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r}\end{pmatrix}\mathbf{S}_2^{-1}\end{pmatrix}\mathbf{U}^{-1} \\ &= \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)\begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r}\end{pmatrix}\left(\mathbf{S}_1^{-1} \oplus \mathbf{S}_2^{-1}\right)\mathbf{U}^{-1},\end{aligned}$$

where $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ is an arbitrary matrix. So, the part (d) of the proof is completed.

(v) Let $a\beta \neq 0$, $a\beta \neq 1$, and $a\alpha \neq 1$.
From the equations in (55), it is easily seen that $\mathbf{Y} = \mathbf{0}$ and $\mathbf{Z} = \mathbf{0}$ thus,

$$\begin{aligned}\mathbf{B} &= \mathbf{U}\begin{pmatrix} \mathbf{S}_1\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q}\end{pmatrix}\mathbf{S}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2\begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r}\end{pmatrix}\mathbf{S}_2^{-1}\end{pmatrix}\mathbf{U}^{-1} \\ &= \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r}\end{pmatrix}\left(\mathbf{S}_1^{-1} \oplus \mathbf{S}_2^{-1}\right)\mathbf{U}^{-1}.\end{aligned}$$

Therefore, the part of the necessity of the proof is completed.

Now it is evident that if $\mathbf{A}$ is represented as in (43) and $\mathbf{B}$ is represented as in (44), (45), (46), (47) or (48) and the scalars $a\alpha, a\beta$ satisfy corresponding conditions, then $\mathbf{K}^2 = \mathbf{K}$. □

Theorem 2.10 is given under the condition $\mathbf{BAB} = \mathbf{AB}^2$. Premultiplying this condition by $\mathbf{A}$ leads to $\mathbf{A}^2\mathbf{B}^2 = (\mathbf{AB})^2$. A similar result can be given below under this new condition.

Theorem 2.11. *Let $\alpha, \beta \in \mathbb{C}^*$ with $\alpha \neq \beta$. Moreover, let $\mathbf{A}$ and $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be an $\{\alpha, \beta\}-$ quadratic matrix and an arbitrary matrix, respectively. Furthermore, let $\mathbf{A}^2\mathbf{B}^2 = (\mathbf{AB})^2$ and $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ with $a, b \in \mathbb{C}^*$. Then $\mathbf{K}$ is an idempotent matrix if and only if there exists a nonsingular matrix $\mathbf{V} \in \mathbb{C}^n$ such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p}\end{pmatrix}\mathbf{V}^{-1}$$

and $\mathbf{B}$ satisfies one of the following cases.

(a) $a\beta = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-1}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ *arbitrary.*

(b) $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-1}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_3 & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1},$$

being $\mathbf{Z}_3 \in \mathbb{C}^{(n-p-r)\times q}$ *arbitrary.*

(c) $a\alpha \neq 1$ *and* $a\beta \neq 1$,

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}.$$

The proof of this theorem is omitted since it is very similar to proof of Theorem 2.10.

Remark 2.12. *Note that the matrices* **B** *given in the last parts of Theorem 2.10 and 2.11 commute with the corresponding matrices* **A** *while there is no such necessity in other results.*

Lastly, let us give the following theorem.

Theorem 2.13. *Let* $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, $\alpha \neq \beta$, *and* $(\alpha, \beta) \notin \{(-1, 1), (1, -1)\}$. *Moreover, let* **A** *and* $\mathbf{B} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *be an* $\{\alpha, \beta\}-$ *quadratic matrix and an arbitrary matrix, respectively. Furthermore, let* $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{B}\mathbf{A}$ *and* $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ *with* $a, b \in \mathbb{C}^*$. *Then* **K** *is an idempotent matrix if and only if there exists a nonsingular matrix* $\mathbf{V} \in \mathbb{C}^n$ *such that*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1} \tag{58}$$

and **B** *satisfies one of the following cases.*

(a) $\beta = 0$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{59}$$

being $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$ *and* $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ *arbitrary.*

(b) $\beta^2 \neq 1$, $\beta \neq 0$, *and* $a\beta = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{W} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{n-p} \end{pmatrix}\mathbf{V}^{-1}, \tag{60}$$

being $\mathbf{W} \in \mathbb{C}^{(p-q)\times(n-p)}$ *arbitrary.*

(c) $\beta^2 = 1$ *and* $a\alpha = 1$.

