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Some Results on the Fundamental Concepts of Fuzzy Set Theory in Intuitionistic Fuzzy Environment by Using α and β cuts

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Abstract. In this paper we have firstly examined the properties of α and β cuts of intuitionistic fuzzy numbers in $\mathbb{R}^n$ with the help of well-known Stacking and Characterization theorems in fuzzy set theory. Then, we have studied the generalized Hukuhara difference in intuitionistic fuzzy environment by using the properties of α and β cuts and support function. Finally, we have extended the strongly generalized differentiability concept from fuzzy set theory to intuitionistic fuzzy environment and proved the related theorems with this concept.

1. Introduction

The fuzzy set theory was firstly introduced by L. Zadeh in 1965 to utilize the vagueness or uncertainty in mathematical models in science and engineering [1]. In this theory, each element x of a fuzzy set in a universal set X is given with a function $\mu : X \to [0, 1]$, called membership function. Hence with the help of this function it is possible to consider both the full and partial membership of an element to the set. In fuzzy set theory, if μ is the membership function of an element in a fuzzy set then $1 - \mu$ is immediately assumed as the non-membership function of the element to the set. Hence the sum of membership and non-membership function in a fuzzy set is always equal to 1. However, due to inadequate or incomplete data or information in models, there may be uncertainty in membership function or non-membership function. For example if there is a lack of information in the membership function itself, then sum of the membership and non-membership function cannot be 1. Hence, some extensions of fuzzy set theory were introduced [2–4]. One of these extensions is Atanassov's intuitionistic fuzzy set (IFS) theory [2].

In 1986, Atanassov [2] introduced the concept of intuitionistic fuzzy sets. In this set, he separately introduced a new degree $\nu : X \to [0, 1]$, called non-membership function, to a classical fuzzy set such that the sum $\mu + \nu$ is less than or equal to 1. Further the difference $1 - (\mu + \nu)$ is regarded as degree of hesitation [2]. Since an intuitionistic fuzzy set contains the information of the membership function, non-membership function and the degree of hesitation of an element, IFS theory can be regarded as a tool which is more flexible and more human consistent reasoning in handling uncertainty due to imprecise information or data. Later Atanassov carried out rigorous researches to develop IFS theory [5–12].

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Intuitionistic fuzzy set and fuzzy set theory have interesting flourishing applications in different fields of science and engineering such as decision-making problems [13, 14]; image processing and pattern recognition [15], medicine [16, 17]; fault analysis [18]. In [19–21], Szmidt et. al showed that intuitionistic fuzzy sets are pretty useful for the models having imperfect data and he demonstrated some applications of IFS in some cases with linguistic variables. More detailed information about applications of fuzzy set and intuitionistic fuzzy set theory can be found in [22–38].

The space of compact and convex sets has a linear structure with respect to Minkowski sum and scalar multiplication [39]. These two operations are associative, commutative and have the set $\{0\}$ as the identity element. This linear structure is that of a cone rather than a vector space since the inverse element of a set with respect to Minkowski sum is a problematic issue. Because if $A = \{a\}$ is not a singleton set, the Minkowski sum of A and $-A$ is not always the identity element $\{0\}$, i.e., $A + (-1)A \neq \{0\}$ [39, 40]. Hence, the inverse element with respect to Minkowski sum may not exist. This is a drawback not only in theory of compact and convex sets but also in theory of fuzzy sets and IFS. Because the α-cut of a fuzzy number, which is a special fuzzy set, is a compact and convex set. That is why inverse element issue may cause problems in finding solutions of algebraic equations with fuzzy numbers and in differentiability concept of fuzzy number valued functions [40]. Hence some alternative difference operations such as Hukuhara difference, generalized Hukuhara difference, generalized difference were proposed [41–44].

The study of Hukuhara was the starting point to handle the inverse element problem. He defined a new difference called Hukuhara difference (H-difference) for compact and convex sets [41]. Later Hukuhara difference of fuzzy sets and Hukuhara derivative (H-derivative) of fuzzy number valued functions were introduced and studied [45–47]. However Hukuhara differentiability has a disadvantage such that a differentiable function may have increasing length of its support [39]. This means that the uncertainty increases as time passes by [42, 43]. That is why, Stefanini et al. [42, 44] proposed and studied some generalizations for Hukuhara difference and derivative concept for compact and convex sets and for fuzzy number valued functions in $\mathbb{R}$. Recently some other differences were also proposed and studied [48].

The main idea of this paper is to study and extend the fundamental concepts in fuzzy set theory to Atanassov's intuitionistic fuzzy set theory. In this work, firstly we study the properties of α and β cuts of intuitionistic fuzzy numbers in n-dimensional Euclidean space $\mathbb{R}^n$ by extending the well-known Characterization and Stacking theorems [39, 49–51] in fuzzy set theory to IFS theory. Then, we study the generalized Hukuhara difference in intuitionistic fuzzy environment by using the properties of α and β cuts and support function [39]. Finally as an application of Hukuhara difference in derivative concept we study strongly generalized Hukuhara derivative and its properties in intuitionistic fuzzy environment.

The general structure of the paper is as follows. In Section 2 we give some notations, definitions and theorems about compact and convex sets, fuzzy sets and intuitionistic fuzzy sets in $\mathbb{R}^n$. In Section 3, we have studied Stacking and Characterization theorems in intuitionistic fuzzy environment. In Section 4 and Section 5, we give the generalizations for generalized Hukuhara difference and strongly generalized Hukuhara derivative in intuitionistic fuzzy environment and studied their properties by using α and β cuts. Finally we give conclusions and summary in Section 6 .

2. Preliminaries

In this section we will give some fundamental definitions and theorems which will be needed in further sections. Throughout this paper we will assume the considered sets in definitions or theorems are non-empty sets. Since α and β cuts of intuitionistic fuzzy numbers are very closely related with compact and convex sets, we will firstly give some definitions and theorems on compact and convex sets.

2.1. Compact and Convex Sets in $\mathbb{R}^n$

Definition 2.1. *[39] A subset A in n-dimensional Euclidean space $\mathbb{R}^n$ is called a compact set if it is a closed and bounded.*

Definition 2.2. *[39] A subset A in n-dimensional Euclidean space $\mathbb{R}^n$ is called a convex set if for every $x, y \in A$ and $\lambda \in [0,1]$; $\lambda x + (1-\lambda)y \in A$ holds.*

We will denote set of all compact and convex set in $\mathbb{R}^n$ by $K_C(\mathbb{R}^n)$.

Definition 2.3. *[39] Let $A, B \in K_C(\mathbb{R}^n)$ and $k \in \mathbb{R}$. The Minkowski addition and scalar multiplication are defined as follows:*

$$\begin{aligned} A+B &= \{a+b : a \in A \text{ and } b \in B\} \\ kA &= \{ka : a \in A, k \in \mathbb{R}\} \end{aligned}$$

Theorem 2.4. *[39] $K_C(\mathbb{R}^n)$ is closed under Minkowski addition and scalar multiplication.*

Minkowski addition and scalar multiplication induce a linear structure on $K_C(\mathbb{R}^n)$ with zero element $\{0\}$. However, if we take $k = -1$ we can see that $(-1)A = \{-a : a \in A\}$ is not always the inverse of A with respect to Minkowski sum. Let us see this in the following example.

Example 2.5. *Let $A = [0, 1]$ then*

$$A + (-1)A = [0, 1] + (-1)[0, 1] = [-1, 1].$$

Hence $A + (-1)A \neq \{0\}$ indeed. Thus, adding -1 times a set does not constitute a natural operation of substraction. To overcome this problem, Hukuhara introduced a new difference operation, called Hukuhara difference. It was defined as follows:

$$A \ominus_H B = C \Leftrightarrow A = B + C.$$

Clearly, $A \ominus_H A = \{0\}$ for all non-empty sets A. However Hukuhara difference of compact and convex sets does not always exist. As seen from the definition of Hukuhara difference, an obvious necessary condition for the existence of $A \ominus_H B$ is that A contains some translate $\{c\} + B$ for some $c \in C$. That is why Hukuhara difference was generalized by Stefanini et al. [42].

Definition 2.6. *[39] Let $A, B \in K_C(\mathbb{R}^n)$. The generalized Hukuhara (gH) difference of A and B is such that*

$$A \ominus_{gH} B = C \Leftrightarrow A = B + C \text{ or } B = A + (-1)C.$$

Let us consider the previous example by using gH-difference.

Example 2.7.

$$\begin{aligned} [0,1] \ominus_{gH} [0,1] &= C \Rightarrow [0,1] = [0,1] + [c_1, c_2] \\ &\Rightarrow [0,1] = [c_1, 1 + c_2] \\ &\Rightarrow c_1 = 0 \text{ and } c_2 = 0 \\ &\Rightarrow C = [0,0] \end{aligned}$$

Note.

1) If $A = B + C$ and $B = A + (-1)C$ both hold simultaneously then C is a singleton set.

2) In the case of compact intervals in $\mathbb{R}$, if we take $A = [a_1, a_2]$ and $B = [b_1, b_2]$ then it can be easily shown that

$$A \ominus_{gH} B = [\min\{a_1 - b_1, a_2 - b_2\}, \max\{a_1 - b_1, a_2 - b_2\}]$$

holds.

Theorem 2.8. *Let $A, B \in K_C(\mathbb{R}^n)$. If $A \ominus_{gH} B = C$ exists then it is unique.*

Theorem 2.9. *Let $A, B \in K_C(\mathbb{R}^n)$. If $A \ominus_{gH} B$ exists then the followings hold:*

1. $A \ominus_{gH} A = 0$.

2. $(A + B) \ominus_{gH} B = A$.
3. $B \ominus_{gH} A$ *exists and* $B \ominus_{gH} A = -(A \ominus_{gH} B) = (-B) \ominus_{gH} (-A)$.
4. $(A - B) + B = C \Leftrightarrow (A - B) = C \ominus_{gH} B$.
5. $A \ominus_{gH} B = B \ominus_{gH} A = C \Leftrightarrow C = \{0\}$ *and* $A = B$.

More useful information can be obtained directly from the concept of support function of non-empty compact and convex sets as well. Let us now give some basics of support functions.

Definition 2.10. *[39] Let A be a non-empty subset of* $\mathbb{R}^n$. *The support function of A is defined for all* $p \in \mathbb{R}^n$ *by*

$$s_A(p) = \sup\{\langle p, a\rangle : a \in A\}.$$

Here $\langle . \rangle$ *is the standard inner product in* $\mathbb{R}^n$.

Note that when A is compact and convex set, the supremum is always attained and the support function is well defined.

Theorem 2.11. *[39] Let* $A, B \in K_C(\mathbb{R}^n)$.

1. *For all* $p \in \mathbb{R}^n$
$$s_{A+B}(p) = s_A(p) + s_B(p).$$
2. *If* $A \subseteq B$ *then, for all* $p \in \mathbb{R}^n$
$$s_A(p) \subseteq s_B(p).$$
3. $A = B$ *if and only if for all* $p \in \mathbb{R}^n$, $s_A(p) = s_B(p)$.
4. *For all* $p \in \mathbb{R}^n$ *and* $t \geq 0$
$$s_{tA}(p) = ts_A(p).$$
5. *For a fixed* $A \in K_C(\mathbb{R}^n)$, s_A *is positively homogeneous, i.e., for all* $t \geq 0$,
$$s_A(tp) = ts_A(p).$$
6. *s is subadditive, i.e., for all* $p_1, p_2 \in \mathbb{R}^n$
$$s_A(p_1 + p_2) \leq s_A(p_1) + s_B(p_2).$$

Note. [39, 42]
1. Since the restriction $s|_{S^{n-1}}$ of s_A to the unit sphere $S^{n-1} = \{p \in \mathbb{R}^n : \left\|p\right\| = 1\}$ is such that for all $0 \neq p \in \mathbb{R}^n$

$$s|_{S^{n-1}}(\frac{p}{\left\|p\right\|}) = \frac{1}{\left\|p\right\|} s_A(p),$$

we can consider s is restricted to S^{n-1}. Here $\|.\|$ is the standard norm in $\mathbb{R}^n$.
2. if s_A is the support function of $A \in K_C(\mathbb{R}^n)$ and s_{-A} is the support function of $-A \in K_C(\mathbb{R}^n)$ then for all $p \in S^{n-1}$,

$$s_{-A}(p) = s_A(-p)$$

3. A non-empty compact and convex subset of $\mathbb{R}^n$ is uniquely characterized by its support function. If $A \in K_C(\mathbb{R}^n)$ is a compact and convex set, then it is characterized by its support function such that

$$A = \{x \in \mathbb{R}^n : \langle p, x\rangle \leq s_A(p);\ \forall p \in \mathbb{R}^n\}$$

4. If $s : \mathbb{R}^n \to \mathbb{R}$ is continuous, positively homogeneous and subadditive then s is a support function of a compact convex set.

