Filomat 33:10 (2019), 3149–3159
https://doi.org/10.2298/FIL1910149S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by Operators Sequence Involving a Certain Class of Polynomials

Sezgin Sucu[a]

[a] *Ankara University, Faculty of Science, Department of Mathematics, TR-06100, Ankara, Turkey.*

Abstract. This paper is devoted to the convergence properties of the operators sequence including a certain class of polynomials in the continuous functions space and weighted continuous functions space. The convergence order for the presented operators sequence is also discussed. At last, operators including Hermite polynomials of variance ν are given as an application.

1. Introduction

Bernstein polynomials were proposed by Bernstein [5] to contribute to the proof of the Weierstrass theorem in virtue of probabilistic considerations. Quantitative properties of Bernstein polynomials together with several generalizations have been studied by many authors ([6], [15], [22], [25], [27]). Henceforth, there have been significant developments in the investigation of approximating functions which are continuous on the interval $[a, b]$ and have specific conditions on the real line. In 1930 Kantorovich [21] made a small modification to Bernstein polynomials for approximating Lebesgue integrable functions.

Szasz-Mirakjan operators which defined with the help of Poisson distribution have been introduced by Mirakjan [23] and were investigated by many authors ([9], [27]). This operator, denoted by S_n, associates to any function $\varphi \in C[0,\infty)$ the operator $S_n(\varphi, .)$, defined by

$$S_n(\varphi; x) = \exp(-nx) \sum_{\nu=0}^{\infty} \frac{(nx)^{\nu}}{\nu!} \varphi\left(\frac{\nu}{n}\right). \tag{1}$$

The polynomials q_ν having the following form

$$g(\omega)\exp(\omega x) = \sum_{\nu=0}^{\infty} q_\nu(x)\,\omega^{\nu} \tag{2}$$

are called Appell polynomials, where g is a holomorphic function in the $|\omega| < R$ $(R > 1)$ and suppose that $g(1) \neq 0$. Szasz type operators containing Appell polynomials were constructed by Jakimovski and Leviatan [16]. If $q_\nu(x) \geq 0$ for $x \in [0,\infty)$, the operators

$$P_n(\varphi; x) = \frac{\exp(-nx)}{g(1)} \sum_{\nu=0}^{\infty} q_\nu(nx)\,\varphi\left(\frac{\nu}{n}\right) \tag{3}$$

2010 *Mathematics Subject Classification*. Primary 41A35; Secondary 41A25
Keywords. Qualitative and quantitative approximation results, Hermite polynomials of variance ν.
Received: 16 November 2018; Accepted: 11 February 2019
Communicated by Ivana Djolović
Email address: ssucu@ankara.edu.tr (Sezgin Sucu)

were introduced in [16] and then the convergence properties of (3) were presented with the help of Szasz's perspective by Jakimovski and Leviatan.

The polynomials q_ν having the following form

$$A(\omega)\exp(xH(\omega)) = \sum_{\nu=0}^{\infty} q_\nu(x)\,\omega^\nu, \ |\omega| < R, \ (R > 1)$$

are called Sheffer polynomials, where A and H are analytic functions. By assuming $q_\nu(x) \geq 0$, $A(1) \neq 0$ and $H'(1) = 1$, the operators defined as

$$V_n(\varphi; x) = \frac{\exp(-nxH(1))}{A(1)} \sum_{\nu=0}^{\infty} q_\nu(nx)\,\varphi\left(\frac{\nu}{n}\right)$$

were presented by Ismail [12].

The polynomials q_ν having the following form

$$A(\omega)B(x\omega) = \sum_{\nu=0}^{\infty} q_\nu(x)\,\omega^\nu \tag{4}$$

are called Brenke type polynomials [7], where

$$A(\omega) = \sum_{\nu=0}^{\infty} a_\nu \omega^\nu, \ a_0 \neq 0, \tag{5}$$

$$B(\omega) = \sum_{\nu=0}^{\infty} b_\nu \omega^\nu, \ b_\nu \neq 0 \ (\nu \geq 0) \tag{6}$$

and for $k \in \mathbb{N}_0$

$$q_\nu(x) = \sum_{j=0}^{\nu} a_{\nu-j} b_j x^j. \tag{7}$$

Under the some assumptions, Varma [28] presented the linear positive operators as follows

$$L_n(\varphi; x) = \frac{1}{A(1)B(nx)} \sum_{\nu=0}^{\infty} q_\nu(nx)\,\varphi\left(\frac{\nu}{n}\right). \tag{8}$$

There are also some other articles investigating the convergence result of these type operators sequence involving certain class of polynomials ([1], [4], [11], [14], [17], [18], [19], [20], [26]).

