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A Class of Big (p, q)-Appell Polynomials and Their Associated Difference Equations

H. M. Srivastava[a], B. Y. Yaşar[b], M. A. Özarslan[b]

[a]*Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*

and

Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402*, Taiwan, Republic of China*

[b]*Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, Famagusta, TR-33310 Mersin, Turkey*

Abstract. In the present paper, we introduce and investigate the big (p, q)-Appell polynomials. We prove an equivalance theorem satisfied by the big (p, q)-Appell polynomials. As a special case of the big (p, q)-Appell polynomials, we present the corresponding equivalence theorem, recurrence relation and difference equation for the big q-Appell polynomials. We also present the equivalence theorem, recurrence relation and differential equation for the usual Appell polynomials. Moreover, for the big (p, q)-Bernoulli polynomials and the big (p, q)-Euler polynomials, we obtain recurrence relations and difference equations. In the special case when $p = 1$, we obtain recurrence relations and difference equations which are satisfied by the big q-Bernoulli polynomials and the big q-Euler polynomials. In the case when $p = 1$ and $q \to 1-$, the big (p, q)-Appell polynomials reduce to the usual Appell polynomials. Therefore, the recurrence relation and the difference equation obtained for the big (p, q)-Appell polynomials coincide with the recurrence relation and differential equation satisfied by the usual Appell polynomials. In the last section, we have chosen to also point out some obvious connections between the (p, q)-analysis and the classical q-analysis, which would show rather clearly that, in most cases, the transition from a known q-result to the corresponding (p, q)-result is fairly straightforward.

1. Introduction, Definitions and Preliminaries

The well-known Appell polynomials $P_n(x)$ are given by

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x) \, \frac{t^n}{n!},$$

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Email addresses: harimsri@math.uvic.ca (H. M. Srivastava), banu.yilmaz@emu.edu.tr, banuyilmazgm@gmail.com (B. Y. Yaşar), mehmetali.ozarslan@emu.edu.tr (M. A. Özarslan)

where $A(t)$ is the determining function of the Appell polynomials satisfying the following condition:

$$\frac{A'(t)}{A(t)} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!} \qquad \Big(A(0) \neq 0\Big).$$

For the Appell polynomials $P_n(x)$, we have

$$\frac{d}{dx}\Big(P_n(x)\Big) = nP_{n-1}(x) \qquad \left(\frac{d}{dx} =: D_x\right),$$

which shows that the lowering operator L_n^- for the Appell polynomials $P_n(x)$ is, in fact, the derivative operator:

$$L_n^- := \frac{1}{n}\frac{d}{dx}.$$

Some recurrence relation and differential equation satisfied by the Appell polynomials were obtained by He and Ricci [6]. By means of the lowering and raising operators, they used the factorization method (see, for details, [8])

$$L_{n+1}^- \, L_n^+\Big(P_n(x)\Big) = P_n(x)$$

and obtained the differential equation. In the classical factorization method, in order to obtain the differential equation, one needs to find the lowering and raising operators. This method is applicable for some such Appell polynomials as the Bernoulli polynomials, the Euler polynomials, the 2*D*-Bernoulli polynomials, the 2*D*-Euler, and the Hermite-based Appell polynomials. Also, for some extensions of the Appell polynomials the recurrence relations and differential equations were obtained (see, for example, [3], [23] and [24]). A generalization of this method was given in [17] and a set of finite-order differential equations was obtained for the Appell polynomials with the kth iteration of the lowering and the kth iteration of the raising operators by

$$\Big(\theta_{n+k}^{-(k)} \, \theta_n^{+(k)}\Big)\Big(P_n(x)\Big) = P_n(x).$$

In some different calculus, the raising operator cannot be defined for the Appell polynomials. For instance, in the basic (or q-) calculus, some difference equations satisfied by the Appell polynomials were obtained without using the raising operator [13]. Moreover, in the (p,q)-calculus, the raising operator cannot be defined for the Appell polynomials. Therefore, some different techniques should be applied in order to obtain the difference equations satisfied by the (p,q)-Appell polynomials.

We now introduce some basic definitions, notations and conventions about the (p,q)-calculus. By assuming (for simplicity) that $0 < q < p \leqq 1$, we first recall the (p,q)-derivative of a function f defined by (see [9] and [19])

$$\Big(D_{p,q}\, f\Big)(x) := \begin{cases} \dfrac{f(px) - f(qx)}{(p-q)x} & (x \neq 0;\ 0 < q < p \leqq 1) \\[2ex] f'(0) & (x = 0;\ 0 < q < p \leqq 1), \end{cases}$$

so that, for the familiar q-derivative operator D_q, we have

$$\Big(D_{1,q}\, f\Big)(x) = \Big(D_q\, f\Big)(x) := \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x} & (x \neq 0;\ 0 < q < 1) \\[2ex] f'(0) & (x = 0;\ 0 < q < 1). \end{cases}$$

Clearly, we have the following relationship between the (p,q)-derivative operator $D_{p,q}$ and the q-derivative operator D_q:

$$\left(D_{p,q}\, f\right)(x) = \left(D_{\frac{q}{p}}\, f\right)(px) \qquad \text{and} \qquad \left(D_q\, f\right)(x) = \left(D_{p,pq}\, f\right)\left(\frac{x}{p}\right) \qquad (x \neq 0;\ 0 < q < p \leqq 1).$$

Moreover, it is easily seen that

$$\lim_{q\to p}\left\{\left(D_{p,q}\, f\right)(x)\right\} = f'(px) \qquad \text{and} \qquad \lim_{q\to 1-}\left\{\left(D_q\, f\right)(x)\right\} = f'(x)$$

for a function f which is differentiable in a given subset of $\mathbb{R}$.

The (p,q)-analog $[n]_{p,q}$ of a number n is defined by

$$[n]_{p,q} = \begin{cases} \dfrac{p^n - q^n}{p-q} & (n \in \mathbb{N};\ 0 < q < p \leqq 1) \\ 0 & (n = 0), \end{cases}$$

so that, for the familiar q-number $[n]_q$, we have

$$[n]_q = [n]_{1,q} \qquad \text{and} \qquad [n]_{p,q} = p^{n-1}\, [n]_{\frac{q}{p}}.$$

The (p,q)-factorial $[n]_{p,q}!$ is defined by

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}\, [2]_{p,q}\, [3]_{p,q} \cdots [n]_{p,q} & (n \in \mathbb{N}) \\ 1 & (n = 0), \end{cases}$$

so that, for the q-factorial $[n]_q!$, we have

$$[n]_q! = [n]_{1,q}! := [1]_q\, [2]_q\, [3]_q \cdots [n]_q \qquad \text{and} \qquad [n]_{p,q}! = p^{\binom{n}{2}}\, [n]_{p,\frac{q}{p}}!.$$

Finally, the (p,q)-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q}$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}!\, [n-k]_{p,q}!} \qquad (n,k \in \mathbb{N};\ 0 \leqq k \leqq n),$$

so that, for the q-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_{1,q} := \frac{[n]_q!}{[k]_q!\, [n-k]_q!}.$$

For any constants $\mathcal{A}$ and $\mathcal{B}$, we have the following linearity property of the (p,q)-derivative operator $D_{p,q}$:

$$\left(D_{p,q}(\mathcal{A}f + \mathcal{B}g)\right)(x) = \mathcal{A}\left(D_{p,q}\, f\right)(x) + \mathcal{B}\left(D_{p,q}\, g\right)(x).$$

The product and quotient rules in the (p,q)-calculus are given by (see [19])

$$\begin{aligned}\left(D_{p,q}(fg)\right)(x) &= f(px)\left(D_{p,q}\, g\right)(x) + g(qx)\left(D_{p,q}\, f\right)(x) \\ &= g(px)\left(D_{p,q}\, f\right)(x) + f(qx)\left(D_{p,q}\, g\right)(x) \qquad (1)\end{aligned}$$

and

$$\left(D_{p,q}\left(\frac{f}{g}\right)\right)(x) = \frac{g(qx)\left(D_{p,q}\, f\right)(x) - f(qx)\left(D_{p,q}\, g\right)(x)}{g(px)g(qx)}$$
$$= \frac{g(px)\left(D_{p,q}\, f\right)(x) - f(px)\left(D_{p,q}\, g\right)(x)}{g(px)g(qx)}, \tag{2}$$

respectively. The special case of each of the above rules when $p = 1$ holds true for the q-derivative operator D_q.

In the present paper, we define the big (p, q)-Appell polynomials by

$$A_{p,q}(t)E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,p,q}(x)\ \frac{t^n}{[n]_{p,q}!},$$

where $E_{p,q}(x)$ is given by

$$E_{p,q}(x) := \sum_{n=0}^{\infty} q^{\binom{n}{2}}\ \frac{x^n}{[n]_{p,q}!} \qquad \left(0 < \left|\frac{q}{p}\right| < 1;\ |x| < 1\right)$$

and $A_{p,q}(t)$ is the determining function of the big (p, q)-Appell polynomials given by

$$A_{p,q}(t) = \sum_{n=0}^{\infty} a_{n,p,q}\ \frac{t^n}{[n]_{p,q}!} \qquad \left(A_{p,q}(0) \neq 0\right).$$

The big (p, q)-Appell polynomials satisfy the following relation:

$$\left(D_{p,q}\ P_{n,p,q}\right)(x) = \frac{[n]_{p,q}}{q}\ P_{n-1,p,q}(qx).$$

Hence the lowering operator $L^-_{n,p,q}$ is defined here

$$L^-_{n,p,q} := \frac{q}{[n]_{p,q}!}\ D_{p,q}.$$

In the special cases of $A_{p,q}(t)$, we introduce the big (p, q)-Bernoulli and the big (p, q)-Euler polynomials.

In the case when

$$A_{p,q}(t) = \frac{t}{E_{p,q}\left(\frac{t}{q}\right) - 1},$$

we have the big (p, q)-Bernoulli polynomials given by

$$\left(\frac{t}{E_{p,q}\left(\frac{t}{q}\right) - 1}\right)E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} B_{n,p,q}(x)\ \frac{t^n}{[n]_{p,q}!},$$

where the big (p, q)-Bernoulli numbers are given by

$$\frac{t}{E_{p,q}\left(\frac{t}{q}\right) - 1} = \sum_{k=0}^{\infty} B_{k,p,q}\ \frac{t^k}{[k]_{p,q}!}.$$

In the case when

$$A_{p,q}(t) = \frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right) + 1},$$

we have the big (p, q)-Euler polynomials given by

$$\left(\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right) + 1}\right) E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!},$$

where the big (p, q)-Euler numbers are given by

$$\left(\frac{[2]_{p,q}}{E_{p,q}\left(\frac{[2]_{p,q}}{q} t\right) + 1}\right) E_{p,q}\left(\frac{t}{q}\right) = \sum_{k=0}^{\infty} E_{k,p,q} \frac{t^k}{[k]_{p,q}!}.$$

Also, upon replacing p by $\frac{1}{p}$ and q by $\frac{1}{q}$ in the generating functions of the big (p, q)-Bernoulli polynomials and the big (p, q)-Euler polynomials, we get the corresponding definitions of the (p, q)-Bernoulli polynomials and the (p, q)-Euler polynomials, respectively:

$$\left(\frac{t}{e_{p,q}(qt) - 1}\right) e_{p,q}(qxt) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} p^{\binom{n}{2}} B_{n,\frac{1}{p},\frac{1}{q}}(x) \frac{t^n}{[n]_{p,q}!}$$

and

$$\left(\frac{[2]_{p,q}}{e_{p,q}(qt) + 1}\right) e_{p,q}(qxt) = pq \sum_{n=0}^{\infty} q^{\binom{n}{2}} p^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(x) \frac{t^n}{[n]_{p,q}!},$$

where

$$E_{\frac{1}{p},\frac{1}{q}}(qt) = e_{p,q}(qt), \ [2]_{\frac{1}{p},\frac{1}{q}} = \frac{[2]_{p,q}}{pq}, \ [n]_{\frac{1}{p},\frac{1}{q}}! = \frac{[n]_{p,q}!}{p^{\binom{n}{2}} q^{\binom{n}{2}}}$$

and

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \qquad \left(0 < \left|\frac{q}{p}\right| < 1; \ |x| < 1\right).$$

We note that, in the special case $p = 1$ in the definition of the big (p, q)-Appell polynomials, we are led to the big q-Appell polynomials given by

$$A_q(t) E_q\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,q}(x) \frac{t^n}{[n]_q!},$$

where

$$E_q(x) := \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1 - q) q^k x\right) \qquad \left(0 < |q| < 1; \ x \in \mathbb{C}\right)$$

and

$$A_q(t) = \sum_{n=0}^{\infty} a_{n,q} \frac{t^n}{[n]_q!} \qquad \left(a_{0,q} \neq 0\right).$$

The big q-Bernoulli polynomials are defined by

$$\left(\frac{t}{E_q\left(\frac{t}{q}\right)-1}\right)E_q\left(\frac{xt}{q}\right)=\sum_{n=0}^{\infty}B_{n,q}(x)\,\frac{t^n}{[n]_q!}$$

and the big q-Euler polynomials are defined by

$$\left(\frac{[2]_q}{E_q\left(\frac{t}{q}\right)+1}\right)E_q\left(\frac{xt}{q}\right)=\sum_{n=0}^{\infty}E_{n,q}(x)\,\frac{t^n}{[n]_q!}.$$

For a detailed analysis of the q-Appell polynomials and related q-polynomials, we refer the reader to [11], [12] and [20].