By using the properties of support function one can define the gH-difference of A and $B \in K_C(\mathbb{R}^n)$ as in the following definition.

Definition 2.12. *[39] Let s_A, s_B, s_C and $s_{(-1)C}$ be support functions of A, B, C and $(-1)C \in K_C(\mathbb{R}^n)$ respectively. Then*

$$s_C(p) = \begin{cases} s_A(p) - s_B(p); & A = B + C \\ s_B(-p) - s_A(-p); & B = A + (-1)C \end{cases}$$

Theorem 2.13. *[42] Let s_A and s_B be the support functions of $A, B \in K_C(\mathbb{R}^n)$ and let $s_1 = s_A - s_B$ and $s_2 = s_B - s_A$. Then the followings hold:*

1. *If s_1 and s_2 are both subadditive, then $A \ominus_{gH} B$ exists and $A \ominus_{gH} B$ is a singleton set.*
2. *If s_1 is subadditive and s_2 is not, then $A = B + C$ and $s_C = s_A - s_B$.*
3. *If s_2 is subadditive and s_1 is not, then $B = A + (-1)C$ and $s_C = s_{-B} - s_{-A}$.*
4. *If s_1 and s_2 are neither subadditive, then $A \ominus_{gH} B$ does not exist.*

Definition 2.14. *[39] Let $x \in \mathbb{R}^n$ and A be a non-empty subset of $\mathbb{R}^n$. The distance from x to A is determined by*

$$d(x, A) = \inf\{\|x - a\| : a \in A\}.$$

Here $\|.\|$ is the standard norm in $\mathbb{R}^n$

We will call the subsets

$$N_r^n(A) = \{x \in \mathbb{R}^n : d(x, A) < r\}$$

and

$$\overline{N}_r^n(A) = \{x \in \mathbb{R}^n : d(x, A) \leq r\}$$

as open and closed r-neighbourhood (ball) of $A \subseteq \mathbb{R}^n$, respectively.

Definition 2.15. *[39] Let A and B be non-empty subsets of $\mathbb{R}^n$. Let N_1^n denote the closed unit ball in $\mathbb{R}^n$.*

1. *The Hausdorff separation of A to B is defined by*

   $$d_H^*(A, B) = \sup\{d(a, B) : a \in A\}$$

   or, equivalently,

   $$d_H^*(A, B) = \inf\{r > 0 : A \subseteq B + rN_1^n\}$$

2. *The Hausdorff separation of A to B is defined by*

   $$d_H^*(B, A) = \sup\{d(b, A) : b \in B\}$$

   or, equivalently,

   $$d_H^*(B, A) = \inf\{r > 0 : B \subseteq A + rN_1^n\}$$

Note that $d_H^*(A, B) \neq d_H^*(B, A)$ in general.

Definition 2.16. *[39] Let A and B be non-empty subsets of $\mathbb{R}^n$. The Hausdorff distance of A and B is defined as*

$$d_H(A, B) = \max\{d_H^*(A, B), d_H^*(B, A)\}$$

Note. [39]

1. Hausdorff distance defines a metric on the set of non-empty compact subsets of $\mathbb{R}^n$.
2. Hausdorff metric is related to the support function of compact and convex sets A and B by

$$d_H(A, B) = \sup\{\left|s_A(p) - s_B(p)\right| : p \in S^{n-1}\}.$$

2.2. Fuzzy and Intuitionistic Fuzzy Sets in $\mathbb{R}^n$

Definition 2.17. *[39] Let $\mu_A : \mathbb{R}^n \to [0,1]$ be a function. The set*

$$\tilde{A} = \{(x, \mu_A(x)) : x \in \mathbb{R}^n,\ \mu_A : \mathbb{R}^n \to [0,1]\}$$

is called a fuzzy set in $\mathbb{R}^n$.

We will denote set of all fuzzy sets in $\mathbb{R}^n$ by $F(\mathbb{R}^n)$.

Definition 2.18. *[39] Let $\tilde{A} \in F(\mathbb{R}^n)$. The α-cut of $\tilde{A}$ is denoted by $A(\alpha)$ such that for $\alpha \in (0,1]$,*

$$A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\}$$

and especially for $\alpha = 0$,

$$A(0) = cl\left(\bigcup_{\alpha \in (0,1]} A(\alpha)\right).$$

Here "cl" means the closure of the set.

Definition 2.19. *[39] A fuzzy set $\tilde{A} \in F(\mathbb{R}^n)$ satisfying the following properties is called a fuzzy number in $\mathbb{R}^n$*

1. *$\tilde{A}$ is a normal set; i.e., $A(1) \neq \emptyset$.*
2. *$A(0)$ is a bounded set in $\mathbb{R}^n$.*
3. *$\mu_A : \mathbb{R}^n \to [0,1]$ is an upper semi-continuous function; i.e., $\forall k \in [0,1]$, $\{x \in \mathbb{R}^n : \mu_A(x) < k\}$ is open.*
4. *$\mu_A : \mathbb{R}^n \to [0,1]$ is a quasi-concave function; i.e., for $\forall x, y \in A(\alpha)$ and $\lambda \in [0,1]$*

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}.$$

We will denote set of all fuzzy numbers in $\mathbb{R}^n$ by $F_N(\mathbb{R}^n)$.

Theorem 2.20. *[39] Let $\tilde{A} \in F_N(\mathbb{R}^n)$ be a fuzzy number and $A(\alpha)$ be its α-cut. Then the following properties hold:*

1. *For every $\alpha \in [0,1]$, $A(\alpha)$ is a non-empty compact and convex set in $\mathbb{R}^n$.*
2. *For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$; $A(1) \subseteq A(\alpha_2) \subseteq A(\alpha_1) \subseteq A(0)$ is satisfied.*
3. *Let $(\alpha_n) \subseteq [0,1]$ be a non-decreasing sequence converging to α then*

$$A(\alpha) = \bigcap_{n=1}^{\infty} A(\alpha_n).$$

4. *Let $(\alpha_n) \subseteq [0,1]$ be a non-increasing sequence converging to 0 then*

$$A(0) = cl\left(\bigcup_{n=1}^{\infty} A(\alpha_n)\right).$$

Definition 2.21. *[39] Let $\tilde{A} \in F_N(\mathbb{R}^n)$ and let $S^{n-1} = \{p \in \mathbb{R}^n : \|p\| = 1\}$. The function $s_{A(\alpha)} : S^{n-1} \to \mathbb{R}$ defined by*

$$s_{A(\alpha)}(p) = \sup\{< p, x >: x \in A(\alpha)\}$$

is called the support function of a fuzzy number $\tilde{A}$. Here $< . >$ is the standard inner product in $\mathbb{R}^n$.

Theorem 2.22. *[39] Let $\tilde{A}$ and $\tilde{B} \in F_N(\mathbb{R}^n)$. Then the support function*

1. *$s_{A(\alpha)}$ is uniformly bounded on S^{n-1}.*
2. *$s_{A(\alpha)}$ satisfies Lipschitz condition in p for every $\alpha \in [0,1]$*

3. *For every* $\alpha \in [0,1]$ *,*

$$d_H(A(\alpha), B(\alpha)) = \sup\left\{\left|s_{A(\alpha)}(p) - s_{B(\alpha)}(p)\right| : p \in S^{n-1}\right\}.$$

Theorem 2.23. *[39] Let* $\tilde{A} \in F_N(\mathbb{R}^n)$*. Then* $s_{A(\alpha)}$ *is a non-increasing and left continuous function in* α *for each* $p \in S^{n-1}$*.*

Definition 2.24. *[2] Let* $\mu_A, \nu_A : \mathbb{R}^n \to [0,1]$ *be two functions such that for each* $x \in \mathbb{R}^n$*,* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *holds. The set*

$$\tilde{A}^i = \{(x, \mu_A(x), \nu_A(x)) : x \in \mathbb{R}^n; \mu_A, \nu_A : \mathbb{R}^n \to [0,1]\}$$

is called an intuitionistic fuzzy set in $\mathbb{R}^n$*. Here* μ_A *and* ν_A *are called membership and non-membership functions, respectively.*

We will denote set of all intuitionistic fuzzy sets in $\mathbb{R}^n$ by $IF(\mathbb{R}^n)$.

Definition 2.25. *[2] Let* $\tilde{A}^i \in IF(\mathbb{R}^n)$*. The* α*-cut of* $\tilde{A}^i$ *is defined as follows:*
For $\alpha \in (0,1]$

$$A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\},$$

and for $\alpha = 0$

$$A(0) = cl\left(\bigcup_{\alpha\in(0,1]} A(\alpha)\right).$$

Definition 2.26. *[2] Let* $\tilde{A}^i \in IF(\mathbb{R}^n)$*. The* β*-cut of* $\tilde{A}^i$ *is defined as follows:*
For $\beta \in [0,1)$

$$A^*(\beta) = \{x \in \mathbb{R}^n : \nu_A(x) \leq \beta\},$$

and for $\beta = 1$

$$A^*(1) = cl\left(\bigcup_{\beta\in[0,1)} A^*(\beta)\right).$$

Definition 2.27. *[2] Let* $\tilde{A}^i \in IF(\mathbb{R}^n)$*. For* α *and* $\beta \in [0,1]$ *with* $0 \leq \alpha + \beta \leq 1$*, the set*

$$A(\alpha, \beta) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$$

is called (α, β)*-cut of* $\tilde{A}^i$

Theorem 2.28. *[2] Let* $\tilde{A}^i \in IF(\mathbb{R}^n)$*. Then*

$$A(\alpha, \beta) = A(\alpha) \cap A^*(\beta)$$

holds.

Definition 2.29. *An intuitionistic fuzzy set* $\tilde{A}^i \in IF(\mathbb{R}^n)$ *satisfying the following properties is called an intuitionistic fuzzy number in* $\mathbb{R}^n$

1. *$\tilde{A}^i$ is a normal set, i.e., $A(1) \neq \emptyset$ and $A^*(0) \neq \emptyset$.*
2. *$A(0)$ and $A^*(1)$ are bounded sets in $\mathbb{R}^n$.*
3. *$\mu_A : \mathbb{R}^n \to [0,1]$ is an upper semi-continuous function; i.e., $\forall k \in [0,1], \{x \in A : \mu_A(x) < k\} isanopenset$.*
4. *$\nu_A : \mathbb{R}^n \to [0,1]$ is a lower semi-continuous function; i.e., $\forall k \in [0,1], \{x \in A : \nu_A(x) > k\} isanopenset$.*
5. *The membership function μ_A is quasi-concave; i.e., $\forall \lambda \in [0,1], \forall x,\ y \in \mathbb{R}^n$*

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y))$$

6. *The non-membership function ν_A is quasi-convex; i.e.,$\forall \lambda \in [0,1], \forall x,\ y \in \mathbb{R}^n$*

$$\nu_A(\lambda x + (1-\lambda)y) \leq \max(\nu_A(x), \nu_A(y)); \forall \lambda \in [0,1],$$

We will denote the set of all intuitionistic fuzzy numbers in $\mathbb{R}^n$ by $IF_N(\mathbb{R}^n)$.

Theorem 2.30. *[52, 53] Let $\tilde{A}^i, \tilde{B}^i \in IF_N(\mathbb{R}^n)$. Let us define*

$$\begin{aligned} D_1(\tilde{A}^i, \tilde{B}^i) &= \sup\{d_H(A(\alpha), B(\alpha)) : \alpha \in [0,1]\} \\ D_2(\tilde{A}^i, \tilde{B}^i) &= \sup\{d_H(A(\beta), B(\beta)) : \beta \in [0,1]\} \end{aligned}$$

The function

$$D_\infty(\tilde{A}^i, \tilde{B}^i) = \max\left\{D_1(\tilde{A}^i, \tilde{B}^i), D_2(\tilde{B}^i, \tilde{A}^i)\right\}$$

defines a metric on $IF_N(R^n)$. Hence $(IF_N(R^n), D_\infty)$ is a metric space.