The following operators are the main concern of this paper

$$\mathcal{T}_n(f; x) = \frac{1}{\lambda(2nx+1)} \sum_{\nu=0}^{\infty} q_\nu(nx)\, f\left(\frac{\nu}{n}\right), \tag{9}$$

where q_ν are polynomials [2] which have following form

$$\lambda\left(2x\omega + \omega^2\right) = \sum_{\nu=0}^{\infty} q_\nu(x)\,\omega^\nu \tag{10}$$

and λ is analytic function such that

$$\lambda(\omega) = \sum_{\nu=0}^{\infty} a_\nu \omega^\nu \ (a_\nu \neq 0). \tag{11}$$

Let us confine ourselves to the polynomials (10) which have the following properties

$$\begin{array}{ll} (i) & \lambda:\mathbb{R}\longrightarrow(0,\infty),\ q_{\nu}(x)\geq 0,\\ (ii) & \text{For } |\omega|<R\ (R>1),\ (10) \text{ and } (11) \text{ converge.}\end{array} \tag{12}$$

With reference to the (12), $\mathcal{T}_n$ operators defining by (9) are positive.

The objective of the following work is to investigate approximation of the operators defined as (9) in the continuous functions space and weighted continuous functions space. Moreover, the degree of convergence of a function is obtained.

The organization of this work is the following. Firstly, we succinctly acquire some essential knowledge about operators (9) and express our main results about convergence of operators (9) in continuous functions space and weighted continuous functions space. Furthermore, we obtain the rate of convergence through the instrument of modulus of continuity, Peetre's K functional and second order modulus of continuity in continuous functions space and give the order of approximation under favour of weighted modulus of continuity in weighted continuous functions space. In section 3, operators including Hermite polynomials of variance ν are given as an example.

2. Approximation Properties of $\mathcal{T}_n$ Operators

The following set E

$$E=\left\{f:\frac{f(x)}{1+x^2}\text{ is convergent as }x\to\infty\right\}$$

will be critical in our result.

We first prove two auxiliary lemmas.

Lemma 2.1. *The operators* (9) *admit*

$$\begin{aligned}
\mathcal{T}_n(1;x) &= 1,\\
\mathcal{T}_n(s;x) &= 2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)}x+2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)\,n},\\
\mathcal{T}_n\left(s^2;x\right) &= 4\frac{\lambda''(2nx+1)}{\lambda(2nx+1)}x^2+2\frac{4\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n}x+4\frac{\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n^2},\\
\mathcal{T}_n\left(s^3;x\right) &= 8\frac{\lambda'''(2nx+1)}{\lambda(2nx+1)}x^3+12\frac{2\lambda'''(2nx+1)+\lambda''(2nx+1)}{\lambda(2nx+1)\,n}x^2\\
&\quad+2\frac{12\lambda'''(2nx+1)+18\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n^2}x\\
&\quad+8\frac{\lambda'''(2nx+1)+3\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n^3},\\
\mathcal{T}_n\left(s^4;x\right) &= 16\frac{\lambda^{(4)}(2nx+1)}{\lambda(2nx+1)}x^4+16\frac{4\lambda^{(4)}(2nx+1)+3\lambda'''(2nx+1)}{\lambda(2nx+1)\,n}x^3\\
&\quad+4\frac{24\lambda^{(4)}(2nx+1)+48\lambda'''(2nx+1)+7\lambda''(2nx+1)}{\lambda(2nx+1)\,n^2}x^2\\
&\quad+2\frac{32\lambda^{(4)}(2nx+1)+120\lambda'''(2nx+1)+64\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n^3}x\\
&\quad+16\frac{\lambda^{(4)}(2nx+1)+6\lambda'''(2nx+1)+7\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n^4}.
\end{aligned} \tag{13}$$

Proof. Comparing the relations (9) and (10), we establish the first equality of (13). Differentiating the statement (10) with respect to ω, we get

$$\begin{aligned}
\sum_{\nu=1}^{\infty} q_\nu(x)\,\nu\omega^{\nu-1} &= 2\lambda'\left(2x\omega+\omega^2\right)[x+\omega],\\
\sum_{\nu=2}^{\infty} q_\nu(x)\,\nu(\nu-1)\,\omega^{\nu-2} &= 4\lambda''\left(2x\omega+\omega^2\right)[x+\omega]^2+2\lambda'\left(2x\omega+\omega^2\right),\\
\sum_{\nu=3}^{\infty} q_\nu(x)\,\nu(\nu-1)(\nu-2)\,\omega^{\nu-3} &= 8\lambda'''\left(2x\omega+\omega^2\right)[x+\omega]^3+12\lambda''\left(2x\omega+\omega^2\right)[x+\omega],\\
\sum_{\nu=4}^{\infty} q_\nu(x)\,\nu(\nu-1)(\nu-2)(\nu-3)\,\omega^{\nu-4} &= 16\lambda^{(4)}\left(2x\omega+\omega^2\right)[x+\omega]^4+48\lambda'''\left(2x\omega+\omega^2\right)[x+\omega]^2\\
&\quad+12\lambda''\left(2x\omega+\omega^2\right). \qquad (14)
\end{aligned}$$