In the case when $p=1$ and $q\to 1-$, the big (p,q)-Appell polynomials reduce to the above-defined Appell polynomials $P_n(x)$, the big (p,q)-Bernoulli polynomials reduce to the Bernoulli polynomials $B_n(x)$ given by

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{n=0}^{\infty}B_n(x)\,\frac{t^n}{n!},$$

and the big (p,q)-Euler polynomials reduce to the Euler polynomials $E_n(x)$ given by

$$\left(\frac{2}{e^t+1}\right)e^{xt}=\sum_{n=0}^{\infty}E_n(x)\,\frac{t^n}{n!}.$$

It is important to state that the following q-Appell polynomials were introduced by Al-Salam (see [1] and [2]) and were subsequently investigated and characterized by Srivastava [20]:

$$a_q(t)e_q(xt)=\sum_{n=0}^{\infty}A_{n,q}(x)\,\frac{t^n}{[n]_q!}\qquad(0<q<1).$$

In the case when

$$a_q(t)=\frac{t}{e_q(t)-1}$$

and

$$a_q(t)=\frac{2}{e_q(t)+1},$$

we have the q-Bernoulli polynomials and the q-Euler polynomials given by (see [1])

$$\left(\frac{t}{e_q(t)-1}\right)e_q(xt)=\sum_{n=0}^{\infty}b_{n,q}(x)\,\frac{t^n}{[n]_q!}$$

and

$$\left(\frac{2}{e_q(t)+1}\right)e_q(xt)=\sum_{n=0}^{\infty}e_{n,q}(x)\,\frac{t^n}{[n]_q!},$$

respectively. In the above generating functions, the q-exponential function $e_q(x)$ is given by

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{\left(1-(1-q)q^k x\right)} \qquad \left(0 < \left|q\right| < 1;\ |x| < \frac{1}{1-q}\right).$$

For the big q-Appell polynomials and the q-Appell polynomials, we have

$$A_q(t)E_q\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,q}(x)\ \frac{t^n}{[n]_q!}$$

and

$$a_q(t)e_q(xt) = \sum_{n=0}^{\infty} A_{n,q}(x)\ \frac{t^n}{[n]_q!},$$

respectively. Thus, if we replace q by $\frac{1}{q}$ in the generating function of the big q-Appell polynomials and replace x by qx in the the generating function of the q-Appell polynomials, we find that

$$A_{\frac{1}{q}}(t)e_q(qxt) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} P_{n,\frac{1}{q}}(x)\ \frac{t^n}{[n]_q!}$$

and

$$a_q(t)e_q(qxt) = \sum_{n=0}^{\infty} A_{n,q}(qx)\ \frac{t^n}{[n]_q!}.$$

Taking

$$A_{\frac{1}{q}}(t) = a_q(t) = \sum_{n=0}^{\infty} a_{n,q}\ \frac{t^n}{[n]_q!},\ a_q(0) \neq 0,$$

we have

$$q^{\binom{n}{2}} P_{n,\frac{1}{q}}(x) = A_{n,q}(qx).$$

For the big q-Bernoulli polynomials and the usual q-Bernoulli polynomials, respectively, we have

$$\left(\frac{t}{e_q(qt)-1}\right)e_q(qxt) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} B_{n,\frac{1}{q}}(x)\ \frac{t^n}{[n]_q!}$$

and

$$\left(\frac{qt}{e_q(qt)-1}\right)e_q(qxt) = \sum_{n=0}^{\infty} b_{n,q}(x)\ \frac{q^n t^n}{[n]_q!}.$$

For the big q-Euler polynomials and the usual q-Euler polynomials, respectively, we have

$$\left(\frac{[2]_q}{e_q(qt)+1}\right)e_q(qxt) = q\sum_{n=0}^{\infty} q^{\binom{n}{2}} E_{n,\frac{1}{q}}(x)\ \frac{t^n}{[n]_q!}$$

and

$$\left(\frac{2}{e_q(qt)+1}\right)e_q(qxt)=\sum_{n=0}^{\infty} e_{n,q}(x)\ \frac{q^n t^n}{[n]_q!}.$$

Comparing both sides of the generating function of the big q-Bernoulli polynomials with that of the q-Bernoulli polynomials, and the generating function of the big q-Euler polynomials with that of the q-Euler polynomials, we have the following relations:

$$q^{\binom{n}{2}+1-n}\ B_{n,\frac{1}{q}}(x)=b_{n,q}(x)$$

and

$$\frac{2}{[2]_q}\ q^{\binom{n}{2}+1-n}E_{n,\frac{1}{q}}(x)=e_{n,q}(x),$$

respectively.

The difference equations satisfied by the q-Appell polynomials were obtained in [13]. Some relations satisfied by the generalized q-Bernoulli polynomials and the generalized q-Euler polynomials were obtained in [15]. Some relations satisfied by the q-extensions of the Apostol type polynomials were given in [14].

In an earlier work [10], the Apostol type (p,q)-Bernoulli polynomials of order α and the Apostol type (p,q)-Euler polynomials of order α were defined by

$$\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x,y;u;\lambda)\ \frac{t^n}{[n]_{p,q}!}=\left(\frac{t}{\lambda e_{p,q}(t)-1}\right)^{\alpha}\ e_{p,q}(xt)\ E_{p,q}(yt)$$

and

$$\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x,y;u;\lambda)\ \frac{t^n}{[n]_{p,q}!}=\left(\frac{[2]_{p,q}}{\lambda e_{p,q}(t)+1}\right)^{\alpha}\ e_{p,q}(xt)\ E_{p,q}(yt),$$

respectively. Moreover, the Apostol type (p,q)-Frobenius-Euler polynomials were introduced in [10] and some new identities satisfied by the Apostol type (p,q)-Frobenius-Euler polynomials were obtained in [10] (see also the recent works [7], [22] and [21]).

Our present investigation is motivated by the generating function, which was introduced in [18] and used in [16] for solving some symmetry identities and multiplication formulas as follows:

$$f_{a,b}(x;t;k,\beta):=\frac{2^{1-k}t^k e^{xt}}{\beta^b e^t-a^b}=\sum_{n=0}^{\infty}P_{n,\beta}(x;k,a,b)\ \frac{t^n}{n!}$$

$$\left(|t|<2\pi\ \text{ when }\ \beta=a;\ |t|<\left|\beta\log\left(\frac{b}{a}\right)\right|\ \text{ when }\ \beta\neq a;\right.$$

$$\left.\alpha,k\in\mathbb{N}_0;\ a,b\in\mathbb{R}\setminus\{0\};\ \beta\in\mathbb{C}\right).$$

In another work [4], the following unified (p,q)-analogs of the Apostol-Bernoulli polynomials, Apostol-Euler polynomials and the Apostol-Genocchi polynomials of order α were defined:

$$\begin{aligned}\mathcal{Y}_{a,b}^{(\alpha)}(x,y;z;k,\beta;p,q)&=\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b;p,q)\ \frac{z^n}{[n]_{p,q}!}\\&=\left(\frac{2^{1-k}\ z^k}{\beta^b e_{p,q}(z)-a^b}\right)^{\alpha}\ e_{p,q}(xz)\ E_{p,q}(yz)\end{aligned}$$

$$\left(|z| < 2\pi \ \text{ when } \ \beta = a; \ |z| < \left|\beta \log\left(\frac{b}{a}\right)\right| \ \text{ when } \ \beta \neq a; \ \alpha, k \in \mathbb{N}_0; \ a, b \in \mathbb{R} \setminus \{0\}; \ \beta \in \mathbb{C}\right).$$

In this paper, by using the theory of the (p, q)-calculus, we obtain recurrence relations and difference equations satisfied by the big (p, q)-Appell polynomials. In the special cases, we obtain the corresponding recurrence relations and difference equations satisfied by the big (p, q)-Bernoulli polynomials and the big (p, q)-Euler polynomials. Since the big (p, q)-Appell polynomials reduce to the Appell polynomials in the case when $p = 1$ and $q \to 1-$, the recurrence relations and difference equations satisfied by the big (p, q)-Appell polynomials coincide with the corresponding recurrence relations and differential equations satisfied by the Appell polynomials.

The organization of this paper is as follows. In Section 2, we introduce the big (p, q)-Appell polynomials. We prove an equivalence theorem satisfied by the big (p, q)-Appell polynomials and obtain recurrence relations and difference equations satisfied by the big (p, q)-Appell polynomials. Upon specializing the parameters p and q, the equivalence theorem, recurrence relations and difference equations satisfied by the big (p, q)-Appell polynomials are shown to reduce to the corresponding equivalence theorem, recurrence relations and differential equations satisfied by the Appell polynomials. In Section 3, we introduce the big (p, q)-Bernoulli polynomials and obtain some recurrence relations and difference equations satisfied by the big (p, q)-Bernoulli polynomials and specialize our results to deduce the corresponding recurrence relations and difference (or differential) equations for the big q-Bernoulli polynomials as well as the Bernoulli polynomials. In Section 4, we obtain some properties of the big (p, q)-exponential functions. In Section 5, we introduce the big (p, q)-Euler polynomials and obtain some properties of the big (p, q)-Euler polynomials. In particular, we obtain some recurrence relations and difference equations satisfied by the big (p, q)-Euler polynomials and consider their special cases as in Section 2. Finally, in Section 6, we present some concluding remarks and observations and we also point out some obvious connections between the (p, q)-analysis and the classical q-analysis, exhibiting the fact that the additional parameter p is redundant.

2. The Big (p, q)-Appell Polynomials

In this Section, we introduce the big (p, q)-Appell polynomials and prove an equivalence theorem satisfied by the big (p, q)-Appell polynomials. We also obtain recurrence relations and difference equations satisfied by the big (p, q)-Appell polynomials and consider their special cases.

Definition 1. A polynomial sequence $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ is said to be a big (p, q)-Appell sequence if it satisfies the following property:

$$\left(D_{p,q,x}(P_{n,p,q})\right)(x) = \frac{[n]_{p,q}}{q} \, P_{n-1,p,q}(qx). \tag{3}$$

Theorem 1. *The following statements are all equivalent to one another*:

(i) *Let* $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ *be a big* (p, q)*-Appell sequence defined by* (3).

(ii) *The big* (p, q)*-Appell sequence* $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ *possesses an explicit form given by*

$$P_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,p,q} \, q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k. \tag{4}$$

(iii) *The big* (p, q)*-Appell sequence* $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ *has a generating function*

$$A_{p,q}(t)E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,p,q}(x) \, \frac{t^n}{[n]_{p,q}!} \tag{5}$$

where

$$A_{p,q}(t) = \sum_{k=0}^{\infty} a_{k,p,q} \, \frac{t^k}{[k]_{p,q}!}. \tag{6}$$

Proof. In order to prove the assertion (i) ⇒ (ii), we let $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ be a big (p,q)-Appell sequence. Then we can write

$$P_{n,p,q}(x) = \sum_{k=0}^{n} a_{n,k,p,q}\,[x]_k^{p,q}, \tag{7}$$

where

$$[x]_k^{p,q} = x\left(\frac{q}{p}\,x\right)\left(\frac{q^2}{p^2}\,x\right)\cdots\left(\frac{q^{k-1}}{p^{k-1}}\,x\right) = \frac{q^{\binom{k}{2}}}{p^{\binom{k}{2}}}\,x^k. \tag{8}$$

Applying the (p,q)-derivative operator $D_{p,q,x}$ on both sides of (7) and using the fact that

$$D_{p,q,x}[x]_k^{p,q} = [k]_{p,q}\frac{q^{\binom{k}{2}}}{p^{\binom{k}{2}}}x^{k-1}, \tag{9}$$

we get

$$\frac{[n]_{p,q}}{q}\,P_{n-1,p,q}(qx) = \sum_{k=1}^{n} a_{n,k,p,q}\,\frac{q^{\binom{k}{2}}}{p^{\binom{k}{2}}}\,x^{k-1}[k]_{p,q}.$$

We also have

$$P_{n-1,p,q}(qx) = \frac{q}{[n]_{p,q}}\sum_{k=1}^{n} a_{n,k,p,q}\,\frac{q^{\binom{k}{2}}}{p^{\binom{k}{2}}}x^{k-1}[k]_{p,q}, \tag{10}$$

which, upon replacing k by $k+1$ in (10), yields

$$P_{n-1,p,q}(qx) = \frac{q}{[n]_{p,q}}\sum_{k=0}^{n-1} a_{n,k+1,p,q}\,\frac{q^{\binom{k+1}{2}}}{p^{\binom{k+1}{2}}}x^k[k+1]_{p,q}. \tag{11}$$

Replacing n by $n+1$ and replacing qx by x in (11), we get

$$P_{n,p,q}(x) = \frac{q}{[n+1]_{p,q}}\sum_{k=0}^{n} a_{n+1,k+1,p,q}\,\frac{q^{\binom{k+1}{2}}}{p^{\binom{k+1}{2}}}\left(\frac{x}{q}\right)^k[k+1]_{p,q}. \tag{12}$$

Comparing (7) and (12), we find that

$$a_{n,k,p,q} = \frac{p^{k-1}}{q}\,\frac{[n]_{p,q}}{[k]_{p,q}}\,a_{n-1,k-1,p,q}. \tag{13}$$

Iterating the equation (13) k times, we obtain

$$a_{n,k,p,q} = \frac{p^{\binom{k}{2}}}{q^k}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,0,p,q}. \tag{14}$$