3. Characterization and Stacking Theorems for Intuitionistic Fuzzy Numbers

In this section we will give the fundamental theorems characterizing α and β cuts of intuitionistic fuzzy numbers based on the characterization and stacking theorems for fuzzy numbers given in [39, 49–51].

Theorem 3.1. *Let $\tilde{A}^i \in IF_N(\mathbb{R}^n)$ and $\alpha, \beta \in [0,1]$ such that the α and β cuts of $\tilde{A}^i$ given by $A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\}$ and $A^*(\beta) = \{x \in \mathbb{R}^n : \nu_A(x) \leq \beta\}$. Then the followings hold:*

1. *For every $\alpha \in [0,1]$, $A(\alpha)$ is a non-empty compact and convex set in $\mathbb{R}^n$*
2. *If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $A(\alpha_2) \subseteq A(\alpha_1)$.*
3. *If (α_n) is a non-decreasing sequence in $[0,1]$ converging to α then*

$$\bigcap_{n=1}^{\infty} A(\alpha_n) = A(\alpha).$$

4. *If (α_n) is a non-increasing sequence in $[0,1]$ converging to 0 then*

$$cl\left(\bigcup_{n=1}^{\infty} A(\alpha_n)\right) = A(0).$$

5. *For every $\beta \in [0,1]$, $A^*(\beta)$ is a non-empty compact and convex set in $\mathbb{R}^n$.*
6. *If $0 \leq \beta_1 \leq \beta_2 \leq 1$ then $A^*(\beta_1) \subseteq A^*(\beta_2)$.*
7. *If (β_n) is a non-increasing sequence in $[0,1]$ converging to β then*

$$\bigcap_{n=1}^{\infty} A^*(\beta_n) = A^*(\beta).$$

8. *If* (β_n) *is a non-decreasing sequence in* $[0,1]$ *converging to* 1 *then*

$$cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right) = A^*(1).$$

Proof. From [39, 49] we know that $A(\alpha)$ satisfy the properties given in (1.)-(4.). Let us prove the properties given in (5.)-(8.)

5. Since $\tilde{A}^i$ is an intuitionistic fuzzy number, the 0-cut $A^*(0)$ is a non-empty set. Let $x, y \in A^*(\beta)$. So $\nu_A(x) \leq \beta$ and $\nu_A(y) \leq \beta$. Since ν_A is a quasi-convex function, for all $\lambda \in [0,1]$

$$\nu_A(\lambda x + (1-\lambda)y) \leq \max\{\nu_A(x), \nu_A(y)\} \leq \beta \Rightarrow \nu_A(\lambda x + (1-\lambda)y) \leq \beta$$

holds. Hence we obtain that $\lambda x + (1-\lambda)y \in A^*(\beta)$. Therefore $A^*(\beta)$ is a convex set.
Now let us show that $A^*(\beta)$ is a closed set. Since ν_A is a lower semi-continuous function $\{x \in \mathbb{R} : \nu_A(x) > \beta\}$ is an open set for every $\beta \in [0,1]$. Hence its complement $A^*(\beta) = \{x \in \mathbb{R} : \nu_A(x) \leq \beta\}$ is a closed set. Since for all $\beta \in [0,1)$, $A^*(\beta) \subseteq A^*(1)$ and $A^*(1)$ is bounded, $A^*(\beta)$ is bounded as well. As a result $A^*(\beta)$ is a compact and convex set in $\mathbb{R}^n$.
6. Let $0 \leq \beta_1 \leq \beta_2 \leq 1$ and $x \in A^*(\beta_1)$. Since $\nu_A(x) \leq \beta_1 \leq \beta_2$ holds, we obtain that $x \in A^*(\beta_2)$. Hence $A^*(\beta_1) \subseteq A^*(\beta_2)$.
7. Let (β_n) be a non-increasing sequence converging to β and $x \in A^*(\beta)$. Since for all $n \in \mathbb{N}$, $\beta \leq \beta_n$ we can write that $\nu_A(x) \leq \beta_n \Rightarrow x \in A^*(\beta_n)$. So we obtain that $x \in \bigcap_{n=1}^{\infty} A^*(\beta_n)$.
On the other hand let $x \in \bigcap_{n=1}^{\infty} A^*(\beta_n)$ then for all $n \in \mathbb{N}$, $x \in A^*(\beta_n) \Rightarrow \nu_A(x) \leq \beta_n$. Since $\beta_n \to \beta$ as $n \to \infty$, then $\nu_A(x) \leq \beta$. So $x \in A^*(\beta)$. Hence

$$\bigcap_{n=1}^{\infty} A^*(\beta_n) = A^*(\beta).$$

holds.
8. Let (β_n) be a non-decreasing sequence converging to 1 and $x \in A^*(\beta_n)$. Since $\beta_n \leq 1$ then for all $n \in \mathbb{N}$, $A^*(\beta_n) \subseteq A^*(1)$. Hence we can write that $cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right) \subseteq A^*(1)$.
Let $x \in A^*(1)$ then there exists a sequence (x_n) in $\bigcup_{\beta\in[0,1)} A^*(\beta)$ converging to x. Without loss of generality, for $\beta_1 \leq \beta_2 \leq \beta_3 \leq, ..., \leq \beta_n, ..., \leq 1$ we can construct the sequence such that $x_1 \in A^*(\beta_1)$, $x_2 \in A^*(\beta_2)$, $x_3 \in A^*(\beta_3)$,..., $x_n \in A^*(\beta_n)$,.... Then we can write that $(x_n) \subseteq cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right)$. So by the property of the closure of a set we obtain that $x \in cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right)$. So $A^*(1) \subseteq cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right)$.
Therefore

$$A^*(1) = cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right)$$

holds.

□

Corollary 3.2. *Let* $\tilde{A}^i \in IF_N(\mathbb{R})$. *Then* α *and* β *cuts of* $\tilde{A}^i$ *are closed and bounded intervals such that*

$$A(\alpha) = [A_1(\alpha), A_2(\alpha)]$$

and

$$A^*(\beta) = [A_1^*(\beta), A_2^*(\beta)].$$

Here
$A_1(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$,
$A_2(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$,
$A_1^*(\beta) = \inf\{x \in \mathbb{R} : \nu_A(x) \leq \beta\}$,
$A_2^*(\beta) = \sup\{x \in \mathbb{R} : \nu_A(x) \leq \beta\}$.

Theorem 3.3. *Let $\tilde{A}^i \in IF_N(\mathbb{R})$ be an intuitionistic fuzzy number and $\alpha, \beta \in [0,1]$. Let α and β cuts of $\tilde{A}^i$ be given by $A(\alpha) = [A_1(\alpha), A_2(\alpha)]$ and $A^*(\beta) = [A_1^*(\beta), A_2^*(\beta)]$, respectively. Then the functions $A_1 : [0,1] \to \mathbb{R}$, $A_2 : [0,1] \to \mathbb{R}$, $A_1^* : [0,1] \to \mathbb{R}$ and $A_2^* : [0,1] \to \mathbb{R}$ satisfy the followings:*

1. *A_1 is a bounded, non-decreasing, left continuous function in $(0,1]$ and it is a right continuous function at 0.*
2. *A_2 is a bounded, non-increasing, left continuous function in $(0,1]$ and it is a right continuous function at 0.*
3. *$A_1(1) \leq A_2(1)$ holds.*
4. *A_1^* is a bounded, non-increasing, right continuous function in $[0,1)$ and it is a left continuous function at 1.*
5. *A_2^* is a bounded, non-decreasing, right continuous function in $[0,1)$ and it is a left continuous function at 1.*
6. *$A_1^*(0) \leq A_2^*(0)$ holds.*

Proof. By [49, 50] we know that (1.)-(3.) are satisfied. Hence we will proceed the proof by considering only β-cuts.

Let $A^*(\beta) = [A_1^*(\beta), A_2^*(\beta)]$ be the β-cut of $\tilde{A}^i$. Since $0 \leq \beta \leq 1$, we can write that $A^*(0) \subseteq A^*(\beta) \subseteq A^*(1)$. This implies that

$$[A_1^*(0), A_2^*(0)] \subseteq [A_1^*(\beta), A_2^*(\beta)] \subseteq [A_1^*(1), A_2^*(1)].$$

Hence we obtain that

$$A_1^*(1) \leq A_1^*(\beta) \leq A_1^*(0) \leq A_2^*(0) \leq A_2^*(\beta) \leq A_2^*(1).$$

Since $A^*(1)$ is a closed and bounded interval then the functions $A_1^*(\beta)$ and $A_2^*(\beta)$ are bounded as well. And from the inequality above we obtain that $A_1^*(0) \leq A_2^*(0)$ holds.
Let $0 \leq \beta_1 \leq \beta_2 \leq 1$ then by the similar argument above this implies that

$$A_1^*(\beta_2) \leq A_1^*(\beta_1) \leq A_2^*(\beta_1) \leq A_2^*(\beta_2).$$

Hence A_1^* is a non-increasing function and A_2^* is a non-decreasing function.
Let (β_n) be a non-increasing sequence converging to β. Then by property (7.) in Theorem 3.1 we can write that

$$\bigcap_{n=1}^{\infty} [A_1^*(\beta_n), A_2^*(\beta_n)] = [A_1^*(\beta), A_2^*(\beta)].$$

By Proposition 2.4.5 in [39], this implies that $A_1^*(\beta_n) \to A_1^*(\beta)$ and $A_2^*(\beta_n) \to A_2^*(\beta)$. Hence by the sequential definition of right continuity, both functions A_1^* and A_2^* are right continuous at $\beta \in [0,1)$.

Let us show the left continuity of the functions A_1^* and A_2^* at 1. Let (β_n) be a non-decreasing sequence converging to 1 then by property (8.) in Theorem 3.1, we can write that

$$cl(\bigcup_{n=1}^{\infty} A^*(\beta_n)) = A^*(1) = [A_1^*(1), A_2^*(1)]$$

Since $A^*(1)$ and $A^*(\beta_n))$ are closed and bounded intervals, by Proposition 2.4.5 in [39] we get $A_1^*(\beta_n) \to A_1^*(1)$ and $A_2^*(\beta_n) \to A_2^*(1)$. So by the sequential definition of left continuity, A_1^* and A_2^* are left continuous at 1. □

Theorem 3.4. *Let $\{M_\alpha \subseteq \mathbb{R}^n : \alpha \in [0,1]\}$ be a family of sets in $\mathbb{R}^n$ satisfying (1.)-(4.) in Theorem 3.1 and $\{M_\beta \subseteq \mathbb{R}^n : \beta \in [0,1]\}$ be a family of set in $\mathbb{R}^n$ satisfying (5.)-(8.) in Theorem 3.1. Let us define the functions $\mu : \mathbb{R}^n \to [0,1]$ and $\nu : \mathbb{R}^n \to [0,1]$ by*

$$\mu(x) = \begin{cases} \sup\{\alpha \in [0,1] : x \in M_\alpha\}, & x \in M_0 \\ 0, & x \notin M_0 \end{cases}$$

$$\nu(x) = \begin{cases} \inf\{\beta \in [0,1] : x \in M_\beta\}, & x \in M_1 \\ 1, & x \notin M_1 \end{cases}$$

Then there exists an intuitionistic fuzzy number $\tilde{A}^i \in IF_N(\mathbb{R}^n)$ with its α and β cuts $A(\alpha)$ and $A^*(\beta)$ satisfying the followings:

1. *For all $\alpha \in [0,1]$, $A(\alpha) = M_\alpha$.*
2. *For all $\beta \in [0,1]$, $A^*(\beta) = M_\beta$.*

Proof. Assume that $\{M_\alpha \subseteq \mathbb{R}^n : \alpha \in [0,1]\}$ is a family of sets in $\mathbb{R}^n$ satisfying (1.)-(4.) in Theorem 3.1 and $\{M_\beta \subseteq \mathbb{R}^n : \beta \in [0,1]\}$ is a family of set in $\mathbb{R}^n$ satisfying (5.)-(8.) in Theorem 3.1. Since for every $x \in \mathbb{R}^n$, $0 \le \mu(x) + \nu(x) \le 1$ holds, the functions μ and ν obviously define a membership and a non-membership function for an intuitionistic fuzzy set, say $\tilde{A}^i$. We want to prove that $\tilde{A}^i$ is an intuitionistic ***fuzzy number*** such that its α and β cuts are M_α and M_β. By [49, 50] we know that μ defines a membership function of a fuzzy number whose α-cuts are M_α satisfying (1.). Hence we will complete the proof by showing (2.)