To obtain the required equalities, we replace x with nx and ω with 1 in the above statements. Finally, considering (14) in the definition of operators given by (9) will conclude the proof of lemma. □

Lemma 2.2. *With λ and $\mathcal{T}_n$ as above, we have*

$$\begin{aligned}
\mathcal{T}_n(s-x;x) &= \left[2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)}-1\right]x+2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)\,n},\\
\mathcal{T}_n\left((s-x)^2;x\right) &= \left[4\frac{\lambda''(2nx+1)-\lambda'(2nx+1)}{\lambda(2nx+1)}+1\right]x^2+2\frac{4\lambda''(2nx+1)-\lambda'(2nx+1)}{\lambda(2nx+1)\,n}x\\
&\quad+4\frac{\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n^2},\\
\mathcal{T}_n\left((s-x)^4;x\right) &= \left[8\frac{2\lambda^{(4)}(2nx+1)-4\lambda'''(2nx+1)+3\lambda''(2nx+1)-\lambda'(2nx+1)}{\lambda(2nx+1)}+1\right]x^4\\
&\quad+4\frac{16\lambda^{(4)}(2nx+1)-12\lambda'''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n}x^3\\
&\quad+4\frac{24\lambda^{(4)}(2nx+1)+24\lambda'''(2nx+1)-23\lambda''(2nx+1)+4\lambda'(2nx+1)}{\lambda(2nx+1)\,n^2}x^2\\
&\quad+2\frac{32\lambda^{(4)}(2nx+1)+104\lambda'''(2nx+1)+16\lambda''(2nx+1)-15\lambda'(2nx+1)}{\lambda(2nx+1)\,n^3}x\\
&\quad+16\frac{\lambda^{(4)}(2nx+1)+6\lambda'''(2nx+1)+7\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)\,n^4}.
\end{aligned}$$

Proof. Because of the linearity of $\mathcal{T}_n$, we can write

$$\mathcal{T}_n(s-x;x)=\mathcal{T}_n(s;x)-x\mathcal{T}_n(1;x).$$

In a similar manner, we may obtain

$$\mathcal{T}_n\left((s-x)^2;x\right)=\mathcal{T}_n\left(s^2;x\right)-2x\mathcal{T}_n(s;x)+x^2\mathcal{T}_n(1;x)$$

and

$$\mathcal{T}_n\left((s-x)^4;x\right)=\mathcal{T}_n\left(s^4;x\right)-4x\mathcal{T}_n\left(s^3;x\right)+6x^2\mathcal{T}_n\left(s^2;x\right)-4x^3\mathcal{T}_n(s;x)+x^4\mathcal{T}_n(1;x).$$

Substituting the equalities given by (13) in the above statements, we conclude the desired result. □

Theorem 2.3. *Suppose that the following*

$$\lim_{y\to\infty}\frac{\lambda'(y)}{\lambda(y)}=\frac{1}{2}\quad\text{and}\quad\lim_{y\to\infty}\frac{\lambda''(y)}{\lambda(y)}=\frac{1}{4}\tag{15}$$

are provided. So, for $f\in C[0,\infty)\cap E$

$$\lim_{n\to\infty}\mathcal{T}_n(f;x)=f(x)$$

converges uniformly on every bounded and closed subset of $[0,\infty)$.

Proof. Keep in view (15) in Lemma 2.1, we infer for $i=0,1,2$

$$\lim_{n\to\infty}\mathcal{T}_n\left(s^i;x\right)=x^i$$

converge uniformly on every bounded and closed subset of $[0,\infty)$. Putting into practice the Korovkin-type property from [3], the proof is completed. □

Remark 2.4. *The following examples serve to illustrate some functions which satisfy conditions* (15)*:*
(*a*) *For* $i\in\mathbb{N}$, $\lambda(y)=y^ie^{\frac{y}{2}}$,
(*b*) *For q is any polynomial with positive coefficients, the functions* $\lambda(y)=q(y)\exp\left(\frac{y}{2}\right)$.

If we make use of the representation $\tilde{C}[0,\infty)$ for the class of uniformly continuous functions on $[0,\infty)$, for $\varphi\in\tilde{C}[0,\infty)$

$$w(\varphi;\rho):=\sup\left\{\left|\varphi(x)-\varphi(y)\right|:x,y\in[0,\infty),\ \left|x-y\right|\le\rho\right\}\tag{16}$$

is titled by modulus of continuity attached to the function φ.