Upon setting

$$a_{n-k,0,p,q} = a_{n-k,p,q}$$

and inserting $a_{n,k,p,q}$ into the equation (7), we get

$$P_{n,p,q}(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,p,q}\,q^{\binom{k}{2}}\left(\frac{x}{q}\right)^k.$$

For proving the assertion (ii) ⇒ (iii), we let

$$P_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,p,q}\, q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k.$$

Upon summing both sides from $n = 0$ to $n = \infty$ and taking $\frac{t^n}{[n]_{p,q}!}$, we have

$$\sum_{n=0}^{\infty} P_{n,p,q}(x)\, \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k,p,q}\, q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k \frac{t^n}{[n]_{p,q}!}. \tag{15}$$

Applying the Cauchy product in (15), we get

$$\begin{aligned}\sum_{n=0}^{\infty} P_{n,p,q}(x)\, \frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q} a_{n,p,q}\, q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k \frac{t^{n+k}}{[n+k]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{a_{n,p,q}}{[n]_{p,q}!}\, \frac{q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k}{[k]_{p,q}!}\, t^{n+k} \\ &= \sum_{n=0}^{\infty} a_{n,p,q}\, \frac{t^n}{[n]_{p,q}!} \sum_{k=0}^{\infty} q^{\binom{k}{2}}\, \frac{\left(\frac{xt}{q}\right)^k}{[k]_{p,q}!}\end{aligned}$$

which, in view of the expansion in (6) and the series expression for the big (p,q)-exponential function, yields

$$\sum_{n=0}^{\infty} P_{n,p,q}(x)\, \frac{t^n}{[n]_{p,q}!} = A_{p,q}(t) E_{p,q}\left(\frac{xt}{q}\right).$$

Finally, in order to demonstrate the assertion (iii) ⇒ (i), we let $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ have the following generating function:

$$A_{p,q}(t) E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,p,q}(x)\, \frac{t^n}{[n]_{p,q}!}.$$

Then, by applying (p,q)-derivative operator with respect to x on both sides of (5), we find that

$$A_{p,q}(t) D_{p,q,x}\Big(E_{p,q}\Big(\frac{xt}{q}\Big)\Big) = \sum_{n=0}^{\infty} D_{p,q,x}\Big(P_{n,p,q}(x)\Big)\, \frac{t^n}{[n]_{p,q}!}. \tag{16}$$

Using the fact that

$$D_{p,q,x}\Big(E_{p,q}\Big(\frac{xt}{q}\Big)\Big) = \frac{t}{q}\, E_{p,q}(xt), \tag{17}$$

in (16), we get

$$\frac{t}{q} A_{p,q}(t) E_{p,q}(xt) = \sum_{n=0}^{\infty} D_{p,q,x}\Big(P_{n,p,q}(x)\Big) \frac{t^n}{[n]_{p,q}!}. \tag{18}$$

Inserting the corresponding series in (18), we obtain

$$\frac{1}{q} \sum_{n=0}^{\infty} P_{n,p,q}(qx)\, \frac{t^{n+1}}{[n]_{p,q}!} = \sum_{n=0}^{\infty} D_{p,q,x}\Big(P_{n,p,q}(x)\Big)\, \frac{t^n}{[n]_{p,q}!}. \tag{19}$$

Shifting the series in (19), we get

$$\frac{1}{q}\sum_{n=0}^{\infty}[n]_{p,q}\, P_{n-1,p,q}(qx)\, \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} D_{p,q,x}\Big(P_{n,p,q}(x)\Big)\, \frac{t^n}{[n]_{p,q}!}. \tag{20}$$

Upon equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$ in (20), we get

$$D_{p,q,x}\Big(P_{n,p,q}(x)\Big) = \frac{[n]_{p,q}}{q}\, P_{n-1,p,q}(qx),$$

which shows that $\{P_{n,p,q}(x)\}_{n\in\mathbb{N}}$ is a big (p,q)-Appell sequence. □

Definition 2. A polynomial sequence $\{P_{n,q}(x)\}_{n\in\mathbb{N}}$ is said to be a big q-Appell sequence if it satisfies the following property:

$$D_{q,x}\Big(P_{n,q}(x)\Big) = \frac{[n]_q}{q}\, P_{n-1,q}(qx). \tag{21}$$

Taking $p = 1$ in Theorem 1, we get the following corollary.

Corollary 1. *The following statements are all equivalent.*

(i) *Let $\{P_{n,q}(x)\}_{n\in\mathbb{N}}$ be a big q-Appell sequence.*

(ii) *$\{P_{n,q}(x)\}_{n\in\mathbb{N}}$ has an explicit form given by*

$$P_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a_{n-k,q}\, q^{\binom{k}{2}} \left(\frac{x}{q}\right)^k. \tag{22}$$

(iii) *$\{P_{n,q}(x)\}_{n\in\mathbb{N}}$ has a generating function given by*

$$A_q(t)E_q\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} P_{n,q}(x)\, \frac{t^n}{[n]_q!}, \tag{23}$$

where

$$A_q(t) = \sum_{k=0}^{\infty} a_{k,q}\, \frac{t^k}{[k]_q!}. \tag{24}$$

Definition 3. A polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ is said to be an Appell sequence if it satisfies the following property:

$$\frac{d}{dx}\Big(P_n(x)\Big) = n\, P_{n-1}(x). \tag{25}$$

Corollary 2. *The following statements are all equivalent.*

(i) *Let $\{P_n(x)\}_{n\in\mathbb{N}}$ be an Appell sequence.*

(ii) *The Appell sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ has an explicit form given by*

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\, x^k. \tag{26}$$

(iii) *The Appell sequence* $\{P_n(x)\}_{n\in\mathbb{N}}$ *has a generating function given by*

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\,\frac{t^n}{n!}, \tag{27}$$

where

$$A(t) = \sum_{k=0}^{\infty} a_k\,\frac{t^k}{k!}. \tag{28}$$

Theorem 2. *The recurrence relation satisfied by the big* (p,q)*-Appell polynomials is given by*

$$\frac{xp^n}{q}\,P_{n,p,q}\left(\frac{q}{p}\,x\right) + P_{n,p,q}(qx)\alpha_{0,p,q} + \sum_{k=0}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \alpha_{n-k,p,q}P_{k,p,q}(qx) = P_{n+1,p,q}(x). \tag{29}$$

Proof. Differentiating both sides of the generating function of the big (p,q)-Appell polynomials with respect to t, we have

$$\begin{aligned} &A_{p,q}(pt)D_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right) + A_{p,q}(t)E_{p,q}(xt)\,\frac{D_{p,q,t}\big(A_{p,q}(t)\big)}{A_{p,q}(t)} \\ &\qquad = \sum_{n=0}^{\infty} P_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}. \end{aligned} \tag{30}$$

In the equation (30), we need to compute the (p,q)-derivative

$$D_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right)$$

by first expanding the (p,q)-exponential function $E_{p,q}\left(\frac{xt}{q}\right)$ and then evaluating the (p,q)-derivative with respect to t. We thus obtain

$$\begin{aligned} D_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right) &= D_{p,q,t}\left(\sum_{n=0}^{\infty} q^{\binom{n}{2}}\,\frac{\left(\frac{xt}{q}\right)^n}{[n]_{p,q}!}\right) \\ &= \sum_{n=0}^{\infty} q^{\binom{n}{2}}\,q^{-n}\,x^n\,\frac{D_{p,q,t}(t^n)}{[n]_{p,q}!}. \end{aligned}$$

Upon inserting the following expression:

$$D_{p,q,t}(t^n) = t^{n-1}[n]_{p,q}$$

and after some series manipulations, we find that

$$D_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right) = \frac{x}{q}E_{p,q}(xt). \tag{31}$$

Now, using (31) in (30), we have

$$\frac{x}{q}\,A_{p,q}(pt)E_{p,q}(xt) + A_{p,q}(t)E_{p,q}(xt)\,\frac{D_{p,q,t}\big(A_{p,q}(t)\big)}{A_{p,q}(t)} = \sum_{n=0}^{\infty} P_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}. \tag{32}$$

Now, if we define

$$\frac{D_{p,q,t}\big(A_{p,q}(t)\big)}{A_{p,q}(t)} := \sum_{n=0}^{\infty} \alpha_{n,p,q} \ \frac{t^n}{[n]_{p,q}!} \tag{33}$$

and consider the generating function of the big (p, q)-Appell polynomials in (32), we find that

$$\frac{x}{q} \sum_{n=0}^{\infty} P_{n,p,q}\left(\frac{q}{p}\ x\right) \ \frac{(pt)^n}{[n]_{p,q}!} + \sum_{k=0}^{\infty} P_{k,p,q}(qx) \ \frac{t^k}{[k]_{p,q}!} \sum_{n=0}^{\infty} \alpha_{n,p,q} \ \frac{t^n}{[n]_{p,q}!}$$
$$= \sum_{n=0}^{\infty} P_{n+1,p,q}(x) \ \frac{t^n}{[n]_{p,q}!}.$$

Applying the Cauchy product, we get

$$\frac{x}{q} \sum_{n=0}^{\infty} P_{n,p,q}\left(\frac{q}{p}\ x\right) \ \frac{p^n t^n}{[n]_{p,q}!} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} P_{k,p,q}(qx)\alpha_{n-k,p,q} \ \frac{t^n}{[n]_{p,q}!}$$
$$= \sum_{n=0}^{\infty} P_{n+1,p,q}(x) \ \frac{t^n}{[n]_{p,q}!}. \tag{34}$$

Equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$ in (34), we have

$$\frac{xp^n}{q} P_{n,p,q}\left(\frac{q}{p}\ x\right) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} P_{k,p,q}(qx)\alpha_{n-k,p,q} = P_{n+1,p,q}(x). \tag{35}$$

Equation (35) can be written as follows:

$$\frac{xp^n}{q} \ P_{n,p,q}\left(\frac{q}{p}\ x\right) + P_{n,p,q}(qx)\alpha_{0,p,q} + \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \alpha_{n-k,p,q} P_{k,p,q}(qx)$$
$$= P_{n+1,p,q}(x).$$

□

Upon setting

$$p = 1 \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{1,q} =: \begin{bmatrix} n \\ k \end{bmatrix}_{q}$$

in Theorem 2, we have the following corollary.

Corollary 3. *A recurrence relation satisfied by the big q-Appell polynomials is given by*

$$\left(\frac{x}{q} + \alpha_{0,q}\right) P_{n,q}(qx) + \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{q} \alpha_{n-k,q} \ P_{k,q}(qx) = P_{n+1,q}(x). \tag{36}$$

By setting

$$p = 1, \quad q \to 1-, \quad \alpha_{0,1,1} =: \alpha_0 \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{1,1} = \binom{n}{k}$$

in Theorem 2, we have the following corollary.

Corollary 4. (see [6]) *A recurrence relation satisfied by the Appell polynomials is given by*

$$(x+\alpha_0)P_n(x)+\sum_{k=0}^{n-1}\binom{n}{k}\alpha_{n-k}P_k(x)=P_{n+1}(x). \tag{37}$$

Theorem 3. *The difference equation satisfied by the big (p,q)-Appell polynomials is given by*

$$\begin{aligned}&\frac{xp^{n+1}}{q}D_{p,q,x}\left(P_{n,p,q}\Big(\frac{q}{p}\,x\Big)\right)+\frac{p^n}{q}\,P_{n,p,q}\left(\frac{q^2}{p}\,x\right)+\alpha_{0,p,q}D_{p,q,x}\Big(P_{n,p,q}(qx)\Big)\\&\qquad+\sum_{k=0}^{n-1}\frac{\alpha_{n-k,p,q}}{[n-k]_{p,q}!}\,q^{n-k}D_{p,q,x}^{n-k+1}\Big(P_{n,p,q}(x)\Big)=\frac{[n+1]_{p,q}}{q}\,P_{n,p,q}(qx).\end{aligned} \tag{38}$$

Proof. To obtain the difference equation (38) satisfied by the big (p,q)-Appell polynomials, we need to find the derivative operator. Indeed, for the big (p,q)-Appell property, we have

$$P_{n-1,p,q}(qx)=\frac{q}{[n]_{p,q}}\,D_{p,q,x}\Big(P_{n,p,q}(x)\Big).$$

Hence, clearly, the derivative operator is given by

$$L^-_{n,p,q}:=\frac{q}{[n]_{p,q}}D_{p,q,x}. \tag{39}$$

Now, using the derivative operator (39) in the recurrence relation satisfied by the big (p,q)-Appell polynomials, $P_{k,p,q}(qx)$ can be written as follows:

$$\begin{aligned}P_{k,p,q}(qx)&=[L^-_{k+1,p,q}L^-_{k+2,p,q}\cdots L^-_{n,p,q}]P_{n,p,q}(x)\\&=\left[\frac{q}{[k+1]_{p,q}}D_{p,q,x}\;\frac{q}{[k+2]_{p,q}}D_{p,q,x}\cdots\frac{q}{[n]_{p,q}}D_{p,q,x}\right]P_{n,p,q}(x)\\&=q^{n-k}\,\frac{[k]_{p,q}!}{[n]_{p,q}!}\,D_{p,q,x}^{n-k}\Big(P_{n,p,q}(x)\Big).\end{aligned} \tag{40}$$

Substituting from (40) into the recurrence relation in (29), we have

$$\begin{aligned}&\frac{xp^n}{q}\,P_{n,p,q}\left(\frac{q}{p}\,x\right)+P_{n,p,q}(qx)\alpha_{0,p,q}+\sum_{k=0}^{n-1}\frac{\alpha_{n-k,p,q}}{[n-k]_{p,q}!}\,q^{n-k}\,D_{p,q,x}^{n-k}\Big(P_{n,p,q}(x)\Big)\\&\quad=P_{n+1,p,q}(x).\end{aligned} \tag{41}$$