Let

$$\tilde{A}^i = \{(x, \mu(x), \nu(x)) : x \in \mathbb{R}^n; \mu, \nu\} : \mathbb{R}^n \to [0,1]\}.$$

Firstly let us show that $A^*(\beta) = M_\beta$. Let $\beta_0 \in [0,1)$ be an arbitrary fixed number and let $x \in M_{\beta_0}$. Since $M_{\beta_0} \subseteq M_1$, then $x \in M_1$. So by the definition of ν and properties of infimum, we can write that

$$\beta_0 \in \{\beta : x \in M_\beta\} \Rightarrow \inf\{\beta \in [0,1] : x \in M_\beta\} \le \beta_0 \Rightarrow \nu(x) \le \beta_0 \Rightarrow x \in A^*(\beta_0) \Rightarrow M_{\beta_0} \subseteq A^*(\beta_0).$$

On the other hand, let $x \in A^*(\beta_0)$ so we can write that $v(x) \le \beta_0 \Rightarrow \inf\{\beta \in [0,1] : x \in M_\beta\} \le \beta_0$. Now let us consider the following two cases:

Case 1. If $\inf\{\beta \in [0,1] : x \in M_\beta\} = \beta_0$ then there exists a non-increasing sequence (β_n) in $\{\beta \in [0,1] : x \in M_\beta\}$ converging to β_0. Hence by the property (7.) in Theorem 3.1 we have

$$\bigcap_{n=1}^{\infty} M_{\beta_n} = M_{\beta_0}.$$

Since for every $n \in \mathbb{N}$, $x \in M_{\beta_n}$ then we obtain that $x \in M_{\beta_0}$. And this implies that $A^*(\beta_0) \subseteq M_{\beta_0}$.

Case 2. If $\inf\{\beta \in [0,1] : x \in M_\beta\} < \beta_0$ then by the definition of infimum there exists a $\beta_1 \in \{\beta \in [0,1] : x \in M_\beta\}$ with $x \in M_{\beta_1}$ such that $\inf\{\beta \in [0,1] : x \in M_\beta\} \le \beta_1 < \beta_0$. Since $\beta_1 < \beta_0$ implies $M_{\beta_1} \subseteq M_{\beta_0}$ we obtain $x \in M_{\beta_0}$. So $A^*(\beta_0) \subseteq M_{\beta_0}$ hold. Hence for an arbitrary fixed $\beta_0 \in [0,1]$ we obtain that $A^*(\beta_0) = M_{\beta_0}$. As a result we obtain that $A^*(\beta) = M_\beta$ for any $\beta \in [0,1]$.

Now let us prove that $\tilde{A}^i$ is an intuitionistic **fuzzy number**. We know that μ satisfies requirements (i.e., normality, quasi-concavity, upper semi-continuity and compactness of its 0-cut) for being a membership function of an intuitionistic fuzzy number. So let us prove that ν satisfies the normality, quasi-convexity, lower semi-continuity and compactness of its 1-cut.

Normality. Since M_β is non-empty by (5.) in Theorem 3.1, there exists at least an element $x_0 \in M_0$ such that $\nu(x_0) = 0$. So $\tilde{A}^i$ is normal.
Quasi-convexity. Since for any $\beta \in [0,1)$, M_β is a quasi-convex set and $A^*(\beta) = M_\beta$ then $A^*(\beta)$ is also a quasi-convex set. So for every $\beta \in [0,1]$ the level set $\{x : \nu(x) \leq \beta\}$ is a convex set. This implies that ν is a quasi-convex function.
Lower semi-continuity. Since $A^*(\beta) = M_\beta$ are closed by (5.) in Theorem 3.1 the complement of $\{x : \nu(x) > \beta\}$ is an open set. This implies that ν is a lower semi-continuous function.
Compactness of its 1-cut. Let (β_n) be a non-decreasing sequence converging to 1 then we can write that

$$A^*(1) = cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right) = cl\left(\bigcup_{n=1}^{\infty} M_{\beta_n}\right) = M_1.$$

So we have $A^*(1) = M_1$. Hence $A^*(1)$ is a compact set.

As a result $\tilde{A}^i$ is an intuitionistic *fuzzy number* such that its α and β cuts are M_α and M_β. □

Theorem 3.5. *Let $A_1 : [0,1] \to \mathbb{R}$, $A_2 : [0,1] \to \mathbb{R}$, $A_1^* : [0,1] \to \mathbb{R}$ and $A_2^* : [0,1] \to \mathbb{R}$ be functions satisfying the followings:*

1. *A_1 is a bounded, non-decreasing, left continuous function in $(0,1]$ and it is a right continuous function at 0.*

2. *A_2 is a bounded, non-increasing, left continuous function in $(0,1]$ and it is a right continuous function at 0.*

3. *$A_1(1) \leq A_2(1)$ holds.*

4. *A_1^* is a bounded, non-increasing, right continuous function in $[0,1)$ and it is a left continuous function at 1.*

5. *A_2^* is a bounded, non-decreasing, right continuous function in $[0,1)$ and it is a left continuous function at 1.*

6. *$A_1^*(0) \leq A_2^*(0)$ holds.*

Then there exists an intuitionistic fuzzy number $\tilde{A}^i$ such that the α and β cuts of $\tilde{A}^i$ are the following closed and bounded intervals $[A_1(\alpha), A_2(\alpha)]$ and $[A_1^*(\beta), A_2^*(\beta)]$, respectively.

Proof. We will prove the theorem with the aid of Theorem 3.4. By [49, 50] under the conditions (1.)-(3.) $A(\alpha) = [A_1(\alpha), A_2(\alpha)]$ holds. Now let us prove the rest. Let $\{M_\beta \subseteq \mathbb{R}^n : \beta \in [0,1]\}$ be a family of set in $\mathbb{R}^n$ such that $M_\beta = [A_1^*(\beta), A_2^*(\beta)]$. Firstly we will show that M_β satisfies (5.)-(8.) in Theorem 3.1.

Let $0 \leq \beta_1 \leq \beta_2 \leq 1$. Since $A_1^*(\beta)$ is a non-increasing function we can write that

$$A_1^*(1) \leq A_1^*(\beta_2) \leq A_1^*(\beta_1) \leq A_1^*(0).$$

And since $A_2^*(\beta)$ is a non-decreasing function we can write that

$$A_2^*(0) \leq A_2^*(\beta_1) \leq A_2^*(\beta_2) \leq A_2^*(1).$$

Since $A_1^*(0) \leq A_2^*(0)$ we obtain that

$$A_1^*(1) \leq A_1^*(\beta_2) \leq A_1^*(\beta_1) \leq A_1^*(0) \leq A_2^*(0) \leq A_2^*(\beta_1) \leq A_2^*(\beta_2) \leq A_2^*(1).$$

And this implies that $M_0 \subseteq M_{\beta_2} \subseteq M_{\beta_1} \subseteq M_1$.

Consider the sequence $(\beta_n) \subseteq [0,1]$ converging to $\beta \in [0,1)$ from above. Since $A_1^*(\beta)$ and $A_2^*(\beta)$ are right continuous on $[0,1)$ we obtain that $A_1^*(\beta_n) \to A_1^*(\beta)$ and $A_2^*(\beta_n) \to A_2^*(\beta)$. Hence

$$\bigcap_{n=1}^{\infty} M_{\beta_n} = \bigcap_{n=1}^{\infty} [A_1^*(\beta_n), A_2^*(\beta_n)] = [A_1^*(\beta), B_2(\beta)] = M_\beta$$

holds.

Let (β_n) converge to 1 from below in $[0,1]$. Since $A_1^*(\beta)$ and $A_2^*(\beta)$ are left continuous at 1 then we obtain that $A_1^*(\beta_n) \to A_1^*(1)$ and $A_2^*(\beta_n) \to A_2^*(1)$. So

$$cl\left(\bigcup_{n=1}^{\infty} M_{\beta_n}\right) = cl\left(\bigcup_{n=1}^{\infty} A^*(\beta_n)\right) = [A_1^*(1), A_2^*(1)] = M_1$$

holds. Hence (5.)-(8.) in Theorem 3.1 are all satisfied by M_β.
As a result, by Theorem 3.4 and [49, 50] there exists an intuitionistic fuzzy number $\tilde{A}^i$ such that the α and β cuts of $\tilde{A}^i$ are given by $[A_1(\alpha), A_2(\alpha)]$ and $[A_1^*(\beta), A_2^*(\beta)]$, respectively. □

4. Generalized Hukuhara Difference In Intuitionistic Fuzzy Environment

Definition 4.1. *Let $\tilde{A}^i, \tilde{B}^i \in IF_N(\mathbb{R}^n)$ and $c \in \mathbb{R} - \{0\}$. Minkowski addition and scalar multiplication of intuitionistic fuzzy numbers in $IF_N(\mathbb{R}^n)$ are defined level wise as follows:*

$$\begin{aligned} \tilde{A}^i + \tilde{B}^i &= \tilde{C}^i \Leftrightarrow C(\alpha) = A(\alpha) + B(\alpha) \text{ and } C^*(\beta) = A^*(\beta) + B^*(\beta) \\ c(\tilde{A}^i) &= \tilde{D}^i \Leftrightarrow D(\alpha) = cA(\alpha) \text{ and } D^*(\beta) = cA^*(\beta) \end{aligned}$$

Theorem 4.2. *$IF_N(\mathbb{R}^n)$ is closed under Minkowski addition and scalar multiplication.*

Proof. Let α and $\beta \in [0,1]$. Let $\tilde{A}^i + \tilde{B}^i = \tilde{C}^i$ and $c(\tilde{A}^i) = \tilde{D}^i$. We will prove $IF_N(\mathbb{R}^n)$ is closed under addition and multiplication given in Definition 4.1. To prove the theorem we will use Theorem 3.4. Since the families of subsets $\{C(\alpha) : \alpha \in [0,1]\}$, $\{D(\alpha) : \alpha \in [0,1]\}$ satisfy the properties (1.)-(4.) in Theorem 3.1 by [39], we only need to show that the families of subsets $\{C^*(\beta) : \beta \in [0,1]\}$ and $\{D^*(\beta) : \beta \in [0,1]\}$ satisfy the properties (5.)-(8.) in Theorem 3.1.

Since $A^*(\beta)$ and $B^*(\beta)$ are compact and convex sets and families of compact and convex sets are closed under Minkowski addition and scalar multiplication [39], the families $\{C^*(\beta) : \beta \in [0,1]\}$ and $\{D^*(\beta) : \beta \in [0,1]\}$ are compact and convex as well. Hence (5.) in Theorem 3.1 is satisfied. Moreover the property (6.) in Theorem 3.1 follows directly from the inclusion property of $A^*(\beta)$ and $B^*(\beta)$.

Now let $(\beta_n) \subseteq [0,1]$ be a non-increasing sequence converging to $\beta \in [0,1]$. By properties of Hausdorff metric we write that

$$\begin{aligned} d_H(C^*(\beta_n), C^*(\beta)) &= d_H(A^*(\beta_n) + B^*(\beta_n), A^*(\beta) + B^*(\beta)) \\ &\leq d_H(A^*(\beta_n), A^*(\beta)) + d_H(B^*(\beta_n), B^*(\beta)). \end{aligned}$$

And since $d_H(A^*(\beta_n), A^*(\beta)) \to 0$ and $d_H(B^*(\beta_n), B^*(\beta)) \to 0$ as $n \to \infty$ then we obtain that $d_H(C^*(\beta_n), C^*(\beta)) \to 0$.