Theorem 2.5. *Suppose that* $f\in\tilde{C}[0,\infty)\cap E$*, then*

$$\left|\mathcal{T}_n(f;x)-f(x)\right|\le 2w\left(f;\left[\mathcal{T}_n\left((s-x)^2;x\right)\right]^{1/2}\right).$$

Proof. In compliance with (13) and also taking advantage of the (16),

$$\begin{aligned}\left|\mathcal{T}_n(f;x)-f(x)\right| &\le \frac{1}{\lambda(2nx+1)}\sum_{\nu=0}^{\infty}q_\nu(nx)\left|f\left(\frac{\nu}{n}\right)-f(x)\right|\\ &\le \left(1+\frac{1}{\rho}\frac{1}{\lambda(2nx+1)}\sum_{\nu=0}^{\infty}q_\nu(nx)\left|\frac{\nu}{n}-x\right|\right)w(f;\rho)\end{aligned}\tag{17}$$

are deduced. When we take into account the infinite series in (17), then the following inequality

$$\begin{aligned}\sum_{\nu=0}^{\infty}q_\nu(nx)\left|\frac{\nu}{n}-x\right| &\le \left\{\sum_{\nu=0}^{\infty}q_\nu(nx)\right\}^{1/2}\left\{\sum_{\nu=0}^{\infty}q_\nu(nx)\left[\frac{\nu}{n}-x\right]^2\right\}^{1/2}\\ &= \lambda(2nx+1)\left\{\left(4\frac{\lambda''(2nx+1)-\lambda'(2nx+1)}{\lambda(2nx+1)}+1\right)x^2\right.\\ &\quad +2\frac{4\lambda''(2nx+1)-\lambda'(2nx+1)}{\lambda(2nx+1)n}x\\ &\quad \left.+4\frac{\lambda''(2nx+1)+\lambda'(2nx+1)}{\lambda(2nx+1)n^2}\right\}^{1/2}\end{aligned}$$

is derived from well known Hölder inequality. Considering the last inequality in (17),

$$\left|\mathcal{T}_n(f;x)-f(x)\right|\leq\left\{1+\frac{1}{\rho}\left[\mathcal{T}_n\left((s-x)^2;x\right)\right]^{1/2}\right\}w(f;\rho)$$

can be composed. With choosing $\rho=\left[\mathcal{T}_n\left((s-x)^2;x\right)\right]^{1/2}$ in the above expression, we get the conclusion. □

For $0<\gamma\leq 1$ and $M>0$,

$$Lip_M(\gamma):=\left\{\varphi\in C_B[0,\infty):\left|\varphi(x)-\varphi(y)\right|\leq M\left|x-y\right|^{\gamma}\right\}$$

is called Lipschitz class of order γ, where $C_B[0,\infty)$ represents the space of all continuous and bounded functions on $[0,\infty)$.

Results of Theorem 2.6 can be used to estimate for the approximation error of $\mathcal{T}_n$ to function φ in $Lip_M(\gamma)$.

Theorem 2.6. *Consider the function φ in $Lip_M(\gamma)$, so then*

$$\left|\mathcal{T}_n(\varphi;x)-\varphi(x)\right|\leq M\left[\mathcal{T}_n\left((s-x)^2;x\right)\right]^{\gamma/2}.$$

Proof. Due to the fact that $\mathcal{T}_n$ operators are monotone,

$$\left|\mathcal{T}_n(\varphi;x)-\varphi(x)\right|\leq M\mathcal{T}_n\left(|s-x|^{\gamma};x\right)$$

can be expressed. If well known Hölder inequality could be applied, the above argument shows that

$$\begin{aligned}\left|\mathcal{T}_n(\varphi;x)-\varphi(x)\right| &\leq M\frac{1}{\lambda(2nx+1)}\left[\sum_{\nu=0}^{\infty}q_\nu(nx)\right]^{\frac{2-\gamma}{2}}\left[\sum_{\nu=0}^{\infty}q_\nu(nx)\left(\frac{\nu}{n}-x\right)^2\right]^{\frac{\gamma}{2}}\\ &\leq M\left[\mathcal{T}_n\left((s-x)^2;x\right)\right]^{\gamma/2},\end{aligned}$$

which proves the result. □

Let us denote the set of all continuous and bounded functions on $[0,\infty)$ that have a continuous and bounded first and second derivative by $C_B^2[0,\infty)$. The following functionals

$$\left\|\varphi\right\|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}\left|\varphi(x)\right|,$$

$$\left\|\psi\right\|_{C_B^2[0,\infty)}=\left\|\psi\right\|_{C_B[0,\infty)}+\left\|\psi'\right\|_{C_B[0,\infty)}+\left\|\psi''\right\|_{C_B[0,\infty)}$$

are a norm, respectively, on $C_B[0,\infty)$ and $C_B^2[0,\infty)$. Let $\varphi\in C_B[0,\infty)$. In order to obtain the estimations we are going to define [8]

$$w_2(\varphi;\rho)=\sup_{0<t\leq\rho}\left\|\varphi(.+2t)-2\varphi(.+t)+\varphi(.)\right\|_{C_B[0,\infty)},$$

which is called the second modulus of continuity attached to the function φ and

$$\mathcal{K}(\varphi;\rho)=\inf_{\psi\in C_B^2[0,\infty)}\left\{\left\|\varphi-\psi\right\|_{C_B[0,\infty)}+\rho\left\|\psi''\right\|_{C_B[0,\infty)}\right\},$$

which is called the functional of Peetre attached to the function φ.