$$\mathbf{B} = \mathbf{V}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}, \tag{61}$$

being $\mathbf{Z}_2 \in \mathbb{C}^{(n-p-r)\times p}$ *arbitrary.*

Proof. Let us write a quadratic matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1},$$

where $p \in \{0,\dots,n\}$ and $\mathbf{U} \in \mathbb{C}^n$ is a nonsingular matrix. We can represent $\mathbf{B}$ as $\mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{T} \end{pmatrix}\mathbf{U}^{-1}$ where $\mathbf{X} \in \mathbb{C}^p$. In view of the hypotheses $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{B}\mathbf{A}$ and $\alpha \neq 0$ we can write

$$\alpha^2\mathbf{X} = \mathbf{X},\ \alpha^2\beta\mathbf{Y} = \beta\mathbf{Y},\ \beta^2\mathbf{Z} = \mathbf{Z},\ \beta^3\mathbf{T} = \beta\mathbf{T}. \tag{62}$$

Let us assume that $\mathbf{K}$ is an idempotent matrix then it follows that

$$\begin{gathered} \left(a\alpha\mathbf{I}_p + b\mathbf{X}\right)^2 + b^2\mathbf{Y}\mathbf{Z} = a\alpha\mathbf{I}_p + b\mathbf{X},\ ab\left(\alpha+\beta\right)\mathbf{Y} + b^2\left(\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{T}\right) = b\mathbf{Y}, \\ ab\left(\alpha+\beta\right)\mathbf{Z} + b^2\left(\mathbf{Z}\mathbf{X} + \mathbf{T}\mathbf{Z}\right) = b\mathbf{Z},\ b^2\mathbf{Z}\mathbf{Y} + \left(a\beta\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = a\beta\mathbf{I}_{n-p} + b\mathbf{T}. \end{gathered} \tag{63}$$

The proof can be split into following cases, depending on the scalar β.

(i) Let $\beta^2 \neq 1$.
From (62), it is seen that $\mathbf{Z} = \mathbf{0}$. Reorganizing the equations of (63), it can be written

$$\begin{gathered} \left(a\alpha\mathbf{I}_p + b\mathbf{X}\right)^2 = a\alpha\mathbf{I}_p + b\mathbf{X},\ \left(a\beta\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = a\beta\mathbf{I}_{n-p} + b\mathbf{T}, \\ ab\left(\alpha+\beta\right)\mathbf{Y} + b^2\left(\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{T}\right) = b\mathbf{Y}. \end{gathered} \tag{64}$$

It is clear that $a\alpha\mathbf{I}_p + b\mathbf{X}$ and $a\beta\mathbf{I}_{n-p} + b\mathbf{T}$ are idempotent matrices from the first and second equations in (64), respectively. Then, there exist $q \in \{0,\dots,p\}$, $r \in \{0,\dots,n-p\}$ and nonsingular matrices $\mathbf{S}_1 \in \mathbb{C}^p$, $\mathbf{S}_2 \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{X} = \mathbf{S}_1\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} \end{pmatrix}\mathbf{S}_1{}^{-1} \quad\text{and}\quad \mathbf{T} = \mathbf{S}_2\begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{S}_2{}^{-1}. \tag{65}$$

Let us write $\mathbf{Y}$ as

$$\mathbf{Y} = \mathbf{S}_1\begin{pmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{Y}_4 \end{pmatrix}\mathbf{S}_2{}^{-1}, \tag{66}$$

where $\mathbf{Y}_1 \in \mathbb{C}^{q\times r}$. Substituting (65) and (66) into the third equation in (64) yields

$$\begin{pmatrix} b\mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & -b\mathbf{Y}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Then $\mathbf{Y}$ reduces to

$$\mathbf{Y} = \mathbf{S}_1\begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{0} \end{pmatrix}\mathbf{S}_2{}^{-1}, \tag{67}$$

where $\mathbf{Y}_2 \in \mathbb{C}^{q\times(n-p-r)}$ and $\mathbf{Y}_3 \in \mathbb{C}^{(p-q)\times r}$ are arbitrary matrices.