In a similar manner as $n \to \infty$ we can also obtain that $d_H(D^*(\beta_n), D^*(\beta)) = |c|\, d_H(A^*(\beta_n), A^*(\beta)) \to 0$. So by Proposition 2.4.5 in [39], (7.) in Theorem 3.1 is satisfied.

Let (β_n) be a non-decreasing sequence converging to 1. Since for all $n \in \mathbb{N}$, $\beta_n \leq 1$ then $A^*(\beta_n) \subseteq A^*(1)$ and $B^*(\beta_n) \subseteq B^*(1)$. Hence for all $n \in \mathbb{N}$, $A^*(\beta_n) + B^*(\beta_n) \subseteq A^*(\beta) + B^*(\beta) \Rightarrow C^*(\beta_n) \subseteq C^*(1)$ and we can write that

$$cl\left(\bigcup_{n=1}^{\infty} C^*(\beta_n)\right) \subseteq C^*(1).$$

Let $x \in A^*(1) + B^*(1)$ then there exist $a \in A^*(1)$ and $b \in B^*(1)$ such that $x = a + b$. By the property of the closure of a set, there exists a sequence (a_n) in $\bigcup_{\beta\in[0,1)} A^*(\beta)$ converging to a and a sequence (b_n) in $\bigcup_{\beta\in[0,1)} B^*(\beta)$ converging to b. Without loss of generality, for $\beta_1 \leq \beta_2 \leq \beta_3 \leq, ..., \leq \beta_n,..., \leq 1$ we can construct the sequences $(a_n) \in A^*(\beta_n)$ and $(b_n) \in B^*(\beta_n)$ such that $a_1 \in A^*(\beta_1)$, $a_2 \in A^*(\beta_2)$, $a_3 \in A^*(\beta_3)$,..., $a_n \in A^*(\beta_n)$, ...and $b_1 \in B^*(\beta_1)$, $b_2 \in B^*(\beta_2)$, $b_3 \in B^*(\beta_3)$,..., $b_n \in B^*(\beta_n)$, ... Then we can write $(a_n + b_n) \subseteq A^*(\beta_n) + B^*(\beta_n)$. So we obtain that $(a_n + b_n) \subseteq cl\left(\bigcup_{n=1}^{\infty} C^*(\beta_n)\right)$. Since as $n \to \infty$, $(a_n) \to a$ and $(b_n) \to b$ then $(a_n + b_n) \to x = a + b$. So $x \in cl\left(\bigcup_{n=1}^{\infty} C^*(\beta_n)\right)$ and this implies that $C^*(1) \subseteq cl\left(\bigcup_{n=1}^{\infty} C^*(\beta_n)\right)$.

Hence

$$C^*(1) = cl\left(\bigcup_{n=1}^{\infty} C^*(\beta_n)\right)$$

holds. In a similar manner we can show that

$$D^*(1) = cl\left(\bigcup_{n=1}^{\infty} D^*(\beta_n)\right)$$

As a result the families of subsets $\{C^*(\beta) : \beta \in [0,1]\}$ and $\{D^*(\beta) : \beta \in [0,1]\}$ satisfy the properties (5.)-(8.) in Theorem 3.1. So by Theorem 3.4 and [39] we obtain that $\{C(\alpha) : \alpha \in [0,1]\}$, $\{C^*(\beta) : \beta \in [0,1]\}$ and $\{D(\alpha) : \alpha \in [0,1]\}$, $\{D^*(\beta) : \beta \in [0,1]\}$ define intuitionistic fuzzy numbers C^i and D^i. Hence $IF_N(\mathbb{R}^n)$ is closed under addition and multiplication given in Definition 4.1. □

Now, we will firstly define the generalized Hukuhara difference for intuitionistic fuzzy numbers in $\mathbb{R}^n$. Then we will observe the properties of it with the help of support function.

Definition 4.3. *Let $\tilde{A}^i, \tilde{B}^i \in IF_N(\mathbb{R}^n)$. The generalized Hukuhara difference (gH-difference) of $\tilde{A}^i$ and $\tilde{B}^i$ is $\tilde{C}^i$, if it exists, such that*

$$\tilde{A}^i \ominus_{gH} \tilde{B}^i = \tilde{C}^i \Longleftrightarrow \tilde{A}^i = \tilde{B}^i + \tilde{C}^i \text{ or } \tilde{B}^i = \tilde{A}^i + (-1)\tilde{C}^i$$

As in Definition 2.12 we can define generalized Hukuhara difference for intuitionistic fuzzy numbers in terms of support functions of α and β cuts.

Definition 4.4. *Let $\tilde{A}^i \in IF(\mathbb{R}^n)$ and let $S^{n-1} = \{p \in \mathbb{R}^n : \|p\| = 1\}$. The support function for an intuitionistic fuzzy number $\tilde{A}^i$ is given by the following functions $s_{A(\alpha)} : S^{n-1} \to \mathbb{R}$ and $s_{A^*(\beta)} : S^{n-1} \to \mathbb{R}$ defined by*

$$s_{A(\alpha)}(p) = \sup\{< p, x >: x \in A(\alpha)\}$$

and

$$s_{A^*(\beta)}(p) = \sup\{< p, x >: x \in A^*(\beta)\}.$$

Here $< . >$ is the standard inner product in $\mathbb{R}^n$. We will call $s_{A(\alpha)}$ and $s_{A^*(\beta)}$ as α and β support functions of $\tilde{A}^i$, respectively.

Definition 4.5. *Let $\tilde{A}^i, \tilde{B}^i, \tilde{C}^i \in IF_N(\mathbb{R}^n)$ such that $\tilde{C}^i = \tilde{A}^i \ominus_{gH} \tilde{B}^i$ holds. Lets$_{A(.)}$, $s_{B(.)}$, $s_{C(.)}$ and $s_{A^*(.)}$, $s_{B^*(.)}$, $s_{C^*(.)}$ be α and β support functions of $\tilde{A}^i$, $\tilde{B}^i$ and $\tilde{C}^i$, respectively. Then $s_{C(\alpha)}$ and $s_{C^*(\beta)}$ is defined by*

$$s_{C(\alpha)}(p) = \begin{cases} i)\ s_{A(\alpha)}(p) - s_{B(\alpha)}(p) \\ \quad or \\ ii)\ s_{(-1)B(\alpha)}(p) - s_{(-1)A(\alpha)}(p) \end{cases}$$

and

$$s_{C^*(\beta)}(p) = \begin{cases} i)\ s_{A^*(\beta)}(p) - s_{B^*(\beta)}(p) \\ or \\ ii)\ s_{(-1)B^*(\beta)}(p) - s_{(-1)A^*(\beta)}(p) \end{cases}$$

respectively.

With the help of Theorem 2.22 and the properties of β-cuts of intuitionistic fuzzy numbers, we can state the following theorem without its proof.

Theorem 4.6. *Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. Then the support function for β-cut of A satisfies the followings:*

1. $s_{A^*(\beta)}$ *is uniformly bounded on* S^{n-1}.
2. $s_{A^*(\beta)}$ *is Lipschitz in p. i.e., for every p, $q \in S^{n-1}$ there exists a real number $L > 0$ such that*

$$\left|s_{A^*(\beta)}(p) - s_{A^*(\beta)}(q)\right| \leq L \left\|p - q\right\|$$

3. *Let $B \in IF(\mathbb{R}^n)$. For each $\beta \in [0, 1]$*

$$d_H(A^*(\beta), B^*(\beta)) = \sup\{\left|s_{A^*(\beta)}(p) - s_{B^*(\beta)}(p)\right| : p \in S^{n-1}\}$$

Theorem 4.7. *Let $\tilde{A}^i \in IF_N(\mathbb{R}^n)$. Then for each $p \in S^{n-1}$*

1. $s_{A(\alpha)}$ *is a non-increasing and left continuous function with respect to α.*
2. $s_{A^*(\beta)}$ *is a non-decreasing and right continuous function with respect to β.*

Proof. 1. It is immediate from [42]

2. Let $0 \leq \beta_1 \leq \beta_2 \leq 1$. Since β-cut of an intuitionistic fuzzy number forms a non-decreasing sequence of sets, $A^*(\beta_1) \leq A^*(\beta_2)$ and hence $s_{A^*(\beta_1)}(p_1) \leq s_{A^*(\beta_2)}(p_2)$ holds. Let $(\beta_n) \subseteq [0, 1]$ be a sequence converging to $\beta \in [0, 1]$ non-increasingly. By (7.) in Theorem 3.1 and (3.) in Theorem 4.6 we have $\left|s_{A^*(\beta)}(p_n) - s_{A^*(\beta)}(p)\right| \leq d_H(A^*(\beta_n), A^*(\beta)) \to 0$. Hence $s_{A^*(\beta)}$ is a right continuous function. □

In this paper we will define the length of an interval $I = [a, b]$ by the function $len : K_C(\mathbb{R}) \to \mathbb{R}^+ \cup 0$ such that

$$len(I) := b - a$$

Theorem 4.8. *Let $\tilde{A}^i$ and $\tilde{B}^i \in IF_N(\mathbb{R})$ be intuitionistic fuzzy numbers. Let $A(\alpha) = [A_1(\alpha), A_2(\alpha)]$ and $A^*(\beta) = [A_1^*(\beta), A_2^*(\beta)]$ be α and β cuts of $\tilde{A}^i$, and $B(\alpha) = [B_1(\alpha), B_2(\alpha)]$ and $B^*(\beta) = [B_1^*(\beta), B_2^*(\beta)]$ be α and β cuts of $\tilde{B}^i$. Then $\tilde{A}^i \ominus_{gH} \tilde{B}^i = \tilde{C}^i$ exists if and only if one of the followings holds*

1.

$$\text{(a)} \begin{cases} len(A(\alpha)) \geq len(B(\alpha)), \\ C_1(\alpha) := A_1(\alpha) - B_1(\alpha) \text{ is non-decreasing w.r.t } \alpha, \\ C_2(\alpha) := A_2(\alpha) - B_2(\alpha) \text{ is non-increasing w.r.t } \alpha. \end{cases}$$

and

$$\text{(b)} \begin{cases} len(A^*(\beta)) \geq len(B^*(\beta)) \\ C_1^*(\beta) := A_1^*(\beta) - B_1^*(\beta) \text{ is non-increasing w.r.t } \beta, \\ C_2^*(\beta) := A_2^*(\beta) - B_2^*(\beta) \text{ is non-decreasing w.r.t } \beta, \end{cases}$$

or

2.

(a) $\begin{cases} len(A(\alpha)) \geq len(B(\alpha)), \\ C_1(\alpha) := A_1(\alpha) - B_1(\alpha) \textit{ is non-decreasing w.r.t } \alpha, \\ C_2(\alpha) := A_2(\alpha) - B_2(\alpha) \textit{ is non-increasing w.r.t } \alpha. \end{cases}$

and

(b) $\begin{cases} len(A^*(\beta)) \leq len(B^*(\beta)) \\ C_1^*(\beta) := A_2^*(\beta) - B_2^*(\beta) \textit{ is non-increasing w.r.t } \beta, \\ C_2^*(\beta) := A_1^*(\beta) - B_1^*(\beta) \textit{ is non-decreasing w.r.t } \beta, \end{cases}$

or

3.

(a) $\begin{cases} len(A(\alpha)) \leq len(B(\alpha)), \\ C_1(\alpha) := A_2(\alpha) - B_2(\alpha) \textit{ is non-decreasing w.r.t } \alpha, \\ C_2(\alpha) := A_1(\alpha) - B_1(\alpha) \textit{ is non-increasing w.r.t } \alpha. \end{cases}$

and

(b) $\begin{cases} len(A^*(\beta)) \geq len(B^*(\beta)) \\ C_1^*(\beta) := A_1^*(\beta) - B_1^*(\beta) \textit{ is non-increasing w.r.t } \beta, \\ C_2^*(\beta) := A_2^*(\beta) - B_2^*(\beta) \textit{ is non-decreasing w.r.t } \beta, \end{cases}$

or

4.