Theorem 2.7. *Consider the function f in $C_B^2[0,\infty)$, so then*

$$\left|\mathcal{T}_n(f;x)-f(x)\right|\leq\sigma_n(x)\left\|f\right\|_{C_B^2[0,\infty)},$$

where

$$\begin{aligned}\sigma_n(x) &= \left\{\frac{4\left[\lambda''(2nx+1)-\lambda'(2nx+1)\right]+\lambda(2nx+1)}{2\lambda(2nx+1)}x^2\right.\\&+\frac{4\lambda''(2nx+1)+[2n-1]\lambda'(2nx+1)-n\lambda(2nx+1)}{n\lambda(2nx+1)}x\\&\left.+2\frac{\lambda''(2nx+1)+[n+1]\lambda'(2nx+1)}{n^2\lambda(2nx+1)}\right\}.\end{aligned}$$

Proof. To obtain the mentioned inequality, as a beginning we use the linearity of $\mathcal{T}_n$ and then apply standard expansion of Taylor

$$\left|\mathcal{T}_n(f;x)-f(x)\right|\leq\left|f'(x)\right|\mathcal{T}_n(s-x;x)+\frac{\left|f''(\eta)\right|}{2}\mathcal{T}_n\left((s-x)^2;x\right),\ \eta\in(x,s).$$

The result

$$\begin{aligned}\left|\mathcal{T}_n(f;x)-f(x)\right| &\leq \left\{\frac{4\left[\lambda''(2nx+1)-\lambda'(2nx+1)\right]+\lambda(2nx+1)}{2\lambda(2nx+1)}x^2\right.\\&+\frac{4\lambda''(2nx+1)+[2n-1]\lambda'(2nx+1)-n\lambda(2nx+1)}{n\lambda(2nx+1)}x\\&\left.+2\frac{\lambda''(2nx+1)+[n+1]\lambda'(2nx+1)}{n^2\lambda(2nx+1)}\right\}\left\|f\right\|_{C_B^2[0,\infty)}\\&= \sigma_n(x)\left\|f\right\|_{C_B^2[0,\infty)}\end{aligned}$$

is obtained after the elementary calculation. □

Theorem 2.8. *Consider the function φ in $C_B[0,\infty)$, so then*

$$\left|\mathcal{T}_n(\varphi;x)-\varphi(x)\right|\leq Cw_2\left(\varphi;\sqrt{\kappa_n(x)}\right)+w\left(\varphi;\left(2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)}-1\right)x+2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right)\tag{18}$$

where

$$\kappa_n(x)=\frac{1}{8}\left\{\mathcal{T}_n\left((s-x)^2;x\right)+\left[\left(2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)}-1\right)x+2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right]^2\right\},$$

and C is constant.

Proof. We start with defining an operator $\mathcal{H}_n$ by

$$\mathcal{H}_n(\varphi;x)=\mathcal{T}_n(\varphi;x)-\varphi\left(2\left[\frac{\lambda'(2nx+1)}{\lambda(2nx+1)}x+\frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right]\right)+\varphi(x).$$

The result

$$\mathcal{H}_n(s-x;x)=0\tag{19}$$

can be deduced from Lemma 2.2. For this purpose, let us represent the function $\psi \in C_B^2[0,\infty)$ in the form

$$\psi(s) = \psi(x) + (s-x)\psi'(x) + \int_x^s (s-u)\psi''(u)\,du.$$

Bearing in mind the above equality and (19)

$$\begin{aligned}
\left|\mathcal{H}_n(\psi;x) - \psi(x)\right| &= \left|\mathcal{H}_n\left(\int_x^s (s-u)\psi''(u)\,du;x\right)\right| \\
&\leq \left|\mathcal{T}_n\left(\int_x^s (s-u)\psi''(u)\,du;x\right)\right| \\
&\quad + \left|\int_x^{2\left[\frac{\lambda'(2nx+1)}{\lambda(2nx+1)}x+\frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right]} \left\{2\left[\frac{\lambda'(2nx+1)}{\lambda(2nx+1)}x + \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right] - u\right\}\psi''(u)\,du\right| \\
&\leq \frac{1}{2}\left\{\mathcal{T}_n\left((s-x)^2;x\right) + \left[\left(2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)} - 1\right)x + 2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right]^2\right\}\left\|\psi''\right\|_{C_B[0,\infty)} \\
&= 4\kappa_n(x)\left\|\psi''\right\|_{C_B[0,\infty)} \qquad (20)
\end{aligned}$$