Applying the (p,q)-derivative operator $D_{p,q}$ with respect to x on both sides of (41) and using the product rule (1) and the fact that

$$D_{p,q,x}\Big(P_{n,p,q}(x)\Big)=\frac{[n]_{p,q}}{q}\,P_{n-1,p,q}(qx),$$

we get

$$\begin{aligned}&\frac{xp^{n+1}}{q}D_{p,q,x}\left(P_{n,p,q}\Big(\frac{q}{p}\,x\Big)\right)+P_{n,p,q}\left(\frac{q^2}{p}\,x\right)D_{p,q,x}\left(\frac{p^n}{q}\,x\right)+\alpha_{0,p,q}\,D_{p,q,x}\Big(P_{n,p,q}(qx)\Big)\\&\qquad+\sum_{k=0}^{n-1}\frac{\alpha_{n-k,p,q}}{[n-k]_{p,q}!}\,q^{n-k}\,D_{p,q,x}^{n-k+1}\Big(P_{n,p,q}(x)\Big)\\&\quad=\frac{[n+1]_{p,q}}{q}\,P_{n,p,q}(qx).\end{aligned} \tag{42}$$

Inserting the following expression:

$$D_{p,q,x}\left(\frac{p^n}{q}\,x\right)=\frac{p^n}{q} \tag{43}$$

in (42), we get

$$\begin{aligned}
&\frac{xp^{n+1}}{q}D_{p,q,x}\left(P_{n,p,q}\Big(\frac{q}{p}\,x\Big)\right)+\frac{p^n}{q}\,P_{n,p,q}\left(\frac{q^2}{p}\,x\right)+\alpha_{0,p,q}\,D_{p,q,x}\Big(P_{n,p,q}(qx)\Big)\\
&\qquad+\sum_{k=0}^{n-1}\frac{\alpha_{n-k,p,q}}{[n-k]_{p,q}!}q^{n-k}\,D_{p,q,x}^{n-k+1}\Big(P_{n,p,q}(x)\Big)\\
&\quad=\frac{[n+1]_{p,q}}{q}P_{n,p,q}(qx).
\end{aligned}$$

□

Taking $p=1$ and

$$\alpha_{0,1,q}=:\alpha_{0,q},\quad [n-k]_{1,q}!=[n-k]_q!\qquad\text{and}\qquad [n+1]_{1,q}=[n+1]_q$$

in Theorem 3, we have the following corollary.

Corollary 5. *The difference equation satisfied by the big q-Appell polynomials is given by*

$$\begin{aligned}
&\frac{x}{q}D_{q,x}\Big(P_{n,q}(qx)\Big)+\frac{1}{q}\,P_{n,q}(q^2x)+\alpha_{0,q}D_{q,x}\Big(P_{n,q}(qx)\Big)\\
&\qquad+\sum_{k=0}^{n-1}\frac{\alpha_{n-k,q}}{[n-k]_q!}q^{n-k}\,D_{q,x}^{n-k+1}\Big(P_{n,q}(x)\Big)=\frac{[n+1]_q}{q}\,P_{n,q}(qx),
\end{aligned} \tag{44}$$

where

$$\frac{d}{dx}=:D_x.$$

Taking $p=1$ and $q\to 1-$ in Theorem 3 and inserting

$$\alpha_{0,1,1}=:\alpha_0,\quad [n-k]_{1,1}!=(n-k)!\qquad\text{and}\qquad [n+1]_{1,1}=(n+1),$$

we have the following corollary.

Corollary 6. (see [6]) *The differential equation satisfied by the Appell polynomials is given by*

$$(x+\alpha_0)\frac{d}{dx}\Big(P_n(x)\Big)+\sum_{k=0}^{n-1}\frac{\alpha_{n-k}}{(n-k)!}\left(\frac{d}{dx}\right)^{n-k+1}\Big(P_n(x)\Big)-nP_n(x)=0. \tag{45}$$

3. The Big (p,q)-Bernoulli Polynomials

In this Section, we introduce the big (p,q)-Bernoulli polynomials. We obtain the recurrence relation and the difference equation satisfied by the big (p,q)-Bernoulli polynomials. We also deduce the corresponding results in the special cases when $p=1$ as well as when $p=1$ and $q\to 1-$.

Definition 4. The big (p,q)-Bernoulli polynomials are defined by

$$\left(\frac{t}{E_{p,q}\left(\frac{t}{q}\right)-1}\right)E_{p,q}\left(\frac{xt}{q}\right)=\sum_{n=0}^{\infty}B_{n,p,q}(x)\,\frac{t^n}{[n]_{p,q}!},\tag{46}$$

where the big (p,q)-Bernoulli numbers $B_{k,p,q}:=B_{k,p,q}(0)$ are given by

$$\frac{t}{E_{p,q}\left(\frac{t}{q}\right)-1}=\sum_{k=0}^{\infty}B_{k,p,q}\,\frac{t^k}{[k]_{p,q}!}.\tag{47}$$

Some of the big (p,q)-Bernoulli numbers are given by

$$\begin{aligned}
&B_{0,p,q}=q,\ B_{1,p,q}=-\frac{q}{[2]_{p,q}},\ \ B_{2,p,q}=\frac{q}{[2]_{p,q}}-\frac{q^2}{[3]_{p,q}},\\
&B_{3,p,q}=-\frac{q^4}{[4]_{p,q}}+\frac{2}{[2]_{p,q}}q^2-q\frac{[3]_{p,q}}{[2]_{p,q}^2},\\
&B_{4,p,q}=-\frac{q^7}{[5]_{p,q}}+\frac{q^4}{[2]_{p,q}}-q^2\frac{[4]_{p,q}}{[2]_{p,q}^2}+\frac{[4]_{p,q}}{[2]_{p,q}[3]_{p,q}}q^3.
\end{aligned}\tag{48}$$

Definition 5. The big q-Bernoulli polynomials are defined by

$$\left(\frac{t}{E_{q}\left(\frac{t}{q}\right)-1}\right)E_{q}\left(\frac{xt}{q}\right)=\sum_{n=0}^{\infty}B_{n,q}(x)\,\frac{t^n}{[n]_{q}!},\tag{49}$$

where the big q-Bernoulli numbers $B_{k,q}:=B_{k,q}(0)$ are given by

$$\frac{t}{E_{q}\left(\frac{t}{q}\right)-1}=\sum_{k=0}^{\infty}B_{k,q}\,\frac{t^k}{[k]_{q}!}.\tag{50}$$

In the case when $p=1$ in (48), we have some big q-Bernoulli numbers given by

$$\begin{aligned}
&B_{0,q}=q,\ B_{1,q}=-\frac{q}{[2]_{q}},\ \ B_{2,q}=\frac{q}{[2]_{q}}-\frac{q^2}{[3]_{q}},\\
&B_{3,q}=-\frac{q^4}{[4]_{q}}+\frac{2}{[2]_{q}}q^2-q\frac{[3]_{q}}{[2]_{q}^2},\\
&B_{4,q}=-\frac{q^7}{[5]_{q}}+\frac{q^4}{[2]_{q}}-q^2\frac{[4]_{q}}{[2]_{q}^2}+\frac{[4]_{q}}{[2]_{q}[3]_{q}}q^3.
\end{aligned}\tag{51}$$

In the case when $p=1$ and $q\to 1-$, we have the Bernoulli numbers given by

$$B_0=1,\ \ B_1=-\frac{1}{2},\ \ B_2=\frac{1}{6},\ \ B_3=0,\ \ B_4=-\frac{1}{30},\ \cdots.\tag{52}$$

Theorem 4. *A recurrence relation satisfied by the big (p,q)-Bernoulli polynomials is given by*

$$\begin{aligned}
&p^{n-1}\left(\frac{px}{q}-1+\frac{q}{p+q}\right)B_{n,p,q}\left(\frac{q}{p}\,x\right)\\
&\qquad-\frac{1}{[n+1]_{p,q}}\sum_{k=0}^{n-1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}B_{k,p,q}\left(\frac{q}{p}\,x\right)p^{k-1}\,q^{n-k}\,B_{n+1-k,p,q}\\
&\quad=B_{n+1,p,q}(x).
\end{aligned}\tag{53}$$

Proof. Taking the (p,q)-derivative with respect to t on both sides of the generating function of the big (p,q)-Bernoulli polynomials and using the rules in (1) and (2), we have

$$\begin{aligned}&\left(\frac{pt}{E_{p,q}\left(\frac{pt}{q}\right)-1}\right)D_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right)\\&\qquad+E_{p,q}(xt)\left(\frac{\left(E_{p,q}(t)-1\right)D_{p,q,t}(t)-qtD_{p,q,t}\left(E_{p,q}\left(\frac{t}{q}\right)-1\right)}{\left(E_{p,q}\left(\frac{pt}{q}\right)-1\right)\left(E_{p,q}(t)-1\right)}\right)\\&=\sum_{n=0}^{\infty}B_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.\end{aligned}\tag{54}$$

Using the fact that

$$D_{p,q,t}\left(E_{p,q}\left(\frac{xt}{q}\right)\right)=\frac{x}{q}\,E_{p,q}(xt),$$

$$D_{p,q,t}(t)=1\qquad\text{and}\qquad D_{p,q,t}\left(E_{p,q}\left(\frac{t}{q}\right)-1\right)=\frac{1}{q}\,E_{p,q}(t)\tag{55}$$

in (54), we have

$$\begin{aligned}&\frac{x}{q}\left(\frac{pt}{E_{p,q}\left(\frac{pt}{q}\right)-1}\right)E_{p,q}(xt)+E_{p,q}(xt)\left(\frac{\left(E_{p,q}(t)-1\right)-tE_{p,q}(t)}{\left(E_{p,q}\left(\frac{pt}{q}\right)-1\right)\left(E_{p,q}(t)-1\right)}\right)\\&\qquad=\sum_{n=0}^{\infty}B_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.\end{aligned}\tag{56}$$

Equation (56) can be written as follows:

$$\begin{aligned}&\frac{x}{q}\left(\frac{pt}{E_{p,q}\left(\frac{pt}{q}\right)-1}\right)E_{p,q}(xt)+\frac{1}{E_{p,q}\left(\frac{pt}{q}\right)-1}E_{p,q}(xt)\\&\qquad-t\left(\frac{1}{E_{p,q}(\frac{pt}{q})-1}\,E_{p,q}(xt)+\frac{1}{E_{p,q}(\frac{pt}{q})-1}E_{p,q}(xt)\,\frac{1}{E_{p,q}(t)-1}\right)\\&\qquad=\sum_{n=0}^{\infty}B_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!},\end{aligned}$$

which, upon inserting the corresponding series, yields

$$\begin{aligned}&\frac{x}{q}\sum_{n=0}^{\infty}B_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{(pt)^n}{[n]_{p,q}!}+\frac{1}{pt}\sum_{n=0}^{\infty}B_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{(pt)^n}{[n]_{p,q}!}-\frac{1}{p}\sum_{n=0}^{\infty}B_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{(pt)^n}{[n]_{p,q}!}\\&\qquad-\frac{1}{p}\sum_{n=0}^{\infty}B_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{(pt)^n}{[n]_{p,q}!}\,\frac{1}{qt}\sum_{k=0}^{\infty}B_{k,p,q}\,\frac{(qt)^k}{[k]_{p,q}!}\\&=\sum_{n=0}^{\infty}B_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

Applying some series manipulations and the Cauchy product, we get

$$\begin{aligned}&\frac{x}{q}\sum_{n=0}^{\infty}B_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{p^nt^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}p^n\,\frac{B_{n+1,p,q}\left(\frac{q}{p}\,x\right)}{[n+1]_{p,q}}\,\frac{t^n}{[n]_{p,q}!}\\&\qquad-\frac{1}{p}\sum_{n=0}^{\infty}p^n\,B_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{t^n}{[n]_{p,q}!}\\&\qquad-\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}B_{n+1-k,p,q}\left(\frac{q}{p}\,x\right)p^{n-k}\,q^{k-1}\,B_{k,p,q}\,\frac{t^n}{[n+1]_{p,q}!}\\&=\sum_{n=0}^{\infty}B_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.\end{aligned}\tag{57}$$

Equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$ in (57), we get

$$\begin{aligned}&\frac{x}{q}\,B_{n,p,q}\left(\frac{q}{p}\,x\right)p^n+p^n\,\frac{B_{n+1,p,q}\left(\frac{q}{p}\,x\right)}{[n+1]_{p,q}}-p^{n-1}\,B_{n,p,q}\left(\frac{q}{p}\,x\right)\\&\qquad-\frac{1}{[n+1]_{p,q}}\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}B_{n+1-k,p,q}\left(\frac{q}{p}\,x\right)p^{n-k}\,q^{k-1}\,B_{k,p,q}\\&=B_{n+1,p,q}(x).\end{aligned}$$

Rearranging the summation and separating some terms in the summation, we get

$$\begin{aligned}&\frac{x}{q}\,B_{n,p,q}\left(\frac{q}{p}\,x\right)p^n+p^n\,\frac{B_{n+1,p,q}\left(\frac{q}{p}\,x\right)}{[n+1]_{p,q}}-p^{n-1}\,B_{n,p,q}\left(\frac{q}{p}\,x\right)\\&\qquad-\frac{1}{[n+1]_{p,q}}\,B_{n+1,p,q}\left(\frac{q}{p}\,x\right)p^n\,q^{-1}\,B_{0,p,q}-B_{n,p,q}\left(\frac{q}{p}\,x\right)p^{n-1}\,B_{1,p,q}\\&\qquad-\frac{1}{[n+1]_{p,q}}\sum_{k=0}^{n-1}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}B_{k,p,q}\left(\frac{q}{p}\,x\right)p^{k-1}\,q^{n-k}\,B_{n+1-k,p,q}\\&=B_{n+1,p,q}(x).\end{aligned}\tag{58}$$

Upon collecting the like terms, if we rearrange some terms in the summation and set

$$B_{0,p,q}=q,\quad B_{1,p,q}=-\frac{q}{p+q}\qquad\text{and}\qquad[1]_{p,q}=1$$

in (58), we get

$$
\begin{aligned}
&p^{n-1}\left(\frac{px}{q}-1+\frac{q}{p+q}\right)B_{n,p,q}\left(\frac{q}{p}\,x\right)\\
&\qquad -\frac{1}{[n+1]_{p,q}}\sum_{k=0}^{n-1}\begin{bmatrix} n+1\\ k\end{bmatrix}_{p,q} B_{k,p,q}\left(\frac{q}{p}\,x\right)p^{k-1}\,q^{n-k}\,B_{n+1-k,p,q}\\
&\quad = B_{n+1,p,q}(x).
\end{aligned}
$$

□

Setting $p = 1$ in Theorem 4, we have the following corollary.