(a) $\begin{cases} len(A(\alpha)) \leq len(B(\alpha)), \\ C_1(\alpha) := A_2(\alpha) - B_2(\alpha) \textit{ is non-decreasing w.r.t } \alpha, \\ C_2(\alpha) := A_1(\alpha) - B_1(\alpha) \textit{ is non-increasing w.r.t } \alpha. \end{cases}$

and

(b) $\begin{cases} len(A^*(\beta)) \leq len(B^*(\beta)) \\ C_1^*(\beta) := A_2^*(\beta) - B_2^*(\beta) \textit{ is non-increasing w.r.t } \beta, \\ C_2^*(\beta) := A_1^*(\beta) - B_1^*(\beta) \textit{ is non-decreasing w.r.t } \beta, \end{cases}$

Proof. Let $\tilde{A}^i \in IF(\mathbb{R}^n)$. By definition of support function for intuitionistic fuzzy numbers, for every $p \in S^0 = \{-1, 1\}$ and standard inner product in $\mathbb{R}^n$, we can write that

$$s_{A(\alpha)}(p) = \sup\{x \cdot p : x \in [A_1(\alpha), A_2(\alpha)]\},$$

$$s_{A^*(\beta)}(p) = \sup\{x \cdot p : x \in [A_1^*(\beta), A_2^*(\beta)]\},$$

$$s_{B(\alpha)}(p) = \sup\{x \cdot p : x \in [B_1(\alpha), B_2(\alpha)]\},$$

$$s_{B^*(\beta)}(p) = \sup\{x \cdot p : x \in [B_1^*(\beta), B_2^*(\beta)]\}.$$

For $p = -1$, since $s_{A(\alpha)}(p) = -A_1(\alpha)$ and $s_{B(\alpha)}(p) = -B_1(\alpha)$ then we obtain

$$s_{A(\alpha)}(p) - s_{B(\alpha)}(p) = -A_1(\alpha) - (-B_1(\alpha)) = -(A_1(\alpha) - B_1(\alpha)).$$

For $p = 1$, since $s_{A(\alpha)}(p) = A_2(\alpha)$ and $s_{B(\alpha)}(p) = B_2(\alpha)$ then we obtain

$$s_{A(\alpha)}(p) - s_{B(\alpha)}(p) = A_2(\alpha) - B_2(\alpha).$$

Hence,

$$s_{A(\alpha)}(p) - s_{B(\alpha)}(p) = \begin{cases} -(A_1(\alpha) - B_1(\alpha)); & p = -1 \\ A_2(\alpha) - B_2(\alpha); & p = 1 \end{cases}$$

holds. In a similar way we obtain

$$s_{-B(\alpha)}(p) - s_{-A(\alpha)}(p) = \begin{cases} -(A_2(\alpha) - B_2(\alpha)); & p = -1 \\ A_1(\alpha) - B_1(\alpha); & p = 1 \end{cases}$$

On the other hand,
for $p = -1$, we obtain $s_{A^*(\beta)}(p) = -A_1^*(\beta)$ and $s_{B^*(\beta)}(p) = -B_1^*(\beta)$;
for $p = 1$, we obtain $s_{A^*(\beta)}(p) = A_2^*(\beta)$ and $s_{B^*(\beta)}(p) = B_2^*(\beta)$. Hence we can write that

$$s_{A^*(\beta)}(p) - s_{B^*(\beta)}(p) = \begin{cases} -(A_1^*(\beta) - B_1^*(\beta)); & p = -1 \\ A_2^*(\beta) - B_2^*(\beta); & p = 1 \end{cases}$$

and

$$s_{-B^*(\beta)}(p) - s_{-A^*(\beta)}(p) = \begin{cases} -(A_2^*(\beta) - B_2^*(\beta)); & p = -1 \\ A_1^*(\beta) - B_1^*(\beta); & p = 1 \end{cases}$$

By Theorem 4.7 considering monotonicity of $s_{A(\alpha)}(p) - s_{B(\alpha)}(p)$, $s_{-B(\alpha)}(p) - s_{-A(\alpha)}(p)$, $s_{A^*(\beta)}(p) - s_{B^*(\beta)}(p)$ and $s_{-B^*(\beta)}(p) - s_{-A^*(\beta)}(p)$ we obtain the four different cases given in the theorem.
On the other hand, if one of the four cases is satisfied than by Theorem 3.5 it is obvious that $\tilde{C}^i$ exists. □

Definition 4.9. *[23] Let a_1^*, a_1, a_2, a_3 and a_3^* be real numbers such that $a_1^* \leq a_1 \leq a_2 \leq a_3 \leq a_3^*$ holds. A triangular intuitionistic fuzzy number (TIFN) $\tilde{A}^i \in IF_N(\mathbb{R})$ is defined with the following membership and non-membership functions:*

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}; & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}; & a_2 \leq x \leq a_3 \\ 0; & otherwise \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \frac{a_2-x}{a_2-a_1^*}; & a_1^* \leq x \leq a_2 \\ \frac{x-a_2}{a_3^*-a_2}; & a_2 \leq x \leq a_3^* \\ 1; & otherwise \end{cases}$$

It is denoted by $\tilde{A}^i = (a_1, a_2, a_3; a_1^*, a_2, a_3^*)$. And its α and β cuts can be obtain as follows
$A(\alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_3 - a_2)]$ and $A^*(\beta) = [a_2 + \beta(a_2 - a_1^*), a_2 + \beta(a_3^* - a_2)]$.

Example 4.10. *Let $\tilde{A}^i = (6, 9, 10; 5, 9, 11)$ and $\tilde{B}^i = (13, 15, 18; 12, 15, 19)$. Assume $\tilde{A}^i \ominus_{gH} \tilde{B}^i = \tilde{C}^i$ exists.*
The α and β cuts of $\tilde{A}^i$ are

$$\begin{aligned} A(\alpha) &= [A_1(\alpha), A_2(\alpha)] = [6 + 3\alpha, 10 - \alpha], \\ A^*(\beta) &= [A_1^*(\beta), A_2^*(\beta)] = [9 - 4\beta, 9 + 2\beta]. \end{aligned}$$

The α and β cuts of $\tilde{B}^i$ are

$$\begin{aligned} B(\alpha) &= [B_1(\alpha), B_2(\alpha)] = [13 + 2\alpha, 18 - 3\alpha], \\ B^*(\beta) &= [B_1^*(\beta), B_2^*(\beta)] = [15 - 3\beta, 15 + 4\beta]. \end{aligned}$$

Let $C(\alpha) = [C_1(\alpha), C_2(\alpha)]$ and $C^*(\beta) = [C_1^*(\beta), C_2^*(\beta)]$. So we can obtain the followings:

$$\begin{aligned} C(\alpha) := A(\alpha) \ominus_{gH} B(\alpha) &= [\min\{\alpha - 7, 2\alpha - 8\}, \max\{\alpha - 7, 2\alpha - 8\}] \\ &= [2\alpha - 8, \alpha - 7] \\ &= [C_1(\alpha), C_2(\alpha)] \end{aligned}$$

and

$$\begin{aligned} C^*(\beta) := A^*(\beta) \ominus_{gH} B^*(\beta) &= [\min\{-7-\beta, -7-2\beta\}, \max\{-7-\beta, -7-2\beta\}] \\ &= [-7-2\beta, -7-\beta] \\ &= [C_1^*(\beta), C_2^*(\beta)] \end{aligned}$$

Since $C_1(\alpha)$ and $C_2(\alpha)$ are both increasing with respect to α and $C_1^*(\beta)$ and $C_2^*(\beta)$ are both decreasing with respect to β there is no an intuitionistic fuzzy number satisfying $\tilde{A}^i \ominus_{gH} \tilde{B}^i = \tilde{C}^i$.

Remark: *As we see in the Example 4.10 given above that the monotonicity of α and β cuts are essential.*

Corollary 4.11. *Let $\tilde{A}^i = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $\tilde{B}^i = (b_1, b_2, b_3; b_1', b_2, b_3')$ be two triangular intuitionistic fuzzy numbers. $\tilde{A}^i \ominus_{gH} \tilde{B}^i$ exists if and only if one of the following cases is satisfied:*

1. $$\begin{aligned} a_1 - b_1 &\leq a_2 - b_2 \leq a_3 - b_3 \\ a_1' - b_1' &\leq a_2 - b_2 \leq a_3' - b_3' \end{aligned}$$

2. $$\begin{aligned} a_1 - b_1 &\leq a_2 - b_2 \leq a_3 - b_3 \\ a_3' - b_3' &\leq a_2 - b_2 \leq a_1' - b_1' \end{aligned}$$

3. $$\begin{aligned} a_3 - b_3 &\leq a_2 - b_2 \leq a_1 - b_1 \\ a_1' - b_1' &\leq a_2 - b_2 \leq a_3' - b_3' \end{aligned}$$

4. $$\begin{aligned} a_3 - b_3 &\leq a_2 - b_2 \leq a_1 - b_1 \\ a_3' - b_3' &\leq a_2 - b_2 \leq a_1' - b_1' \end{aligned}$$

Proof. *The α and β cuts of $\tilde{A}^i = (a_1, a_2, a_3; a_1', a_2, a_3')$ and $\tilde{B}^i = (b_1, b_2, b_3; b_1', b_2, b_3')$ can be written as follows:*
$A(\alpha) = [A_1(\alpha), A_2(\alpha)] = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)]$,
$B(\alpha) = [B_1(\alpha), B_2(\alpha)] = [b_1 + \alpha(b_2 - b_1), b_3 + \alpha(b_2 - b_3)]$,
$A^*(\beta) = [A_1^*(\beta), A_2^*(\beta)] = [a_2 + \beta(a_1' - a_2), a_2 + \beta(a_3' - a_2)]$,
$B^*(\beta) = [B_1^*(\beta), B_2^*(\beta)] = [b_2 + \beta(b_1' - b_2), b_2 + \beta(b_3' - b_2)]$.
Let us assume Case 1 in Theorem 4.8 is satisfied. So from

$$\begin{aligned} A_1(\alpha) - B_1(\alpha) &= a_1 + \alpha(a_2 - a_1) - b_1 - \alpha(b_2 - b_1) \\ &= a_1 - b_1 + \alpha(a_2 - a_1 - b_2 + b_1)) \end{aligned}$$

we observe that $A_1(\alpha) - B_1(\alpha)$ is a non-decreasing function of α if and only if

$$a_2 - a_1 - b_2 + b_1 \geq 0$$

is satisfied. Hence we obtain that

$$a_1 - b_1 \leq a_2 - b_2.$$

On the other hand, from

$$\begin{aligned} A_2(\alpha) - B_2(\alpha) &= a_3 + \alpha(a_2 - a_3) - b_3 - \alpha(b_2 - b_3) \\ &= a_3 - b_3 + \alpha(a_2 - a_3 - b_2 + b_3)) \end{aligned}$$

we observe that $A_2(\alpha) - B_2(\alpha)$ *is a non-increasing function of* α *if and only if*

$$a_2 - b_2 \le a_3 - b_3.$$

Hence we obtain that

$$a_1 - b_1 \le a_2 - b_2 \le a_3 - b_3$$

Now let us consider the β*-cuts. From*

$$\begin{aligned} A_1^*(\beta) - B_1^*(\beta) &= a_2' + \beta(a_1' - a_2) - b_2' - \beta(b_1' - b_2) \\ &= a_2' - b_2' + \beta(a_1' - a_2 - b_1' + b_2) \end{aligned}$$

we obtain that $A_1^*(\beta) - B_1^*(\beta)$ *is a non-increasing function of* β *if and only if*

$$a_1' - b_1' \le a_2 - b_2$$

holds. On the other hand, from

$$\begin{aligned} A_2^*(\beta) - B_2^*(\beta) &= a_2' + \beta(a_3' - a_2) - b_2' - \beta(b_3' - b_2) \\ &= a_2' - b_2' + \beta(a_3' - a_2 - b_3' + b_2) \end{aligned}$$

we obtain that $A_2^*(\beta) - B_2^*(\beta)$ *is a non-decreasing function of* β *if and only if*

$$a_2 - b_2 \le a_3' - b_3'.$$

Hence we obtain that

$$a_1' - b_1' \le a_2 - b_2 \le a_3' - b_3'$$

As a result $\tilde{A}^i \ominus_{gH} \tilde{B}^i$ *exists if and only if*

$$\begin{aligned} a_1 - b_1 &\le a_2 - b_2 \le a_3 - b_3 \\ a_1' - b_1' &\le a_2 - b_2 \le a_3' - b_3' \end{aligned}$$

hold. In a similar way other cases can be proved by using Theorem 4.8.
On the other hand, if one of the four cases is satisfied than by Theorem 3.5 it is obvious that $\tilde{A}^i \ominus_{gH} \tilde{B}^i$ *exists.* □