Let us define $\mathbf{V} := \mathbf{U}\left(\mathbf{S}_1 \oplus \mathbf{S}_2\right)$. In view of (65) and (67) we obtain that

$$\begin{aligned}\mathbf{A} &= \mathbf{U}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{U}^{-1} \\ &= \mathbf{V}\begin{pmatrix} \mathbf{S}_1{}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2{}^{-1} \end{pmatrix}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\begin{pmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{pmatrix}\mathbf{V}^{-1} \\ &= \mathbf{V}\begin{pmatrix} \alpha\mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I}_{n-p} \end{pmatrix}\mathbf{V}^{-1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{B} &= \mathbf{U}\begin{pmatrix} \mathbf{S}_1\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} \end{pmatrix}\mathbf{S}_1{}^{-1} & \mathbf{S}_1\begin{pmatrix} \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{0} \end{pmatrix}\mathbf{S}_2{}^{-1} \\ \mathbf{0} & \mathbf{S}_2\begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{S}_2{}^{-1} \end{pmatrix}\mathbf{U}^{-1} \\ &= \mathbf{V}\begin{pmatrix} \frac{1-a\alpha}{b}\mathbf{I}_q & \mathbf{0} & \mathbf{0} & \mathbf{Y}_2 \\ \mathbf{0} & \frac{-a\alpha}{b}\mathbf{I}_{p-q} & \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{V}^{-1}\end{aligned}$$

which yields part (a) for $\beta = 0$ and yields part (b) for $\beta \neq 0$ (then from (62), $\mathbf{T} = \mathbf{0}$).

(ii) Let $\beta^2 = 1$.

From the first and second equations in (62) and considering hypotheses $(\alpha, \beta) \notin \{(-1, 1), (1, -1)\}$ and $\alpha \neq \beta$, it is obvious that $\mathbf{X} = \mathbf{0}$ and $\mathbf{Y} = \mathbf{0}$. Reorganizing the equations of (63), it can be written

$$(a\alpha)^2\mathbf{I}_p = a\alpha\mathbf{I}_p,\ \left(a\beta\mathbf{I}_{n-p} + b\mathbf{T}\right)^2 = a\beta\mathbf{I}_{n-p} + b\mathbf{T},\ ab\,(\alpha+\beta)\,\mathbf{Z} + b^2\mathbf{T}\mathbf{Z} = b\mathbf{Z}. \tag{68}$$

It is explicit that $a\alpha = 1$ and $a\beta\mathbf{I}_{n-p} + b\mathbf{T}$ is an idempotent matrix from the first and second equations in (68). So, there exist $r \in \{0, \ldots, n-p\}$ and a nonsingular matrix $\mathbf{S} \in \mathbb{C}^{(n-p)}$ such that

$$\mathbf{T} = \mathbf{S}\begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{S}^{-1}. \tag{69}$$

Let us write $\mathbf{Z}$ as

$$\mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix}, \tag{70}$$

where $\mathbf{Z}_1 \in \mathbb{C}^{r\times p}$. Substituting (69) and (70) into the third equation in (68), it is obtained that

$$\begin{pmatrix} (a\alpha)\,\mathbf{Z}_1 \\ (a\alpha - 1)\,\mathbf{Z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

Since $a\alpha = 1$, $\mathbf{Z}$ turns into $\mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{0} \\ \mathbf{Z}_2 \end{pmatrix}$, where $\mathbf{Z}_2 \in \mathbb{C}^{(n-p-r)\times p}$ is an arbitrary matrix.

Hence, we can easily write

$$\mathbf{A} = \mathbf{U}\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\mathbf{U}^{-1} = \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)\left(\alpha\mathbf{I}_p \oplus \beta\mathbf{I}_{n-p}\right)\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}$$

and

$$\mathbf{B} = \mathbf{U}\begin{pmatrix} \mathbf{0}_p & \mathbf{0} \\ \mathbf{S}\begin{pmatrix} \mathbf{0} \\ \mathbf{Z}_2 \end{pmatrix} & \mathbf{S}\begin{pmatrix} \frac{1-a\beta}{b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \frac{-a\beta}{b}\mathbf{I}_{n-p-r} \end{pmatrix}\mathbf{S}^{-1} \end{pmatrix}\mathbf{U}^{-1}$$

$$= \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)\begin{pmatrix} \mathbf{0}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\alpha-\beta}{\alpha b}\mathbf{I}_r & \mathbf{0} \\ \mathbf{Z}_2 & \mathbf{0} & \frac{-\beta}{\alpha b}\mathbf{I}_{n-p-r} \end{pmatrix}\left(\mathbf{I}_p \oplus \mathbf{S}^{-1}\right)\mathbf{U}^{-1}.$$