Corollary 7. *The recurrence relation satisfied by the big q-Bernoulli polynomials is given by*

$$
\begin{aligned}
&\left(\frac{x}{q}-1+\frac{q}{1+q}\right)B_{n,q}(qx)-\frac{1}{[n+1]_q}\sum_{k=0}^{n-1}\begin{bmatrix} n+1\\ k\end{bmatrix}_{q} B_{k,q}(qx)\;q^{n-k}\;B_{n-k+1,q}\\
&\quad = B_{n+1,q}(x).
\end{aligned}
\tag{59}
$$

Taking $p = 1$ and $q \to 1-$ in Theorem 4, we have the following corollary.

Corollary 8. (see [6]) *The recurrence relation satisfied by the Bernoulli polynomials is given by*

$$
\left(x-\frac{1}{2}\right)B_n(x)-\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_{n-k+1}B_k(x)=B_{n+1}(x).
\tag{60}
$$

Theorem 5. *The difference equation satisfied by the big (p,q)-Bernoulli polynomials is given by*

$$
\begin{aligned}
&\left(\frac{x}{q}p^{n+1}-p^{n-1}+\frac{q}{p+q}p^{n-1}\right)D_{p,q,x}\left(B_{n,p,q}\left(\frac{q}{p}\,x\right)\right)+\frac{p^n}{q}\;B_{n,p,q}\left(\frac{q^2x}{p}\right)\\
&\qquad -\sum_{k=1}^{n-1}\frac{B_{n-k+1,p,q}}{[n-k+1]_{p,q}!}\;q^{2(n-k)}\;p^{k-1}\;D_{p,q,x}^{n-k+1}\left(B_{n,p,q}\left(\frac{x}{p}\right)\right)\\
&\quad = \frac{[n+1]_{p,q}}{q}B_{n,p,q}(qx).
\end{aligned}
\tag{61}
$$

Proof. In the recurrence relation satisfied by the big (p,q)-Bernoulli polynomials, we make use of the derivative operator:

$$
L^{-}_{n,p,q} := \frac{q}{[n]_{p,q}}D_{p,q,x},
$$

so that the term $B_{k,p,q}\left(\frac{q}{p}x\right)$ can be written as follows:

$$
\begin{aligned}
B_{k,p,q}\left(\frac{q}{p}x\right) &= [L^{-}_{k+1,p,q}L^{-}_{k+2,p,q}\cdots L^{-}_{n,p,q}]B_{n,p,q}\left(\frac{x}{p}\right)\\
&= \left[\frac{q}{[k+1]_{p,q}}D_{p,q,x}\frac{q}{[k+2]_{p,q}}D_{p,q,x}\cdots\frac{q}{[n]_{p,q}}D_{p,q,x}\right]B_{n,p,q}\left(\frac{x}{p}\right)\\
&= q^{n-k}\frac{[k]_{p,q}!}{[n]_{p,q}!}D^{n-k}_{p,q,x}\left(B_{n,p,q}\left(\frac{x}{p}\right)\right).
\end{aligned}
\tag{62}
$$

Inserting the term $B_{k,p,q}\left(\frac{q}{p}\ x\right)$ into the recurrence relation in (53), we get

$$\begin{aligned}
&p^{n-1}\left(\frac{px}{q}-1+\frac{q}{p+q}\right)B_{n,p,q}\left(\frac{q}{p}\ x\right)\\
&\qquad-\sum_{k=0}^{n-1}\frac{B_{n-k+1,p,q}}{[n-k+1]_{p,q}!}\ q^{2(n-k)}\ p^{k-1}\ D_{p,q,x}^{n-k}\Big(B_{n,p,q}\Big(\frac{x}{p}\Big)\Big)\\
&=B_{n+1,p,q}(x).
\end{aligned}\tag{63}$$

Taking the (p,q)-derivatives of both sides of (63) with respect to x, we get

$$\begin{aligned}
&\left(\frac{x}{q}p^{n+1}-p^{n-1}+\frac{q}{p+q}p^{n-1}\right)D_{p,q,x}\Big(B_{n,p,q}\Big(\frac{q}{p}\ x\Big)\Big)\\
&\qquad+B_{n,p,q}\left(\frac{q^2x}{p}\right)D_{p,q,x}\left(\frac{x}{q}p^{n}-p^{n-1}+\frac{q}{p+q}p^{n-1}\right)\\
&\qquad-\sum_{k=1}^{n-1}\frac{B_{n-k+1,p,q}}{[n-k+1]_{p,q}!}\ q^{2(n-k)}\ p^{k-1}\ D_{p,q,x}^{n-k+1}\Big(B_{n,p,q}\Big(\frac{x}{p}\Big)\Big)\\
&=\frac{[n+1]_{p,q}}{q}B_{n,p,q}(qx),
\end{aligned}\tag{64}$$

which, upon substituting for the following derivative:

$$D_{p,q,x}\left(\frac{x}{q}p^{n}-p^{n-1}+\frac{q}{p+q}p^{n-1}\right)=\frac{p^n}{q},\tag{65}$$

yields

$$\begin{aligned}
&\left(\frac{x}{q}p^{n+1}-p^{n-1}+\frac{q}{p+q}p^{n-1}\right)D_{p,q,x}\Big(B_{n,p,q}\Big(\frac{q}{p}\ x\Big)\Big)+\frac{p^n}{q}\ B_{n,p,q}\left(\frac{q^2x}{p}\right)\\
&\qquad-\sum_{k=1}^{n-1}\frac{B_{n-k+1,p,q}}{[n-k+1]_{p,q}!}\ q^{2(n-k)}\ p^{k-1}\ D_{p,q,x}^{n-k+1}\Big(B_{n,p,q}\Big(\frac{x}{p}\Big)\Big)\\
&=\frac{[n+1]_{p,q}}{q}B_{n,p,q}(qx).
\end{aligned}$$

□

By setting $p=1$ in Theorem 5, we have the following corollary.

Corollary 9. *The difference equation satisfied by the big q-Bernoulli polynomials is given by*

$$\begin{aligned}
&\left(\frac{x}{q}-1+\frac{q}{1+q}\right)D_{q,x}\Big(B_{n,q}(qx)\Big)+\frac{1}{q}\ B_{n,q}(q^2x)\\
&\qquad-\sum_{k=1}^{n-1}\frac{B_{n-k+1,q}}{[n-k+1]_q!}\ q^{2(n-k)}\ D_{q,x}^{n-k+1}\Big(B_{n,q}(x)\Big)\\
&=\frac{[n+1]_q}{q}\ B_{n,q}(qx).
\end{aligned}\tag{66}$$

By letting $p=1$ and $q\to 1-$ in Theorem 5, we have the following corollary.

Corollary 10. (see [6]) *The differential equation satisfied by the Bernoulli polynomials is given by*

$$\left(x-\frac{1}{2}\right)\ \frac{d}{dx}\Big(B_n(x)\Big)-\sum_{k=1}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}\ \left(\frac{d}{dx}\right)^{n-k+1}\Big(B_n(x)\Big)-nB_n(x)=0.\tag{67}$$

4. Properties of the Big (p, q)-Exponential Functions

In this Section, we obtain several potentially useful properties of the big (p, q)-exponential function and the (p, q)-exponential function. For the convenience in respect of their expansions, we first recall the (p, q)-Gauss binomial expansion. We present the corresponding properties for the big q-exponential and the q-exponential functions in the case when $p = 1$.

The (p, q)-Gauss binomial expansion is given by (see [10])

$$\left([x+y]_{p,q}\right)^n = \prod_{r=0}^{n-1}(q^r x + p^r y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}}\, q^{\binom{n-k}{2}}\, x^{n-k}\, y^k. \tag{68}$$

In the case when $p = 1$ in the (p, q)-binomial expansion (68), we get the following consequence of the (p, q)-Gauss binomial expansion (68).

Corollary 11. *The q-binomial expansion is given by*

$$\left([x+y]_{q}\right)^n = \prod_{r=0}^{n-1}(q^r x + y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q} q^{\binom{n-k}{2}}\, x^{n-k}\, y^k. \tag{69}$$

Theorem 6. *The product of the (p, q)-exponential functions $E_{p,q}(x)$ and $E_{\frac{1}{p},\frac{1}{q}}(y)$ is given by*

$$E_{p,q}(x)E_{\frac{1}{p},\frac{1}{q}}(y) = \sum_{n=0}^{\infty} \frac{\left([x+y]_{p,q}\right)^n}{[n]_{p,q}!}. \tag{70}$$

Proof. Upon replacing p by $\frac{1}{p}$ and q by $\frac{1}{q}$ in $E_{p,q}(x)$, if we take into consideration of the fact that

$$[n]_{\frac{1}{p},\frac{1}{q}}! = \frac{[n]_{p,q}!}{p^{\binom{n}{2}} q^{\binom{n}{2}}}$$

and replace x by y, we find that

$$E_{\frac{1}{p},\frac{1}{q}}(y) = \sum_{k=0}^{\infty} \frac{p^{\binom{k}{2}}}{[k]_{p,q}!} y^k.$$

Also, multiplying $E_{p,q}(x)$ by $E_{\frac{1}{p},\frac{1}{q}}(y)$, we have

$$E_{p,q}(x)E_{\frac{1}{p},\frac{1}{q}}(y) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \sum_{k=0}^{\infty} \frac{p^{\binom{k}{2}}}{[k]_{p,q}!}\, y^k,$$

which, in light of the Cauchy product, yields

$$E_{p,q}(x)E_{\frac{1}{p},\frac{1}{q}}(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{n-k}{2}}\, p^{\binom{k}{2}}\, \frac{x^{n-k} y^k}{[n]_{p,q}!}.$$

Now, by using the (p, q)-binomial expansion, we can write

$$E_{p,q}(x)E_{\frac{1}{p},\frac{1}{q}}(y) = \sum_{n=0}^{\infty} \frac{\left([x+y]_{p,q}\right)^n}{[n]_{p,q}!}.$$

□

Next, by using the fact that

$$E_{\frac{1}{p},\frac{1}{q}}(y) = e_{p,q}(y),$$

we can write the following result from Theorem 6.

Corollary 12. *The product of the (p, q)-exponential functions $E_{p,q}(x)$ and $e_{p,q}(y)$ is given by*

$$E_{p,q}(x)e_{p,q}(y) = \sum_{n=0}^{\infty} \frac{\left([x + y]_{p,q}\right)^n}{[n]_{p,q}!}. \tag{71}$$

If we set $y = -x$ in Corollary 12 and use the fact that $[0]_{p,q} := 1$, we arrive at the following corollary.

Corollary 13. *The (p, q)-exponential functions $E_{p,q}(x)$ and $e_{p,q}(y)$ satisfy the following relation*:

$$E_{p,q}(x)e_{p,q}(-x) = 1. \tag{72}$$

For $p = 1$ in Theorem 6, we have the following result.

Corollary 14. *The q-exponential functions $E_q(x)$ and $E_{\frac{1}{q}}(y)$ satisfy the following relation*:

$$E_q(x)E_{\frac{1}{q}}(y) = \sum_{n=0}^{\infty} \frac{\left([x + y]_q\right)^n}{[n]_{p,q}!}. \tag{73}$$

Theorem 7. *The product of the (p, q)-exponential functions $e_{p,q}(y)$ and $E_{p,q}(x)$ is given by*

$$e_{p,q}(y)E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{\left([x + y]_{p,q}\right)^n}{[n]_{p,q}!}. \tag{74}$$

Proof. Upon inserting the series forms of the (p, q)-exponential functions, we get

$$e_{p,q}(y)\ E_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}}\ \frac{y^n}{[n]_{p,q}!} \sum_{k=0}^{\infty} q^{\binom{k}{2}}\ \frac{x^k}{[k]_{p,q}!},$$

which, in view of the Cauchy product, yields

$$\begin{aligned} e_{p,q}(y)E_{p,q}(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}\ q^{\binom{k}{2}}\ x^k\ y^{n-k}\ \frac{1}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}}\ q^{\binom{n-k}{2}}\ x^{n-k}\ y^k\ \frac{1}{[n]_{p,q}!}. \end{aligned}$$

Finally, by using the (p, q)-binomial theorem (68), we get

$$e_{p,q}(y)E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{\left([x + y]_{p,q}\right)^n}{[n]_{p,q}!}.$$

□

By setting $y = -x$ in Theorem 7 and using the fact that $[0]_{p,q} := 1$, we get the following corollary.