Note that in Example 4.10 $\tilde{C}^i$*is obtained as* $(-8, -6, -7; -9, -7, -8)$ *which is not obviously a triangular intuitionistic fuzzy number.*

Theorem 4.12. *Let* $s_{A(\alpha)}$, $s_{A^*(\beta)}$ *be* α *and* β *support functions of* $\tilde{A}^i \in IF_N(\mathbb{R}^n)$ *and* $s_{B(\alpha)}$, $s_{B^*(\beta)}$ *be* α *and* β *support functions of* $\tilde{B}^i \in IF_N(\mathbb{R}^n)$*. Let us define the following functions* $s_1 = s_{A(\alpha)} - s_{B(\alpha)}$, $s_2 = s_{B(\alpha)} - s_{A(\alpha)}$, $s_1^* = s_{A^*(\beta)} - s_{B^*(\beta)}$ *and* $s_2^* = s_{B^*(\beta)} - s_{A^*(\beta)}$

1. *If*
   (a) s_1 *is subadditive in* $p \in S^{n-1}$ *for all* $\alpha \in [0,1]$ *and non-increasing in* $\alpha \in [0,1]$ *for all* $p \in S^{n-1}$ *and*
   (b) s_1^* *is subadditive in* $p \in S^{n-1}$ *for all* $\beta \in [0,1]$ *and non-decreasing in* β *for all* $p \in S^{n-1}$

then $\tilde{A}^i \ominus_{gH} \tilde{B}^i = \tilde{C}^i$ *exists such that* $\tilde{A}^i = \tilde{B}^i + \tilde{C}^i$ *is satisfied. And* α *and* β *support functions of* $\tilde{C}^i \in IF(\mathbb{R}^n)$ *satisfy* $s_{C(\alpha)} = s_{A(\alpha)} - s_{B(\alpha)}$ *and* $s_{C^*(\beta)} = s_{A^*(\beta)} - s_{B^*(\beta)}$*.*

2. *If*
   (a) s_2 *is subadditive in* $p \in S^{n-1}$ *for all* $\alpha \in [0,1]$ *and non-increasing in* $\alpha \in [0,1]$ *for all* $p \in S^{n-1}$ *and*
   (b) s_2^* *is subadditive in* $p \in S^{n-1}$ *for all* $\beta \in [0,1]$ *and non-decreasing in* β *for all* $p \in S^{n-1}$

then $\tilde{A}^i \ominus_{gH} \tilde{B}^i = \tilde{C}^i$ *exists such that* $\tilde{B}^i = \tilde{A}^i + (-1)\tilde{C}^i$ *is satisfied. And* α *and* β *support functions of* $\tilde{C}^i \in IF(\mathbb{R}^n)$ *satisfy* $s_{C(\alpha)} = s_{-B(\alpha)} - s_{-A(\alpha)}$ *and* $s_{C^*(\beta)} = s_{-B^*(\beta)} - s_{-A^*(\beta)}$.

3. *If*
   (a) s_1 *and* s_2 *are both subadditive in* $p \in S^{n-1}$ *for all* $\alpha \in [0,1]$ *and both non-increasing in* $\alpha \in [0,1]$ *for all* $p \in S^{n-1}$
   and
   (b) s_1^* *and* s_2^* *are both subadditive in* $p \in S^{n-1}$ *for all* $\beta \in [0,1]$ *and both non-decreasing in* β *for all* $p \in S^{n-1}$

then $\tilde{A}^i \ominus_{gH} \tilde{B}^i = \tilde{C}^i$ *exists such that* $\tilde{A}^i = \tilde{B}^i + \tilde{C}^i$ *and* $\tilde{B}^i = \tilde{A}^i + (-1)\tilde{C}^i$ *hold simultaneously.*

Proof.
1. Since for all $\alpha, \beta \in [0,1]$ s_1 and s_1^* are subadditive in p, by Theorem 2.13 $C(\alpha) := A(\alpha) \ominus_{gH} B(\alpha)$ and $C^*(\beta) := A^*(\beta) \ominus_{gH} B^*(\beta)$ exist. Besides $s_{C(\alpha)}$ and $s_{C^*(\beta)}$ are α and β support functions of $C(\alpha)$ and $C^*(\beta)$. Let us now consider the family of sets $\{C(\alpha) : \alpha \in [0,1]\}$ and $\{C^*(\beta) : \beta \in [0,1]\}$. In [42] it is shown that the family $\{C(\alpha) : \alpha \in [0,1]\}$ forms an α-cut. Let us show that $\{C^*(\beta) : \beta \in [0,1]\}$ forms a β-cut. Since $s_{C^*(\beta)}$ is the support function of $C^*(\beta)$ for $\beta \in [0,1)$ we can uniquely define $C^*(\beta)$ by

$$C^*(\beta) = \{x \in \mathbb{R}^n : \langle p, x \rangle \leq s^*_{\tilde{C}^i}(p;\beta), p \in \mathbb{R}^n, \|p\| = \beta\}$$

and for $\beta = 1$

$$C^*(1) = cl\left(\bigcup_{k \in [0,1)} C^*(\beta_k)\right).$$

Since $s_{C^*(\beta)}$ is a support function, for all $\beta \in [0,1]$, $C^*(\beta)$ is a compact and convex set. Since for all $p \in \mathbb{R}^n$ $s_{C^*(\beta)}$ is non-decreasing in β, we get

$$\begin{aligned} \beta_1 &\leq \beta_2 \Rightarrow s_{C^*(\beta_1)} \leq s_{C^*(\beta_2)} \\ &\Rightarrow C^*(\beta_1) \subseteq C^*(\beta_2). \end{aligned}$$

Let (β_k) be a sequence in $[0,1]$ converging to β from above. So for all $k \in \mathbb{N}$, we can obtain that

$$\begin{aligned} \beta &\leq \beta_k \Rightarrow C^*(\beta) \subseteq C^*(\beta_k) \\ &\Rightarrow C^*(\beta) \subseteq \bigcap_{k=1}^{\infty} C^*(\beta_k). \end{aligned}$$

On the other hand let $x \in \bigcap_{k=1}^{\infty} C^*(\beta_k) \Rightarrow \forall\, k \in \mathbb{N}, x \in C^*(\beta_k)$. So for all $p \in \mathbb{R}^n$ with $\|p\| = \beta_k$

$$\langle p, x \rangle \leq s_{C^*(\beta_k)}(p)$$

holds. Especially let us take $p = \frac{\beta_k}{\beta} p_1$ for $p_1 \in \mathbb{R}^n$. So we have

$$\left\langle \frac{\beta_k}{\beta} p_1, x \right\rangle \leq s_{C^*(\beta_k)}(\frac{\beta_k}{\beta} p_1).$$

Since β_k is a non-increasing sequence converging to β, if we take the limit of both sides as $k \to \infty$ we get

$$\langle p_1, x \rangle \leq s_{C^*(\beta)}(p_1).$$

So we obtain $x \in C^*(\beta)$. Hence

$$\bigcap_{k=1}^{\infty} C^*(\beta_k) \subseteq C^*(\beta).$$

As a result

$$\bigcap_{k=1}^{\infty} C^*(\beta_k) = C^*(\beta)$$

is satisfied. In a similar way, for a sequence (β_k) be a sequence in $[0, 1]$ converging to 1 from below it can be shown that

$$cl\left(\bigcup_{k=1}^{\infty} C^*(\beta_k)\right) = C^*(1).$$

Hence by Theorem 3.4 and by [42] we obtain that the family of sets $\{C(\alpha) : \alpha \in [0, 1]\}$ and $\{C^*(\beta) : \beta \in [0, 1]\}$ define an intuitionistic fuzzy number. For (2.) and (3.) proof can be done in a similar way □

5. An Application: Strongly Generalized Hukuhara Differentiability Concept in Intuitionistic Fuzzy Environment

One of the important applications of the Hukuhara difference is in the foundation of generalized derivative concept in fuzzy set theory [43]. In this section we will give the definition of strongly generalized Hukuhara derivative for intuitionistic fuzzy environment. Then we will extend some important theorems from fuzzy set theory to intuitionistic fuzzy set theory.

Definition 5.1. *Let $f : (a, b) \to IF_N(\mathbb{R})$ be an intuitionistic fuzzy number valued function and let $x_0, x_0 + h \in (a, b)$. for all sufficiently small positive real number $h > 0$. f is called strongly generalized Hukuhara (GH) differentiable at x_0 if there exists $f'_{GH}(x_0) \in IF_N(\mathbb{R})$ such that for all sufficiently small positive real number $h > 0$ at least one of the followings is satisfied:*

1. *$f(x_0 + h) \ominus_H f(x_0)$ and $f(x_0) \ominus_H f(x_0 - h)$ exist and the following limits exist such that*

$$\lim_{h\to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_{GH}(x_0)$$

2. *$f(x_0) \ominus_H f(x_0 + h)$ and $f(x_0 - h) \ominus_H f(x_0)$ exist and the following limits exist such that*

$$\lim_{h\to 0^+} \frac{f(x_0) \ominus_H f(x_0 + h)}{-h} = \lim_{h\to 0^+} \frac{f(x_0 - h) \ominus_H f(x_0)}{-h} = f'_{GH}(x_0)$$

3. *$f(x_0 + h) \ominus_H f(x_0)$ and $f(x_0 - h) \ominus_H f(x_0)$ exist and the following limits exist such that*

$$\lim_{h\to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0 - h) \ominus_H f(x_0)}{-h} = f'_{GH}(x_0)$$

4. *$f(x_0) \ominus_H f(x_0 + h)$ and $f(x_0) \ominus_H f(x_0 - h)$ exist and the following limits exist such that*

$$\lim_{h\to 0^+} \frac{f(x_0) \ominus_H f(x_0 + h)}{-h} = \lim_{h\to 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_{GH}(x_0)$$

Lemma 5.2. *Let $f : (a, b) \to IF(\mathbb{R})$ be an intuitionistic fuzzy number valued function such that its α and β cuts given by $f(x; \alpha) = [f_1(x; \alpha), f_2(x; \alpha)]$ and $f^*(x; \beta) = [f_1^*(x; \beta), f_2^*(x; \beta)]$.*

Suppose $f_1(x; \alpha)$, $f_2(x; \alpha)$, $f_1^*(x; \beta)$ and $f_2^*(x; \beta)$ are real valued differentiable functions w.r.t. x uniformly in α and $\beta \in [0, 1]$.

If f is GH-differentiable on (a, b) then

1. *There is no a sign change in the difference $f_2'(x; \alpha_0) - f_1'(x; \alpha_0)$ at a fixed $\alpha_0 \in (0, 1)$.*
   and

2. *There is no a sign change in the difference* $(f_2^*)'(x;\beta_0) - (f_1^*)'(x;\beta_0)$ *at a fixed* $\beta_0 \in (0,1)$.

Proof. From [49] we know (a) holds. Let us prove (2.).