can be written. Owing to the fact Lemma 2.1 and (20)

$$\begin{aligned}
\left|\mathcal{T}_n(\varphi;x) - \varphi(x)\right| &\leq \left|\mathcal{H}_n(\varphi-\psi;x) - (\varphi-\psi)(x)\right| + \left|\mathcal{H}_n(\psi;x) - \psi(x)\right| \\
&\quad + \left|\varphi\left(2\left[\frac{\lambda'(2nx+1)}{\lambda(2nx+1)}x + \frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right]\right) - \varphi(x)\right| \\
&\leq 4\left\|\varphi-\psi\right\|_{C_B[0,\infty)} + 4\kappa_n(x)\left\|\psi''\right\|_{C_B[0,\infty)} \\
&\quad + w\left(\varphi;\left(2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)} - 1\right)x + 2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right)
\end{aligned}$$

can be therefore hold. We can thus see in the light of the property of Peetre's K functional

$$\begin{aligned}
\left|\mathcal{T}_n(\varphi;x) - \varphi(x)\right| &\leq 4\mathcal{K}(\varphi;\kappa_n(x)) + w\left(\varphi;\left(2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)} - 1\right)x + 2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right) \\
&\leq Cw_2\left(\varphi;\sqrt{\kappa_n(x)}\right) + w\left(\varphi;\left(2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)} - 1\right)x + 2\frac{\lambda'(2nx+1)}{\lambda(2nx+1)n}\right).
\end{aligned}$$

This completes the proof. □

Prior to stating and proving the weighted approximation, we first state below the family of functions

$$B_\phi[0,\infty) = \left\{f : \left|f(x)\right| \leq M_f\phi(x),\ M_f \text{ is a constant}\right\},$$

and as a subset of $B_\phi[0,\infty)$ the following class is defined by

$$C_\phi[0,\infty) = B_\phi[0,\infty) \cap C[0,\infty),$$

and as a subset of $C_\phi[0,\infty)$ the following class is defined by

$$C_\phi^k[0,\infty) = \left\{f : f \in C_\phi[0,\infty) \text{ and } \lim_{x\to\infty}\frac{f(x)}{\phi(x)} = k \text{ (a constant)}\right\},$$

where $\phi(x) = 1 + x^2$. The remarkable aspect of this family is that the following functional

$$\left\|f\right\|_{\phi} = \sup_{x\geq 0} \frac{\left|f(x)\right|}{\phi(x)}$$

is a norm on $B_{\phi}[0,\infty)$.

Theorem 2.9. *If $f \in C_{\phi}^{k}[0,\infty)$ is the case then*

$$\lim_{n\to\infty} \left\|\mathcal{T}_n(f;x) - f(x)\right\|_{\phi} = 0.$$

Proof. Considering the identities proved in Lemma 2.1, it immediately follows that

$$\lim_{n\to\infty} \|\mathcal{T}_n(1;x) - 1\|_{\phi} = 0$$

and

$$\lim_{n\to\infty} \|\mathcal{T}_n(s;x) - x\|_{\phi} = 0.$$

A combination of Lemma 2.1 and the definition of norm yields for given $\epsilon > 0$ by choosing $n \in \mathbb{N}$ sufficiently large

$$\sup_{x\in\mathbb{R}^+} \frac{\left|\mathcal{T}_n\left(s^2;x\right) - x^2\right|}{1+x^2} \leq \epsilon \sup_{x\in\mathbb{R}^+} \frac{4x^2 + \frac{10}{n}x + \frac{8}{n^2}}{1+x^2} + \frac{3}{n} \sup_{x\in\mathbb{R}^+} \frac{x}{1+x^2} + \frac{3}{n^2} \sup_{x\in\mathbb{R}^+} \frac{1}{1+x^2}.$$

This proves $\lim_{n\to\infty} \left\|\mathcal{T}_n\left(s^2;x\right) - x^2\right\|_{\phi} = 0$. These facts enable us to complete of the proof by the weighted Korovkin-type theorem [10]. □

We next take up the topic of quantitative result for weighted space [13]. To obtain quantitative approximation result we will need to use

$$\Omega(f;\rho) := \sup_{x\in[0,\infty),|h|\leq\rho} \frac{\left|f(x+h) - f(x)\right|}{(1+h^2)(1+x^2)},\ f \in C_{\phi}^{k}[0,\infty).$$

To prove the result we require the following expression

$$\lim_{\rho\to 0} \Omega(f;\rho) = 0,$$

$$\left|f(t) - f(x)\right| \leq 2\left(1 + \frac{|t-x|}{\rho}\right)\left(1+\rho^2\right)\left(1+x^2\right)\left(1+(t-x)^2\right)\Omega(f;\rho),\ t,x \in [0,\infty). \tag{21}$$

The following theorem, under the suitable restriction

$$\lim_{y\to\infty} \frac{\lambda^{(k)}(y)}{\lambda(y)} = \frac{1}{2^k},\ k = 1,2,3,4$$

on λ, provides an error estimate.