The necessity part of the proof is completed by defining $\mathbf{V}$ as $\mathbf{V} := \mathbf{U}\left(\mathbf{I}_p \oplus \mathbf{S}\right)$.

Now, it is evident that if $\mathbf{A}$ is represented as in (58) and $\mathbf{B}$ is represented as in (59), (60) or (61) and the scalars α, β satisfy corresponding conditions, then $\mathbf{K}^2 = \mathbf{K}$. □

Example 2.14. *Let*

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 6 & -3 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 6 & 0 & 6 & -4 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 2 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & -4 \end{pmatrix}$$

and let us find all ordered pair (a, b) such that $\mathbf{K} = a\mathbf{A} + b\mathbf{B}$ is idempotent, where $a, b \in \mathbb{C}^*$. It is clear that $\mathbf{A}$ is a $\{-1, 2\}-$ quadratic matrix and $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{B}\mathbf{A}$. Moreover,

$$\mathbf{K} = \begin{pmatrix} 5a+2b & 0 & 6a+2b & -3a-2b \\ 0 & 2a & -3a & 0 \\ 0 & 0 & -a & 0 \\ 6a+4b & 0 & 6a+4b & -4a-4b \end{pmatrix}$$

and

$$\mathbf{K}^2 = \begin{pmatrix} 7a^2-4ab-4b^2 & 0 & 6a^2-4ab-4b^2 & -3a^2+4ab+4b^2 \\ 0 & 4a^2 & -3a^2 & 0 \\ 0 & 0 & a^2 & 0 \\ 6a^2-8ab-8b^2 & 0 & 6a^2-8ab-8b^2 & -2a^2+8ab+8b^2 \end{pmatrix}.$$

From the idempotency of $\mathbf{K}$, it is obtained that $(a, b) \in \{(0, 0), (0, -1/2)\}$. Although $\mathbf{A}$ is a $\{-1, 2\}-$ quadratic matrix, the form of the matrix $\mathbf{B}$ is not compatible with the part (a) of Theorem 2.13. Therefore, it is seen that there is no appropriate pair (a, b) to imply that $\mathbf{K}$ is idempotent.

Example 2.15. *Let $\mathbf{A}$ be as in the previous example and let us find $a \in \mathbb{C}^*$ and all matrices $\mathbf{B} \in \mathbb{C}^4$ such that $\mathbf{A}^2\mathbf{B}\mathbf{A} = \mathbf{B}\mathbf{A}$ and $a\mathbf{A} - \mathbf{B}$ is idempotent. A diagonalized form of $\mathbf{A}$ and the matrix $\mathbf{V}$ that diagonalize it are*

$$\mathbf{A} = \mathbf{V}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\mathbf{V}^{-1} \text{ and } \mathbf{V} = \begin{pmatrix} -1 & -1 & 0 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 \end{pmatrix},$$

compatible with (58). *Using the notations of Theorem 2.13, we can assume $\alpha = -1$, $\beta = 2$. Then only part (b) of Theorem 2.13 can be implied, so we get $a = 1/\beta = 1/2$, $n = 4$, $p = 2$, and $q \in \{0, 1, 2\}$. Therefore depending on the appearing and disappearing blocks of $\mathbf{V}^{-1}\mathbf{B}\mathbf{V}$, it can be written the following possible cases:*

POSSIBILITIES OF $\mathbf{V}^{-1}\mathbf{B}\mathbf{V}$		
$q = 0$	$q = 1$	$q = 2$
$\begin{pmatrix} -1/2 & 0 & e & f \\ 0 & -1/2 & g & h \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -3/2 & 0 & 0 & 0 \\ 0 & -1/2 & m & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -3/2 & 0 & 0 & 0 \\ 0 & -3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

where e, f, g, h, m, and s are arbitrary complex numbers.

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