Assume that there is a sign change in the difference $(f_2^*)'(x;\beta_0) - (f_1^*)'(x;\beta_0)$ at a fixed $\beta_0 \in (0,1)$. Then at $\beta_0 \in (0,1)$, $(f_2^*)'(x;\beta_0) - (f_1^*)'(x;\beta_0) \geq 0$ and $(f_2^*)'(x;\beta_0) - (f_1^*)'(x;\beta_0) \leq 0$ both hold. So we obtain than $(f_1^*)'(x;\beta_0) = (f_2^*)'(x;\beta_0)$. This means that $(f_{GH}^*)'(x;\beta_0)$ is a singleton set. Since for $0 \leq \beta < \beta_0$ we know that $(f_{GH}^*)'(x;\beta) \subseteq (f_{GH}^*)(x;\beta_0)$ then $(f_{GH}^*)'(x;\beta)$ is a singleton set as well. So it follows that $(f_1^*)'(x;\beta) - (f_2^*)'(x;\beta) = 0$. And this is a contradiction with the assumption that the difference $(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta)$ changes sign at $\beta_0 \in (0,1)$. □

Theorem 5.3. *Let* $f : (a,b) \to IF_N(\mathbb{R})$ *be an intuitionistic fuzzy number valued function. Let* α *and* β *cuts of* f *be given by* $f(x;\alpha) = [f_1(x;\alpha), f_2(x;\alpha)]$ *and* $f^*(x;\beta) = [f_1^*(x;\beta), f_2^*(x;\beta)]$. *Suppose that the functions* $f_1(x;\alpha)$, $f_2(x;\alpha)$, $f_1^*(x;\beta)$, *and* $f_2^*(x;\beta)$ *are differentiable (in classical sense) w.r.t.* x *uniformly in* α *and* $\beta \in [0,1]$. *Then if the function* $f(x)$ *is GH-differentiable at* $x \in (a,b)$ *then one of the followings is satisfied:*

1. $\begin{cases} \text{(a) } f_1'(x;\alpha) \text{ is non-decreasing w.r.t. } \alpha,\ f_2'(x;\alpha) \text{ is non-increasing w.r.t. } \alpha \text{ and } f_1'(x;1) \leq f_2'(x;1) \\ \text{and} \\ \text{(b) } (f_1^*)'(x;\beta) \text{is non-increasing w.r.t. } \beta,\ (f_2^*)'(x;\beta) \text{ is non-decreasing w.r.t. } \beta \text{ and } (f_1^*)'(x;0) \leq (f_2^*)'(x;0) \end{cases}$

or

2. $\begin{cases} \text{(a) } f_1'(x;\alpha) \text{ is non-decreasing w.r.t. } \alpha,\ f_2'(x;\alpha) \text{ is non-increasing w.r.t. } \alpha \text{ and } f_1'(x;1) \leq f_2'(x;1) \\ \text{and} \\ \text{(b) } (f_2^*)'(x;\beta) \text{ is non-increasing w.r.t. } \beta,\ (f_1^*)'(x;\beta) \text{is non-decreasing w.r.t. } \beta \text{ and } (f_2^*)'(x;0) \leq (f_1^*)'(x;0) \end{cases}$

or

3. $\begin{cases} \text{(a) } f_2'(x;\alpha) \text{ is non-decreasing w.r.t. } \alpha,\ f_1'(x;\alpha) \text{ is non-increasing w.r.t. } \alpha \text{ and } f_2'(x;1) \leq f_1'(x;1) \\ \text{and} \\ \text{(b) } (f_1^*)'(x;\beta) \text{ is non-increasing w.r.t. } \beta,\ (f_2^*)'(x;\beta) \text{is non-decreasing w.r.t. } \beta \text{ and } (f_1^*)'(x;0) \leq (f_2^*)'(x;0) \end{cases}$

or

4. $\begin{cases} \text{(a) } f_2'(x;\alpha) \text{ is non-decreasing w.r.t. } \alpha,\ f_1'(x;\alpha) \text{ is non-increasing w.r.t. } \alpha \text{ and } f_2'(x;1) \leq f_1'(x;1) \\ \text{and} \\ \text{(b) } (f_2^*)'(x;\beta) \text{ is non-increasing w.r.t. } \beta,\ (f_1^*)'(x;\beta) \text{is non-decreasing w.r.t. } \beta \text{ and } (f_2^*)'(x;0) \leq (f_1^*)'(x;0) \end{cases}$

Moreover we have
$f_{GH}'(x;\alpha) = [\min\{f_1'(x;\alpha), f_2'(x;\alpha)\}, \max\{f_1'(x;\alpha), f_2'(x;\alpha)\}]$
and
$(f_{GH}^*)'(x;\beta) = [\min\{(f_1^*)'(x;\beta), (f_2^*)'(x;\beta)\}, \max\{(f_1^*)'(x;\beta), (f_2^*)'(x;\beta)\}]$.

Proof. Suppose that the functions $f_1(x;\alpha)$, $f_2(x;\alpha)$, $f_1^*(x;\beta)$, and $f_2^*(x;\beta)$ are differentiable (in classical sense) w.r.t. x, uniformly in α and $\beta \in [0,1]$. Since f is GH-differentiable on (a,b) we obtain the followings:

$$\begin{aligned} f_{GH}'(x;\alpha) &= [f_1'(x;\alpha), f_2'(x;\alpha)] \\ &\quad \text{or} \\ f_{GH}'(x;\alpha) &= [f_2'(x;\alpha), f_1'(x;\alpha)] \end{aligned}$$

and

$$\begin{aligned} (f_{GH}')^*(x;\beta) &= [(f_1^*)'(x;\beta), (f_2^*)'(x;\beta)] \\ &\quad \text{or} \\ (f_{GH}')^*(x;\beta) &= [(f_2^*)'(x;\beta), (f_1^*)'(x;\beta)] \end{aligned}$$

So we can write that

$$f'_{GH}(x;\alpha) = [\min\{f'_1(x;\alpha), f'_2(x;\alpha)\}, \max\{f'_1(x;\alpha), f'_2(x;\alpha)\}]$$

and

$$(f'_{GH})^*(x;\beta) = [\min\{(f_1^*)'(x;\beta), (f_2^*)'(x;\beta)\}, \max\{(f_1^*)'(x;\beta), (f_2^*)'(x;\beta)\}].$$

By [43, 44] it is known that for any fixed $\alpha \in (0,1)$ the difference

$$f'_2(x;\alpha) - f'_1(x;\alpha)$$

does not change sign on (a,b). So at a fixed $\alpha \in [0,1]$

$$f'_2(x;\alpha) - f'_1(x;\alpha) \geq 0$$

or

$$f'_2(x;\alpha) - f'_1(x;\alpha) \leq 0$$

is satisfied. And by Lemma 5.2 it is known that for any fixed $\beta \in (0,1)$ the difference

$$(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta)$$

does not change sign on (a,b). So at a fixed $\beta \in [0,1]$

$$(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta) \geq 0$$

or

$$(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta) \leq 0$$

is satisfied. By considering these facts we obtain the following cases:

1.
$$f'_2(x;\alpha) - f'_1(x;\alpha) \geq 0$$
$$(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta) \geq 0$$

2.
$$f'_2(x;\alpha) - f'_1(x;\alpha) \geq 0$$
$$(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta) \leq 0$$

3.
$$f'_2(x;\alpha) - f'_1(x;\alpha) \leq 0$$
$$(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta) \geq 0$$

4.
$$f'_2(x;\alpha) - f'_1(x;\alpha) \leq 0$$
$$(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta) \leq 0$$

Let us complete the proof by considering the first case. The proof for the other cases can be done in the same way.

Case 1. Let us assume that $f_2'(x;\alpha) - f_1'(x;\alpha) \geq 0$ and $(f_2^*)'(x;\beta) - (f_1^*)'(x;\beta) \geq 0$ are satisfied at $x \in (a,b)$. Since f is GH-differentiable on (a,b),

$$\begin{aligned} f_{GH}'(x;\alpha) &= [\min\{f_1'(x;\alpha), f_2'(x;\alpha)\}, \max\{f_1'(x;\alpha), f_2'(x;\alpha)\}] \\ &= [f_1'(x;\alpha), f_2'(x;\alpha)] \end{aligned}$$

and

$$\begin{aligned} (f_{GH}')^*(x;\beta) &= [\min\{(f_1^*)'(x;\beta), (f_2^*)'(x;\beta)\}, \max\{(f_1^*)'(x;\beta), (f_2^*)'(x;\beta)\}] \\ &= [(f_1^*)'(x;\beta), (f_2^*)'(x;\beta)] \end{aligned}$$

hold.

Let $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Since $f_{GH}'(x;\alpha_2) \subseteq f_{GH}'(x;\alpha_1)$ then we obtain

$$f_1'(x;\alpha_1) \leq f_1'(x;\alpha_2)$$

and

$$f_2'(x;\alpha_2) \leq f_2'(x;\alpha_1).$$

Hence $f_1'(x;\alpha)$ is non-decreasing w.r.t. α, $f_2'(x;\alpha)$ is non-increasing w.r.t. α and $f_1'(x;1) \leq f_2'(x;1)$ holds.

Let $0 \leq \beta_1 \leq \beta_2 \leq 1$. Since $(f_{GH}')^*(x;\beta_1) \subseteq (f_{GH}')^*(x;\beta_2)$ then we obtain

$$(f_1^*)'(x;\beta_2) \leq (f_1^*)'(x;\beta_1)$$

and

$$(f_2^*)'(x;\beta_1) \leq (f_2^*)'(x;\beta_2).$$

Hence $(f_1^*)'(x;\beta)$ is non-increasing w.r.t. β, $(f_2^*)'(x;\beta)$ is non-decreasing w.r.t. β and $(f_1^*)'(x;0) \leq (f_2^*)'(x;0)$ holds. □

Theorem 5.4. *Let $f : (a,b) \to IF(\mathbb{R})$ be an intuitionistic fuzzy number valued function and $x_0 \in (a,b)$. If*

1. *For every $\alpha, \beta \in [0,1]$, $\lim\limits_{x\to x_0} f(x;\alpha) = [A_1(\alpha), A_2(\alpha)]$ and $\lim\limits_{x\to x_0} f(x;\beta) = [A_1^*(\beta), A_2^*(\beta)]$ and*
2. *$[A_1(\alpha), A_2(\alpha)]$ and $[A_1^*(\beta), A_2^*(\beta)]$ satisfy (1-7) in Theorem 3.1.*

then there exists an intuitionistic fuzzy number $\tilde{A}^i$ such that the α and β cuts of $\tilde{A}^i$ are $[A_1(\alpha), A_2(\alpha)]$ and $[A_1^*(\beta), A_2^*(\beta)]$, respectively; and that $\lim\limits_{x\to x_0} f(x) = \tilde{A}^i$.

Proof: Assume the conditions (1.) and (2.) are satisfied. Then by Theorem 3.4 we can deduce that there exists an intuitionistic fuzzy number $\tilde{A}^i$ such that the α and β cuts of $\tilde{A}^i$ are $[A_1(\alpha), A_2(\alpha)]$ and $[A_1^*(\beta), A_2^*(\beta)]$, respectively. Furthermore, since

$$\lim_{x\to x_0} f(x;\alpha) = A(\alpha) \Rightarrow \lim_{x\to x_0} D_1(f(x;\alpha), A(\alpha)) = 0$$

and

$$\lim_{x\to x_0} f(x;\beta) = A^*(\beta) \Rightarrow \lim_{x\to x_0} D_2(f^*(x;\beta), A^*(\beta)) = 0$$

we can write that $\lim\limits_{x\to x_0} D_\infty(f(x), \tilde{A}^i) = 0$. Hence we obtain that $\lim\limits_{x\to x_0} f(x) = \tilde{A}^i$.

6. Conclusion and Summary

In this paper, we have studied and extended some important definitions and results from fuzzy set theory to intuitionistic fuzzy set theory. We have firstly extended the well-known stacking and characterization theorems to intuitionistic fuzzy sets in $\mathbb{R}^n$. For this, we have proved that the α and β cuts of an intuitionistic fuzzy number in $\mathbb{R}^n$ are compact and convex sets satisfying the properties (1.)-(8.) in Theorem 3.1. With the help of this theorem, we have proved that the endpoints of the α and β cuts of an intuitionistic fuzzy number in $\mathbb{R}^n$satisfies (1.)-(6.) in Theorem 3.2. In Theorem 3.3, we have shown that any two families of sets satisfying (1.) -(8.) in Theorem 3.1 define an intuitionistic fuzzy number. Hence we have shown that the family of closed and bounded intervals satisfying the properties in Theorem 3.1 define an intuitionistic fuzzy number in $\mathbb{R}^n$.

By using Minkowski sum and scalar multiplication of compact and convex sets, we have defined the sum and scalar multiplication for the intuitionistic fuzzy numbers with the help of α and β cuts. And we have proved that the family of intuitionistic fuzzy numbers in $\mathbb{R}^n$ is closed under these two operations. With the help of these results, we have examined the fundamental theorems of Hukuhara and generalized Hukuhara differences. we have given the necessary and sufficient condition for the existence of the gH-difference of intuitionistic fuzzy numbers by using the α and β support functions of intuitionistic fuzzy numbers. Lastly, as an application of Hukuhara difference, we have studied the strongly generalized differentiability concept in intuitionistic fuzzy environment.

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