Theorem 2.10. *Consider the function f in $C_{\phi}^{k}[0,\infty)$, so then*

$$\sup_{x\in[0,\infty)} \frac{\left|\mathcal{T}_n(f;x) - f(x)\right|}{(1+x^2)^3} \leq K\left(1 + \frac{1}{n}\right)\Omega\left(f;\frac{1}{\sqrt{n}}\right)$$

holds for some constant K which is independent of n.

Proof. It follows directly from Lemma 2.1 and expression (21) that

$$\begin{aligned}\left|\mathcal{T}_n(f;x)-f(x)\right| &\leq \frac{1}{\lambda(2nx+1)}\sum_{\nu=0}^{\infty}q_\nu(nx)\left|f\left(\frac{\nu}{n}\right)-f(x)\right| \\ &\leq 2\left(1+\rho^2\right)\left(1+x^2\right)\Omega(f;\rho)\frac{1}{\lambda(2nx+1)}\sum_{\nu=0}^{\infty}q_\nu(nx)\left(1+\frac{\left|\frac{\nu}{n}-x\right|}{\rho}\right)\left(1+\left(\frac{\nu}{n}-x\right)^2\right) \\ &= 2\left(1+\rho^2\right)\left(1+x^2\right)\Omega(f;\rho)\frac{1}{\lambda(2nx+1)}\left\{\sum_{\nu=0}^{\infty}q_\nu(nx)+\sum_{\nu=0}^{\infty}q_\nu(nx)\left(\frac{\nu}{n}-x\right)^2\right. \\ &\quad \left.+\frac{1}{\rho}\sum_{\nu=0}^{\infty}q_\nu(nx)\left|\frac{\nu}{n}-x\right|+\frac{1}{\rho}\sum_{\nu=0}^{\infty}q_\nu(nx)\left|\frac{\nu}{n}-x\right|\left(\frac{\nu}{n}-x\right)^2\right\}.\end{aligned}$$

If well known Hölder inequality could be applied for the above series, the mentioned relation therefore gives

$$\begin{aligned}\left|\mathcal{T}_n(f;x)-f(x)\right| &\leq 2\left(1+\rho^2\right)\left(1+x^2\right)\Omega(f;\rho) \\ &\quad\times\left\{1+\mathcal{T}_n\left((s-x)^2;x\right)+\frac{1}{\rho}\sqrt{\mathcal{T}_n\left((s-x)^2;x\right)}\right. \\ &\quad\left.+\frac{1}{\rho}\sqrt{\mathcal{T}_n\left((s-x)^2;x\right)\mathcal{T}_n\left((s-x)^4;x\right)}\right\}. \qquad (22)\end{aligned}$$

Taking into consideration Lemma 2.2, we readily find

$$\begin{aligned}\mathcal{T}_n\left((s-x)^2;x\right) &\leq O(1)\left(x^2+x+1\right), \\ \mathcal{T}_n\left((s-x)^4;x\right) &\leq O\left(\frac{1}{n}\right)\left(x^4+x^3+x^2+x+1\right).\end{aligned}$$

If these expressions are combined with (22) and by choosing $\rho=\frac{1}{\sqrt{n}}$, we therefore have

$$\sup_{x\in[0,\infty)}\frac{\left|\mathcal{T}_n(f;x)-f(x)\right|}{\left(1+x^2\right)^3}\leq K\left(1+\frac{1}{n}\right)\Omega\left(f;\frac{1}{\sqrt{n}}\right).$$

□

3. Example

In the present section, our plan is to obtain an explicit example of operators (9) including certain set of polynomials ensuring all restrictions (12) and assumptions (15).

Example 3.1. *Generating function of the Hermite polynomials $H_k^{(\nu)}$ of variance ν is given by [24]*

$$e^{-\frac{\nu\omega^2}{2}+x\omega}=\sum_{k=0}^{\infty}\frac{H_k^{(\nu)}(x)}{k!}\omega^k,$$

where $H_k^{(\nu)}$ can be calculated by the following sum

$$H_k^{(\nu)}(x)=\sum_{r=0}^{\left[\frac{k}{2}\right]}\left(-\frac{\nu}{2}\right)^r\frac{k!}{r!\,(k-2r)!}x^{k-2r}.$$

Let $\lambda(\omega) = e^{\frac{\omega}{2}}$. One can show that the polynomials $H_k^{(\nu)}$ satisfy the relation (10) *for λ. The polynomials $H_k^{(\nu)}$ with $\nu < 0$ will satisfy the expressions* (12) *and* (15). *Hence, the explicit form of $\mathcal{T}_n$ can be obtained by applying* (10)

$$\mathcal{T}_n^*(f;x) = e^{-\left(nx+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{H_k^{(\nu)}\left(\sqrt{-\nu}nx\right)}{k!\,(-\nu)^{\frac{k}{2}}} f\left(\frac{k}{n}\right).$$