Corollary 15. *The (p, q)-exponential functions $e_{p,q}(y)$ and $E_{p,q}(x)$ satisfy the following relation:*

$$e_{p,q}(-x)E_{p,q}(x) = 1. \tag{75}$$

Putting $p = 1$ in Theorem 7, we have the following corollary.

Corollary 16. *The q-exponential functions $E_q(x)$ and $E_{\frac{1}{q}}(y)$ satisfy the following relation:*

$$e_q(y)E_q(x) = \sum_{n=0}^{\infty} \frac{\left([x+y]_q\right)^n}{[n]_{p,q}!}. \tag{76}$$

By applying Corollary 13 and Corollary 15, the product of $E_{p,q}(x)$ and $e_{p,q}(-x)$ is seen to be commutative. We are thus led to the following corollary.

Corollary 17. *For the (p, q)-exponential functions $e_{p,q}(y)$ and $E_{p,q}(x)$, it is asserted that*

$$E_{p,q}(x)e_{p,q}(-x) = e_{p,q}(-x)E_{p,q}(x) = 1. \tag{77}$$

5. The Big (p, q)-Euler Polynomials

In this section, we introduce the big (p, q)-Euler polynomials and derive the recurrence relation as well as the difference equation satisfied by these big (p, q)-Euler polynomials. In the case when $p = 1$, we give the corresponding recurrence relation and difference equation satisfied by the big q-Euler polynomials. In the case when $p = 1$ and $q \to 1-$, the big (p, q)-Euler polynomials reduce to the Euler polynomials. Therefore, the recurrence relation and difference equation satisfied by the big (p, q)-Euler polynomials reduce to the corresponding recurrence relation and differential equation satisfied by the Euler polynomials.

Definition 6. The big (p, q)-Euler polynomials are defined by

$$\left(\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right)+1}\right)E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} E_{n,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}, \tag{78}$$

where the big (p, q)-Euler numbers are given by

$$\frac{[2]_{p,q}E_{p,q}\left(\frac{t}{q}\right)}{E_{p,q}\left(\frac{[2]_{p,q}}{q}\,t\right)+1} = \sum_{k=0}^{\infty} E_{k,p,q}\,\frac{t^k}{[k]_{p,q}!}. \tag{79}$$

Some of the big (p,q)-Euler numbers are given below:

$$\begin{aligned}
E_{0,p,q} &= \frac{[2]_{p,q}}{2},\ E_{2,p,q} = \frac{[2]_{p,q}}{2q}\left(1-\frac{[2]_{p,q}^2}{2}\right),\\
E_{4,p,q} &= \frac{[2]_{p,q}}{2}q^2\left(1-\frac{[2]_{p,q}^4}{2}\right)-\frac{[4]_{p,q}![2]_{p,q}}{4q^2}\left(1-\frac{[2]_{p,q}^2}{2}\right),\\
E_{6,p,q} &= \frac{[2]_{p,q}}{2}\ q^9-\frac{[2]_{p,q}^6}{2}\ q^9\ E_{0,p,q}-\frac{[5]_{p,q}\ [6]_{p,q}\ [2]_{p,q}^3}{2}\ q^2\ E_{2,p,q}\\
&\quad -\frac{[5]_{p,q}\ [6]_{p,q}\ [2]_{p,q}}{2}\ q^{-1}\ E_{4,p,q},\\
E_{8,p,q} &= \frac{[2]_{p,q}}{2}\ q^{20}-\frac{[2]_{p,q}^8}{2}\ q^{20}\ E_{0,p,q}-\frac{[7]_{p,q}\ [8]_{p,q}\ [2]_{p,q}^5}{2}\ q^9\ E_{2,p,q}\\
&\quad -\frac{[5]_{p,q}\ [6]_{p,q}\ [7]_{p,q}\ [8]_{p,q}\ [2]_{p,q}^3}{2[3]_{p,q}\ [4]_{p,q}}\ q^2\ E_{4,p,q}\\
&\quad -\frac{[7]_{p,q}\ [8]_{p,q}\ [2]_{p,q}}{2}\ q^{-1}\ E_{6,p,q}.
\end{aligned} \tag{80}$$

Definition 7. The big q-Euler polynomials are defined by

$$\left(\frac{[2]_q}{E_q\left(\frac{t}{q}\right)+1}\right)E_q\left(\frac{xt}{q}\right)=\sum_{n=0}^{\infty}E_{n,q}(x)\ \frac{t^n}{[n]_q!}, \tag{81}$$

where the big q-Euler numbers are given by

$$\frac{[2]_qE_q\left(\frac{t}{q}\right)}{E_q\left(\frac{[2]_q}{q}t\right)+1}=\sum_{k=0}^{\infty}E_{k,q}\ \frac{t^k}{[k]_q!}. \tag{82}$$

In the case when $p=1$, we have the following big q-Euler numbers:

$$\begin{aligned}
E_{0,q} &= \frac{[2]_q}{2},\ E_{2,q} = \frac{[2]_q}{2q}\left(1-\frac{[2]_q^2}{2}\right),\\
E_{4,q} &= \frac{[2]_q}{2}q^2\left(1-\frac{[2]_q^4}{2}\right)-\frac{[4]_q![2]_q}{4q^2}\left(1-\frac{[2]_q^2}{2}\right), E_{6,q} = \frac{[2]_q}{2}\ q^9-\frac{[2]_q^6}{2}\ q^9\ E_{0,q}-\frac{[5]_q\ [6]_q\ [2]_q^3}{2}\ q^2\ E_{2,q}\\
&\quad -\frac{[5]_q\ [6]_q\ [2]_q}{2}\ q^{-1}\ E_{4,q},\\
E_{8,q} &= \frac{[2]_p}{2}\ q^{20}-\frac{[2]_p^8}{2}\ q^{20}\ E_{0,q}-\frac{[7]_q\ [8]_q\ [2]_q^5}{2}\ q^9\ E_{2,q}\\
&\quad -\frac{[5]_q\ [6]_q\ [7]_q\ [8]_q\ [2]_q^3}{2[3]_q\ [4]_q}\ q^2\ E_{4,q}\\
&\quad -\frac{[7]_q\ [8]_q\ [2]_q}{2}\ q^{-1}\ E_{6,q}.
\end{aligned} \tag{83}$$

Upon letting $p=1$ and $q\to 1-$, we have the Euler numbers given by

$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\cdots,\ \ E_{2n+1}=0 \quad (n\in\mathbb{N}). \tag{84}$$

Theorem 8. *The big (p,q)-Euler polynomials $E_{n,p,q}(x)$ satisfy the following relation*:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{k}{2}-k}\, x^k\, E_{n-k,p,q}(0) = E_{n,p,q}(x). \tag{85}$$

Proof. By using the generating function of the big (p,q)-Euler polynomials, we have

$$\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right)+1}\, E_{p,q}\left(\frac{xt}{q}\right) = \sum_{n=0}^{\infty} E_{n,p,q}(x)\, \frac{t^n}{[n]_{p,q}!},$$

where

$$\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right)+1} = \sum_{n=0}^{\infty} E_{n,p,q}(0)\, \frac{t^n}{[n]_{p,q}!}.$$

Thus, upon inserting the corresponding series, we get

$$\sum_{n=0}^{\infty} E_{n,p,q}(0)\, \frac{t^n}{[n]_{p,q}!} \sum_{k=0}^{\infty} q^{\binom{k}{2}}\, \frac{(\frac{xt}{q})^k}{[k]_{p,q}!} = \sum_{n=0}^{\infty} E_{n,p,q}(x)\, \frac{t^n}{[n]_{p,q}!},$$

which, in light of the Cauchy product, yields

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{k}{2}-k}\, x^k\, E_{n-k,p,q}(0)\, \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} E_{n,p,q}(x)\, \frac{t^n}{[n]_{p,q}!}.$$

Finally, by equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we get

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{k}{2}-k}\, x^k\, E_{n-k,p,q}(0) = E_{n,p,q}(x).$$

□

Taking $p = 1$ in Theorem 8, we get the following corollary.

Corollary 18. *For the big q-Euler polynomials $E_{n,q}(x)$, it is asserted that*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q} q^{\binom{k}{2}-k}\, x^k\, E_{n-k,q}(0) = E_{n,q}(x). \tag{86}$$

If we set $p = 1$ and $q \to 1-$ in Theorem 8, we get the following corollary.

Corollary 19. *For the Euler polynomials $E_n(x)$, it is asserted that*

$$\sum_{k=0}^{n} \binom{n}{k} x^k\, E_{n-k}(0) = E_n(x). \tag{87}$$

Theorem 9. *For the big (p,q)-Euler numbers $E_{n,p,q}$, it is asserted that*

$$E_{n,p,q}\left(\frac{1}{[2]_{p,q}}\right) = [2]_{p,q}^{-n}\, E_{n,p,q}. \tag{88}$$

Proof. Our demonstration of Theorem 9 follows easily upon replacing t by $[2]_{p,q}t$ and taking $x = \frac{1}{[2]_{p,q}}$ in generating function of the big (p,q)-Euler polynomials. The details involved are being omitted here. □

Taking $p = 1$ in Theorem 9, we have the following corollary.

Corollary 20. *The big q-Euler polynomials satisfy the following relation*:

$$E_{n,q}\left(\frac{1}{[2]_q}\right) = [2]_q^{-n} E_{n,q}. \tag{89}$$

By letting $p = 1$ and $q \to 1-$ in Theorem 9, we have the following corollary.

Corollary 21. (see [5] and [6]) *For Euler polynomials $E_n(x)$, it is asserted that*

$$E_n\left(\frac{1}{2}\right) = 2^{-n}\, E_n. \tag{90}$$

By first setting $x = 1$ and replacing t by $-t$ in generating function of the big (p,q)-Euler polynomials and then using the fact that

$$E_{p,q}\left(-\frac{t}{q}\right) e_{p,q}\left(\frac{t}{q}\right) = 1,$$

we are led to the following corollary.

Corollary 22. *For the big (p,q)-Euler polynomials, it is asserted that*

$$\left(\frac{[2]_{p,q}}{E_{p,q}\left(-\frac{t}{q}\right)+1}\right) E_{p,q}\left(-\frac{t}{q}\right) = \frac{[2]_{p,q}}{e_{p,q}\left(\frac{t}{q}\right)+1} = \sum_{n=0}^{\infty} E_{n,p,q}(1)\, \frac{(-t)^n}{[n]_{p,q}!}. \tag{91}$$

Theorem 10. *The big (p,q)-Euler polynomials satisfy the following relation*:

$$E_{n,p,q}(1) = (-1)^n\, q^{\binom{n}{2}+1-2n}\, p^{\binom{n}{2}+1}\, E_{n,\frac{1}{p},\frac{1}{q}}(0). \tag{92}$$

Proof. By applying Corollary 22, we have

$$\frac{[2]_{p,q}}{e_{p,q}\left(\frac{t}{q}\right)+1} = \sum_{n=0}^{\infty} E_{n,p,q}(1)\, \frac{(-t)^n}{[n]_{p,q}!}.$$

Replacing p by $\frac{1}{p}$ and q by $\frac{1}{q}$, and taking into consideration of the fact that

$$[2]_{\frac{1}{p},\frac{1}{q}} = \frac{[2]_{p,q}}{pq},\ e_{\frac{1}{p},\frac{1}{q}}(qt) = E_{p,q}(qt) \ \text{ and } [n]_{\frac{1}{p},\frac{1}{q}}! = \frac{[n]_{p,q}!}{p^{\binom{n}{2}} q^{\binom{n}{2}}},$$

we get

$$\frac{1}{pq}\frac{[2]_{p,q}}{E_{p,q}(qt)+1} = \sum_{n=0}^{\infty} (-1)^n\, q^{\binom{n}{2}} p^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(1)\, \frac{t^n}{[n]_{p,q}!}. \tag{93}$$

Taking $x = 0$ in the generating function of the big (p, q)-Euler polynomials, we get

$$\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right)+1} = \sum_{n=0}^{\infty} E_{n,p,q}(0)\ \frac{t^n}{[n]_{p,q}!}$$

which, when used in (93), yields

$$\sum_{n=0}^{\infty} E_{n,p,q}(0)\ \frac{q^{2n}\ t^n}{[n]_{p,q}!} = pq \sum_{n=0}^{\infty} (-1)^n\ q^{\binom{n}{2}} p^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(1)\ \frac{t^n}{[n]_{p,q}!}.$$

Equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we get

$$q^{2n}\ E_{n,p,q}(0) = (-1)^n\ pq q^{\binom{n}{2}} p^{\binom{n}{2}}\ E_{n,\frac{1}{p},\frac{1}{q}}(1).$$

We thus find that

$$E_{n,p,q}(0) = (-1)^n\ pq^{\binom{n}{2}+1-2n}\ p^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(1).$$

Replacing p by $\frac{1}{p}$ and q by $\frac{1}{q}$, we get

$$E_{n,\frac{1}{p},\frac{1}{q}}(0) = (-1)^n\ \frac{1}{q^{\binom{n}{2}+1-2n} p^{\binom{n}{2}+1}} E_{n,p,q}(1),$$

so that

$$E_{n,p,q}(1) = (-1)^n\ pq^{\binom{n}{2}+1-2n}\ p^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(0).$$

□

Taking $p = 1$ in Theorem 10, we get the following corollary.