References

[1] R. Aktaş, B. Çekim, F. Taşdelen, A Kantorovich-Stancu type generalization of Szasz operators including Brenke type polynomials, J. Funct. Spaces Appl. (2013) Art. ID 935430 9 pp.
[2] W.A. Al-Salam, On a characterization of a certain set of orthogonal polynomials, Boll. Un. Mat. Ital. 19(4) (1964) 448-450.
[3] F. Altomare, M. Campiti, Korovkin-type approximation theory and its applications, Appendix A by Michael Pannenberg and Appendix B by Ferdinand Beckhoff, de Gruyter Studies in Mathematics, 17. Walter de Gruyter & Co., Berlin, 1994.
[4] Ç. Atakut, İ. Büyükyazıcı, Approximation by Kantorovich-Szász type operators based on Brenke type polynomials, Numer. Funct. Anal. Optim. 37(12) (2016) 1488-1502.
[5] S.N. Bernstein, Démonstration du théoréme de Weierstrass fondée surle calcul des probabilités, Commun. Soc. Math. Kharkow 2(13) (1912) 1-2.
[6] E.W. Cheney, A. Sharma, On a generalization of Bernstein polynomials, Riv. Mat. Univ. Parma 2(5) (1964) 77-84.
[7] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[8] R.A. Devore, G.G. Lorentz, Constructive approximation, Springer-Verlag, Berlin, 1993.
[9] J. Favard, Sur les multiplicateurs d'interpolation (French), J. Math. Pures Appl. 23(9) (1944) 219-247.
[10] A.D. Gadzhiev, The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous to that of P. P. Korovkin, Soviet Math. Dokl. 15(5) (1974) 1433-1436.
[11] V. Gupta, T.M. Rassias, P.N. Agrawal, A.M. Acu, Recent advances in constructive approximation theory, Springer Optimization and Its Applications, vol. 138, Springer, Cham, 2018.
[12] M.E.H. Ismail, On a generalization of Szász operators, Mathematica (Cluj) 39 (1974) 259-267.
[13] N. Ispir, On modified Baskakov operators on weighted spaces, Turkish J. Math. 25(3) (2001) 355-365.
[14] G. İçöz, S. Varma, S. Sucu, Approximation by operators including generalized Appell polynomials, Filomat 30(2) (2016) 429-440.
[15] A. Jakimovski, D. Leviatan, Generalized Bernstein polynomials, Math. Z. 93 (1966) 416-426.
[16] A. Jakimovski, D. Leviatan, Generalized Szasz operators for the approximation in the infinite interval, Mathematica (Cluj) 11 (1969) 97-103.
[17] A. Kajla, P.N. Agrawal, Approximation properties of Szász type operators based on Charlier polynomials, Turkish J. Math. 39(6) (2015) 990-1003.
[18] A. Kajla, P.N. Agrawal, Szász-Durrmeyer type operators based on Charlier polynomials, Appl. Math. Comput. 268 (2015) 1001-1014.
[19] A. Kajla, P.N. Agrawal, Szász-Kantorovich type operators based on Charlier polynomials, Kyungpook Math. J. 56(3) (2016) 877-897.
[20] A. Kajla, Blending type approximation by generalized Szász type operators based on Charlier polynomials, Creat. Math. Inform. 27(1) (2018) 49-56.
[21] L.V. Kantorovich, Sur certains développements suivant les polynǒmes de la forme de S. Bernstein, I, II, C. R. Acad. Sci. USSR 20 (1930) 563-568 595-600.
[22] W. Meyer-König, K. Zeller, Bernsteinsche Potenzreihen, Studia Math. 19 (1960) 89-94.
[23] G.M. Mirakjan, Approximation of continuous functions with the aid of polynomials (Russian), Dokl. Akad. Nauk SSSR 31 (1941) 201-205.
[24] S. Roman, The umbral calculus. Pure and Applied Mathematics, 111. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
[25] D.D. Stancu, Approximation of functions by a new class of linear polynomial operators, Rev. Roumaine Math. Pures Appl. 13(8) (1968) 1173-1194.
[26] S. Sucu, G. İçöz, S. Varma, On some extensions of Szasz operators including Boas-Buck-type polynomials, Abstr. Appl. Anal. (2012) Art. ID 680340 15 pp.
[27] O. Szasz, Generalization of S. Bernstein's polynomials to the infinite interval, J. Research Nat. Bur. Standards 45 (1950) 239-245.
[28] S. Varma, S. Sucu, G. İçöz, Generalization of Szasz operators involing Brenke type polynomials, Comput. Math. Appl. 64(2) (2012) 121-127.