Corollary 23. *For big q-Euler polynomials, it is asserted that*

$$E_{n,q}(1) = (-1)^n\ q^{\binom{n}{2}+1-2n}\ E_{n,\frac{1}{q}}(0). \tag{94}$$

Upon letting $p = 1$ and $q \to 1-$ in Theorem 10, we get the following corollary.

Corollary 24. (see [5]) *The Euler polynomials $E_n(x)$ satisfies the following relation:*

$$E_n(1) = (-1)^n\ E_n(0). \tag{95}$$

Theorem 11. *The big (p, q)-Euler numbers satisfies the following relation:*

$$\frac{(-1)^n\ pq^{1-2n}}{[2]_{p,q}^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}+\binom{n-k}{2}}\ q^{\binom{n-k}{2}+k}\ E_{n-k,\frac{1}{p},\frac{1}{q}}(1) = E_{n,p,q}(0). \tag{96}$$

Proof. Taking $x = 0$ in the generating function of the big (p, q)-Euler polynomials, we get

$$\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right)+1} = \sum_{n=0}^{\infty} E_{n,p,q}(0)\ \frac{t^n}{[n]_{p,q}!}, \tag{97}$$

where

$$E_{p,q}(0) := 1.$$

Now, using the fact that

$$E_{p,q}\left(\frac{t}{q[2]_{p,q}}\right)e_{p,q}\left(-\frac{t}{q[2]_{p,q}}\right)=1$$

in (97), we get

$$\left(\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right)+1}\right)E_{p,q}\left(\frac{t}{q[2]_{p,q}}\right)e_{p,q}\left(-\frac{t}{q[2]_{p,q}}\right)=\sum_{n=0}^{\infty}E_{n,p,q}(0)\ \frac{t^n}{[n]_{p,q}!}. \tag{98}$$

Using the generating function of the big (p,q)-Euler numbers, we can write

$$\left(\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right)+1}\right)E_{p,q}\left(\frac{t}{q[2]_{p,q}}\right)=\sum_{n=0}^{\infty}E_{n,p,q}\ \frac{\left(\frac{t}{[2]_{p,q}}\right)^n}{[n]_{p,q}!}. \tag{99}$$

Upon inserting (99) and the series of $e_{p,q}(-\frac{t}{q[2]_{p,q}})$ in (98), we get

$$\sum_{n=0}^{\infty}E_{n,p,q}\frac{\left(\frac{t}{[2]_{p,q}}\right)^n}{[n]_{p,q}!}\sum_{k=0}^{\infty}p^{\binom{k}{2}}\ \frac{\left(-\frac{t}{q[2]_{p,q}}\right)^k}{[k]_{p,q}!}=\sum_{n=0}^{\infty}E_{n,p,q}(0)\ \frac{t^n}{[n]_{p,q}!}.$$

Applying the Cauchy product, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}\ q^{-k}[2]_{p,q}^{-n}\ E_{n-k,p,q}\ \frac{t^n}{[n]_{p,q}!}=\sum_{n=0}^{\infty}E_{n,p,q}(0)\ \frac{t^n}{[n]_{p,q}!}. \tag{100}$$

Equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$ in (100), we get

$$\frac{1}{[2]_{p,q}^n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}\ q^{-k}\ E_{n-k,p,q}=E_{n,p,q}(0). \tag{101}$$

Finally, by using the fact that

$$E_{n,p,q}(1):=E_{n,p,q}=pq^{1-2n}q^{\binom{n}{2}}\ p^{\binom{n}{2}}(-1)^nE_{n,\frac{1}{p},\frac{1}{q}}(0),$$

which was obtained in (92), we get

$$\frac{(-1)^n\ pq^{1-2n}}{[2]_{p,q}^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}+\binom{n-k}{2}}\ q^{\binom{n-k}{2}+k}\ E_{n-k,\frac{1}{p},\frac{1}{q}}(0)=E_{n,p,q}(0).$$

□

Taking $p=1$ in Theorem 11, we have the following corollary.

Corollary 25. *For the big q-Euler numbers, it is asserted that*

$$\frac{(-1)^nq^{1-2n}}{[2]_q^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}q^{\binom{n-k}{2}+k}\ E_{n-k,\frac{1}{q}}(1)=E_{n,q}(0). \tag{102}$$

Upon setting $p = 1$ and $q \to 1-$ in Theorem 11, we have the following corollary.

Corollary 26. *The Euler numbers E_n satisfies the following relation*:

$$\frac{(-1)^n}{2^n}\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(1) = E_n(0). \tag{103}$$

For $x = 1$ in Theorem 8, we get the following corollary.

Corollary 27. *For the big (p, q)-Euler polynomials, it is asserted that*

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{k}{2}-k}\, E_{n-k,p,q}(0) = E_{n,p,q}(1). \tag{104}$$

Taking $p = 1$ in Corollary 27, we get the following consequence.

Corollary 28. *The big q-Euler polynomials satisfy the following relation*:

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q} q^{\binom{k}{2}-k}\, E_{n-k,q}(0) = E_{n,q}(1). \tag{105}$$

By letting $p = 1$ and $q \to 1-$ in Corollary 27, we get the following corollary.

Corollary 29. *The Euler polynomials $E_n(x)$ satisfies the following relation*:

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0) = E_n(1). \tag{106}$$

Theorem 12. *A recurrence relation satisfied by the big (p, q)-Euler polynomials is given by*

$$\begin{aligned}&\frac{p^n}{q}\left(x-\frac{1}{2}\right)E_{n,p,q}\left(\frac{q}{p}\,x\right)+\frac{p^n}{q[2]_{p,q}}\sum_{k=0}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} e_{n-k,p,q}\,E_{k,p,q}\left(\frac{q}{p}\,x\right)\\&\qquad = E_{n+1,p,q}(x),\end{aligned} \tag{107}$$

where the coefficients $e_{k,p,q}$ are given by

$$\begin{aligned}e_{k,p,q} &:= \left(\frac{q}{p}\right)^k E_{k,p,q}(0)\\&= \frac{(-1)^k(pq)^{1-k}}{[2]_{p,q}^k}\sum_{l=0}^{k}\begin{bmatrix} k \\ l \end{bmatrix}_{p,q} p^{\binom{l}{2}+\binom{k-l}{2}}\, q^{\binom{k-l}{2}+l}\, E_{k-l,\frac{1}{p},\frac{1}{q}}(0).\end{aligned} \tag{108}$$

Proof. Taking the (p, q) derivatives with respect to t on both sides of the generating function of the big (p, q)-Euler polynomials and using the rules in (1) and (2), and rearranging some terms, we have

$$\begin{aligned}&\frac{x}{q}\left(\frac{[2]_{p,q}}{E_{p,q}\left(\frac{pt}{q}\right)+1}\right)E_{p,q}(xt)\\&\qquad + E_{p,q}(xt)\left(-\frac{p+q}{q}\frac{1}{E_{p,q}\left(\frac{pt}{q}\right)+1}+\frac{p+q}{q}\frac{1}{\left(E_{p,q}\left(\frac{pt}{q}\right)+1\right)\left(E_{p,q}(t)+1\right)}\right)\\&= \sum_{n=0}^{\infty}E_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{109}$$

Using the fact that

$$E_{p,q}(t)\,e_{p,q}(-t)=1$$

in (109) and rearranging some terms again, we can write

$$\begin{aligned}
&\frac{x}{q}\,\frac{[2]_{p,q}}{E_{p,q}\left(\frac{pt}{q}\right)+1}\,E_{p,q}(xt)-\frac{p+q}{q[2]_{p,q}}\,\frac{[2]_{p,q}}{E_{p,q}\left(\frac{pt}{q}\right)+1}\,E_{p,q}(xt)\\
&\qquad+\frac{p+q}{q}\,\frac{1}{E_{p,q}\left(\frac{pt}{q}\right)+1}E_{p,q}(xt)\,\frac{[2]_{p,q}}{[2]_{p,q}\big(E_{p,q}(t)+1\big)}\,E_{p,q}\left(\frac{1}{[2]_{p,q}}t\right)e_{p,q}\left(-\frac{1}{[2]_{p,q}}t\right)\\
&\quad=\sum_{n=0}^{\infty}E_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!},
\end{aligned}$$

which, upon inserting the corresponding series, yields

$$\begin{aligned}
&\frac{x}{q}\sum_{n=0}^{\infty}E_{n,p,q}\left(\frac{q}{p}\,x\right)\frac{(pt)^n}{[n]_{p,q}!}-\frac{1}{q}\sum_{n=0}^{\infty}E_{n,p,q}\left(\frac{q}{p}\,x\right)\frac{(pt)^n}{[n]_{p,q}!}\\
&\quad+\frac{1}{q}\sum_{n=0}^{\infty}E_{n,p,q}\left(\frac{q}{p}\,x\right)\frac{(pt)^n}{[n]_{p,q}!}\,\frac{1}{[2]_{p,q}}\sum_{k=0}^{\infty}E_{k,p,q}\left(\frac{1}{[2]_{p,q}}\right)\frac{(qt)^k}{[k]_{p,q}!}\sum_{l=0}^{\infty}p^{\binom{l}{2}}\,\frac{(-\frac{1}{[2]_{p,q}}t)^l}{[l]_{p,q}!}\\
&=\sum_{n=0}^{\infty}E_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Applying the Cauchy product, we get

$$\begin{aligned}
&\frac{x}{q}\sum_{n=0}^{\infty}p^n\,E_{n,p,q}\left(\frac{q}{p}\,x\right)\frac{t^n}{[n]_{p,q}!}-\frac{1}{q}\sum_{n=0}^{\infty}p^n\,E_{n,p,q}\left(\frac{q}{p}\,x\right)\frac{t^n}{[n]_{p,q}!}\\
&\qquad+\frac{1}{q[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\sum_{l=0}^{k}\begin{bmatrix}k\\l\end{bmatrix}_{p,q}\frac{(-1)^l}{[2]_{p,q}^l}\,q^{k-l}\,p^{n-k+\binom{l}{2}}\\
&\qquad\cdot E_{k-l,p,q}\left(\frac{1}{[2]_{p,q}}\right)E_{n-k,p,q}\left(\frac{q}{p}\,x\right)\frac{t^n}{[n]_{p,q}!}\\
&\quad=\sum_{n=0}^{\infty}E_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.
\end{aligned}\tag{110}$$

Now, using the fact that

$$E_{n,p,q}\Big(\frac{1}{[2]_{p,q}}\Big)=[2]_{p,q}^{-n}\,E_{n,p,q}$$

in (110) with

$$E_{n,p,q}(1):=E_{n,p,q}=(-1)^n\,q^{\binom{n}{2}+1-2n}\,p^{\binom{n}{2}+1}\,E_{n,\frac{1}{p},\frac{1}{q}}(0),$$

we get

$$E_{n,p,q}\left(\frac{1}{[2]_{p,q}}\right)=(-1)^n\,[2]_{p,q}^{-n}\,q^{\binom{n}{2}+1-2n}\,p^{\binom{n}{2}+1}\,E_{n,\frac{1}{p},\frac{1}{q}}(0).\tag{111}$$

Upon setting $n = k - l$ and substituting for $E_{k-l,p,q}\left(\frac{1}{[2]_{p,q}}\right)$ from (111), we get

$$\begin{aligned}
&\frac{x}{q}\sum_{n=0}^{\infty} p^n\, E_{n,p,q}\left(\frac{q}{p}\,x\right)\frac{t^n}{[n]_{p,q}!} - \frac{1}{q}\sum_{n=0}^{\infty} p^n\, E_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{t^n}{[n]_{p,q}!} \\
&\qquad + \frac{1}{q[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \sum_{l=0}^{k}\begin{bmatrix} k \\ l \end{bmatrix}_{p,q} p^{\binom{l}{2}}\, p^{n-k+1}\, q^{-k+l+1}\,[2]_{p,q}^{-k} \\
&\qquad\quad \cdot p^{\binom{k-l}{2}}\, q^{\binom{k-l}{2}}\, E_{k-l,\frac{1}{p},\frac{1}{q}}(0) E_{n-k,p,q}\left(\frac{q}{p}\,x\right)\,\frac{t^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} E_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.
\end{aligned} \tag{112}$$

In this last equation (112), by defining $e_{k,p,q}$ by

$$e_{k,p,q} := \left(\frac{q}{p}\right)^k E_{k,p,q}(0) = \frac{(-1)^k (pq)^{1-k}}{[2]_{p,q}^k}\sum_{l=0}^{k}\begin{bmatrix} k \\ l \end{bmatrix}_{p,q} p^{\binom{k-l}{2}+\binom{l}{2}}\, q^{\binom{k-l}{2}+l}\, E_{k-l,\frac{1}{p},\frac{1}{q}}(0),$$

we get

$$\begin{aligned}
&\frac{x}{q}\sum_{n=0}^{\infty} p^n\, E_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{t^n}{[n]_{p,q}!} - \frac{1}{q}\sum_{n=0}^{\infty} p^n\, E_{n,p,q}\left(\frac{q}{p}\,x\right)\,\frac{t^n}{[n]_{p,q}!} \\
&\qquad + \frac{1}{q[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^n\, e_{k,p,q}\, E_{n-k,p,q}\left(\frac{q}{p}\,x\right)\,\frac{t^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} E_{n+1,p,q}(x)\,\frac{t^n}{[n]_{p,q}!}.
\end{aligned} \tag{113}$$

Equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$ of both sides of (113), we get

$$\begin{aligned}
&\frac{p^n}{q}\,(x-1)\,E_{n,p,q}\left(\frac{q}{p}\,x\right) + \frac{p^n}{q[2]_{p,q}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} e_{k,p,q}E_{n-k,p,q}\left(\frac{q}{p}\,x\right) \\
&= E_{n+1,p,q}(x).
\end{aligned} \tag{114}$$

Equation (114) can be written as follows:

$$\frac{p^n}{q}\,(x-1)E_{n,p,q}\left(\frac{q}{p}\,x\right) + \frac{p^n}{q[2]_{p,q}}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} e_{n-k,p,q}\, E_{k,p,q}\left(\frac{q}{p}\,x\right) = E_{n+1,p,q}(x).$$

We can also write

$$\begin{aligned}
&\frac{p^n}{q}\,(x-1)E_{n,p,q}\left(\frac{q}{p}\,x\right) + \frac{p^n}{q[2]_{p,q}}\, e_{0,p,q}\, E_{n,p,q}\left(\frac{q}{p}\,x\right) \\
&\qquad + \frac{p^n}{q[2]_{p,q}}\sum_{k=0}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} e_{n-k,p,q}\, E_{k,p,q}\left(\frac{q}{p}\,x\right) \\
&= E_{n+1,p,q}(x).
\end{aligned} \tag{115}$$

Now, by the definition of $e_{k,p,q}$, we have

$$e_{0,p,q} = pqE_{0,\frac{1}{p},\frac{1}{q}}(1). \tag{116}$$

Thus, by setting $n = 0$ in Corollary 27 and replacing p by $\frac{1}{p}$, and q by $\frac{1}{q}$, we get

$$E_{0,\frac{1}{p},\frac{1}{q}}(1) = E_{0,\frac{1}{p},\frac{1}{q}}(0). \tag{117}$$

Therefore, we have

$$e_{0,p,q} = pqE_{0,\frac{1}{p},\frac{1}{q}}(0). \tag{118}$$

In order to compute $E_{0,\frac{1}{p},\frac{1}{q}}(0)$, by taking $x = 0$ in the generating function of the big (p, q)-Euler polynomials, we get

$$\frac{[2]_{p,q}}{E_{p,q}\left(\frac{t}{q}\right) + 1} = \sum_{n=0}^{\infty} E_{n,p,q}(0) \frac{t^n}{[n]_{p,q}!}, \tag{119}$$

By replacing p by $\frac{1}{p}$ and q by $\frac{1}{q}$ in (119), we get

$$\frac{[2]_{p,q}}{pq\Big(e_{p,q}(qt) + 1\Big)} = \sum_{n=0}^{\infty} p^{\binom{n}{2}} q^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(0) \frac{t^n}{[n]_{p,q}!}. \tag{120}$$

Equation (120) can be written as follows:

$$\frac{[2]_{p,q}}{pq} = \Big(e_{p,q}(qt) + 1\Big) \sum_{n=0}^{\infty} p^{\binom{n}{2}} q^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(0) \frac{t^n}{[n]_{p,q}!}. \tag{121}$$

Inserting the corresponding series in (121), we get

$$\frac{[2]_{p,q}}{pq} = \left[\sum_{k=0}^{\infty} p^{\binom{k}{2}} \frac{q^k t^k}{[k]_{p,q}!} + 1\right] \sum_{n=0}^{\infty} p^{\binom{n}{2}} q^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(0) \frac{t^n}{[n]_{p,q}!}, \tag{122}$$

which, by an appeal to the Cauchy product, yields

$$\begin{aligned}\frac{[2]_{p,q}}{pq} = &\sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}+\binom{n-k}{2}} q^{\binom{n-k}{2}+k} E_{n-k,\frac{1}{p},\frac{1}{q}}(0) \frac{t^n}{[n]_{p,q}!} \\ &+ \sum_{n=0}^{\infty} p^{\binom{n}{2}} q^{\binom{n}{2}} E_{n,\frac{1}{p},\frac{1}{q}}(0) \frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{123}$$

Equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$ and taking $n = 0$ in (123), we find that

$$E_{0,\frac{1}{p},\frac{1}{q}}(0) = \frac{[2]_{p,q}}{2pq}, \tag{124}$$

which, when inserted into (118), yields

$$e_{0,p,q} = \frac{[2]_{p,q}}{2}. \tag{125}$$

Now, inserting $e_{0,p,q}$ into (115), we get

$$\begin{aligned}&\frac{p^n}{q}\left(x - \frac{1}{2}\right) E_{n,p,q}\left(\frac{q}{p}\, x\right) + \frac{p^n}{q[2]_{p,q}} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} e_{n-k,p,q}\, E_{k,p,q}\left(\frac{q}{p}\, x\right) \\ &\quad = E_{n+1,p,q}(x).\end{aligned}$$

□

Taking $p = 1$ in Theorem 12, we have the following corollary.

Corollary 30. *The recurrence relation satisfied by the big q-Euler polynomials is given by*

$$\frac{1}{q}\left(x-\frac{1}{2}\right)E_{n,q}(qx)+\frac{1}{q[2]_q}\sum_{k=0}^{n-1}\begin{bmatrix}n\\k\end{bmatrix}_q e_{n-k,q}\,E_{k,q}(qx)=E_{n+1,q}(x), \tag{126}$$

where

$$e_{k,q}:=q^kE_{k,q}(0)=\frac{(-1)^kq^{1-k}}{[2]_q^k}\sum_{l=0}^{k}\begin{bmatrix}k\\l\end{bmatrix}_q q^{\binom{k-l}{2}+l}\,E_{k-l,\frac{1}{q}}(0). \tag{127}$$

Taking $p = 1$ and $q \to 1-$ in Theorem 12, we deduce the following corollary.

Corollary 31. (see [6]) *The recurrence relation satisfied by the Euler polynomials $E_n(x)$ is given by*

$$\left(x-\frac{1}{2}\right)E_n(x)+\frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}e_{n-k}\,E_k(x)=E_{n+1}(x), \tag{128}$$

where

$$e_k:=E_k(0)=\frac{(-1)^k}{2^k}\sum_{l=0}^{k}\binom{k}{l}E_{k-l}(0). \tag{129}$$

Theorem 13. *The difference equation satisfied by the big (p,q)-Euler polynomials $E_{n,p,q}(x)$ is given by*

$$\begin{aligned}&\frac{p^n}{q}\left(px-\frac{1}{2}\right)D_{p,q,x}\left(E_{n,p,q}\left(\frac{q}{p}\,x\right)\right)+\frac{p^n}{q}\,E_{n,p,q}\left(\frac{q^2}{p}\,x\right)\\&\qquad+\frac{p^n}{[2]_{p,q}}\sum_{k=1}^{n-1}q^{n-k-1}\frac{e_{n-k,p,q}}{[n-k]_{p,q}!}\,D_{p,q,x}^{n-k+1}\left(E_{n,p,q}\left(\frac{x}{p}\right)\right)\\&\quad=\frac{[n+1]_{p,q}}{q}\,E_{n,p,q}(qx).\end{aligned} \tag{130}$$

Proof. The proof of Theorem 13 is similar to the proof which we have already presented for the big (p,q)-Bernoulli polynomials. Since the derivative operator is given by

$$L^-_{n,p,q}:=\frac{q}{[n]_{p,q}}D_{p,q,x},$$

we write the following expression:

$$E_{k,p,q}\left(\frac{q}{p}\,x\right)=q^{n-k}\frac{[k]_{p,q}!}{[n]_{p,q}!}\,D_{p,q,x}^{n-k}\left(E_{n,p,q}\left(\frac{x}{p}\right)\right),$$

which, when inserted into (107), yields

$$\begin{aligned}&\frac{p^n}{q}\left(x-\frac{1}{2}\right)E_{n,p,q}\left(\frac{q}{p}\,x\right)+\frac{p^n}{[2]_{p,q}}\sum_{k=0}^{n-1}q^{n-k-1}\,\frac{e_{n-k,p,q}}{[n-k]_{p,q}!}\,D_{p,q,x}^{n-k}\left(E_{n,p,q}\left(\frac{x}{p}\right)\right)\\&\quad=E_{n+1,p,q}(x).\end{aligned} \tag{131}$$

Taking the (p,q)-derivatives of both sides of (131) with respect to x, we get the difference equation (130) asserted by Theorem 13. □

Taking $p = 1$ in Theorem 13, we have the following corollary.

Corollary 32. *The difference equation satisfied by the big q-Euler polynomials $E_{n,q}(x)$ is given by*

$$\begin{aligned}
&\frac{1}{q}\left(x-\frac{1}{2}\right)D_{q,x}\Big(E_{n,q}(qx)\Big)+\frac{1}{q}\,E_{n,q}(q^2x)\\
&\qquad+\frac{1}{q+1}\sum_{k=1}^{n-1}q^{n-k-1}\,\frac{e_{n-k,q}}{[n-k]_q!}\,D_{q,x}^{n-k+1}\Big(E_{n,q}(x)\Big)\\
&=\frac{[n+1]_q}{q}\,E_{n,q}(qx).
\end{aligned} \tag{132}$$

Letting $p = 1$ and $q \to 1-$ in Theorem 13, we have the following corollary.

Corollary 33. (see [6]) *The differential equation satisfied by the Euler polynomials $E_n(x)$ is given by*

$$\left(x-\frac{1}{2}\right)\frac{d}{dx}\Big(E_n(x)\Big)+\frac{1}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}\left(\frac{d}{dx}\right)^{n-k+1}\Big(E_n(x)\Big)-nE_n(x)=0. \tag{133}$$

6. Concluding Remarks and Observations

In our present investigation, we have introduced and studied the various properties and characteristics of the big (p, q)-Appell polynomials. In particular, we have derived an equivalence theorem satisfied by big (p, q)-Appell polynomials. By appropriately specializing our main results involving the big (p, q)-Appell polynomials, we have deduced the corresponding equivalence theorem, recurrence relation and difference equation for the big q-Appell polynomials. We have also presented the equivalence theorem, recurrence relation and differential equation for the usual Appell polynomials. Moreover, for the big (p, q)-Bernoulli polynomials and the big (p, q)-Euler polynomials, we have derived the recurrence relations and the difference equations. When $p = 1$, we have given the recurrence relations and the difference equations which are satisfied by the big q-Bernoulli polynomials and the big q-Euler polynomials. In the case when $p = 1$ and $q \to 1-$, the big (p, q)-Appell polynomials reduce to the usual Appell polynomials. Therefore, the recurrence relation and the difference equation which we have obtained for the big (p, q)-Appell polynomials coincide with the recurrence relation and the differential equation which are satisfied by the usual Appell polynomials.

We now choose to point out some obvious connections between the (p, q)-analysis and the classical q-analysis. Here, in this last section on concluding remarks and observations, we reiterate the fact that the results for the (p, q)-analogues, such as those which we have considered in this article for $0 < q < p \leqq 1$, can easily be deduced from the corresponding (possibly known) results for the familiar q-analogues (with $0 < q < 1$) by applying some obvious parametric and argument variations, the additional parameter p being redundant. Indeed, as observed earlier by Srivastava *et al.* [25], a considerably large number of authors have used the so-called (p, q)-analysis by introducing a seemingly redundant parameter p in the classical q-analysis. Also, as we have indicated already in Section 1, the so-called (p, q)-number $[n]_{p,q}$ is given (for $0 < q < p \leqq 1$) by

$$\begin{aligned}
[n]_{p,q} &:= \begin{cases} \dfrac{p^n-q^n}{p-q} & (n\in\{1,2,3,\cdots\})\\[2mm] 0 & (n=0)\end{cases}\\
&=: p^{n-1}\,[n]_{\frac{q}{p}},
\end{aligned} \tag{134}$$

where, for the q-number $[n]_q$, we have

$$\begin{aligned}[n]_q &:= \frac{1-q^n}{1-q} \\ &= p^{1-n}\left(\frac{p^n-(pq)^n}{p-(pq)}\right) \\ &= p^{1-n}\,[n]_{p,pq}.\end{aligned} \tag{135}$$

Furthermore, the so-called (p,q)-derivative or the so-called (p,q)-difference of a suitable function $f(z)$ is denoted by $\left(D_{p,q}\,f\right)(z)$ and defined, in a given subset of $\mathbb{C}$, by

$$\left(D_{p,q}\,f\right)(z) = \begin{cases} \dfrac{f(pz)-f(qz)}{(p-q)\,z} & (z \neq 0;\ 0<q<p \leqq 1) \\[2ex] f'(0) & (z=0;\ 0<q<p\leqq 1), \end{cases} \tag{136}$$

so that, clearly, we have the following connection with the familiar q-derivative or the q-difference $(D_q\,f)(z)$:

$$\left(D_{p,q}\,f\right)(z) = \left(D_{\frac{q}{p}}\,f\right)(pz) \qquad \text{and} \qquad \left(D_q\,f\right)(z) = \left(D_{p,pq}\,f\right)\left(\frac{z}{p}\right) \qquad (z\in\mathbb{C};\ 0<q<p\leqq 1). \tag{137}$$

These last equations (134), (135), (136) and (137) exhibit the fact that, in most cases, the (p,q)-analogues which have been considered in this article as well as in other earlier investigations for $0<q<p\leqq 1$ can easily be deduced from the corresponding (possibly known) q-analogues (with $0<q<1$) by applying some obvious parametric and argument variations of the kind which we have mentioned above and in Section 1, the additional parameter p being redundant.

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