



On the Rates of Convergence to Symmetric Stable Laws for Distributions of Normalized Geometric Random Sums

Tran Loc Hung^a, Phan Tri Kien^a

^aUniversity of Finance and Marketing

Abstract. Let X_1, X_2, \dots be a sequence of independent, identically distributed random variables. Let v_p be a geometric random variable with parameter $p \in (0, 1)$, independent of all $X_j, j \geq 1$. Assume that $\varphi : \mathbb{N} \mapsto \mathbb{R}^+$ is a positive normalized function such that $\varphi(n) = o(1)$ when $n \rightarrow +\infty$. The paper deals with the rate of convergence for distributions of randomly normalized geometric random sums $\varphi(v_p) \sum_{j=1}^{v_p} X_j$ to symmetric stable laws in term of Zolotarev's probability metric.

1. Introduction

Klebanov et al. [13] showed that the solution of Zolotarev's problem must be a geometric random sum and they introduced concepts of geometrically infinitely divisible (GID) random variables and geometrically strictly stable (GSS) random variables (see [13], [14] and [28] for more details). Up to the present this problem has attracted much attention. Moreover, weak limit theorems together with rates of convergence for distributions of geometric random sums have many applications to risk theory, stochastic finance, queuing theory, etc. (see [1], [2], [12], [4], [9], [8], [15], [16], [17], [18], [20], [21], [22], [23] and references therein).

According to results presented by Butzer and Hahn [5], a positive function $\varphi : \mathbb{N} \mapsto \mathbb{R}^+$ such that $\varphi(n) = o(1)$ when $n \rightarrow +\infty$, is called *normalized function*. One of extensions of concept of infinitely divisible (ID) random variables (see for instance [26], page 28) is concept of φ -decomposable random variables (or distributions) introduced by Butzer and Hahn [5], that is a random variable Z is said to be φ -decomposable, if for each $n \in \mathbb{N}$ there exist independent random variables $Z_j, Z_j = Z_{j,n}, 1 \leq j \leq n$, such that the distribution \mathbb{P}_Z of a random variable Z may be represented as

$$\mathbb{P}_Z = \mathbb{P}_{\varphi(n) \sum_{j=1}^n Z_{j,n}}. \quad (1)$$

Note that in (1) when $\varphi(n) \equiv 1$ for each $n \in \mathbb{N}$, the φ -decomposable random variable Z will become a ID distributed random variable (see for instance [22, Definition 1.2.1, p. 18]). Based on concept of φ -decomposability, using Trotter-operator method original by Trotter [29], several limit theorems together

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Email addresses: t1hung@ufm.edu.vn (Tran Loc Hung), phankien@ufm.edu.vn (Phan Tri Kien)

with rates of convergence for normalized sums $\varphi(n) \sum_{j=1}^n X_j$ had been established in [5], where X_1, X_2, \dots is a compatible sequence of independent random variables. The appropriating limiting distributions are distributions of φ -random variables including the standard normal distributed random variable X^* leading to Central limit theorem (CLT) and the random variable X_0 generated at a point x_0 which leads to Weak law of large numbers (WLLN) (see [5] for more details).

The interesting problem had been extended for normalized random sums $\varphi(N_\lambda) \sum_{j=1}^{N_\lambda} X_j$ by Butzer and Schulz in [6] and [7], where N_λ is a positive, integer – valued random variable such that $N_\lambda \xrightarrow{P} +\infty$ when $\lambda \rightarrow +\infty$. The weak convergence concerns the o -rates and \mathcal{O} - rates in limit theorems for normalized random sums $\varphi(N_\lambda) \sum_{j=1}^{N_\lambda} X_j$ are established in [6] and [7] for a desired martingale difference sequence (MDS). Note that the concept of φ -decomposability defined in (1) could be extended to randomly indexed random variables since the range of the index random variable N_λ is a subset of \mathbb{N} . In fact, for any φ -decomposable random variable Z one has by (1)

$$\mathbb{P}_Z = \mathbb{P}_{\varphi(N_\lambda) \sum_{j=1}^{N_\lambda} Z_j}, \quad \lambda \in \mathbb{R}^+, \tag{2}$$

where $Z_j, j \geq 1$ are i.i.d random variables, and N_λ is assumed independent of all $Z_j, j \geq 1$. Since N_λ is independent of $Z_j, j \geq 1$, from the usual rules for conditional expectations, it follows that

$$\mathbb{P}_Z = \sum_{n=1}^{\infty} \mathbb{P}(N_\lambda = n) \mathbb{P}_{\varphi(n) \sum_{j=1}^n Z_j}. \tag{3}$$

The problem to be considered in this paper is to investigate weak limit theorems together with convergence rates for desired normalized geometric random sums of independent, identically distributed (i.i.d.) random variables whose applications are widely in various fields.

Throughout the paper, for $p \in (0, 1)$, let v_p be a geometric random variable with parameter $p \in (0, 1)$, denoted by $v_p \sim Geo(p)$, having probability mass function

$$\mathbb{P}(v_p = j) = p(1 - p)^{j-1}, \quad j = 1, 2, \dots$$

Extending the known definition of φ -decomposability in [5], a concept of φ -Geometrically decomposable (GD) random variables is introduced as follows

Definition 1.1. A random variable Z is said to be φ -Geometric decomposable, if there exist i.i.d. random variables $Z_j, 1 \leq j \leq n$, such that the distribution \mathbb{P}_Z of Z can be represented as

$$\mathbb{P}_Z = \sum_{n=1}^{\infty} \mathbb{P}(v_p = n) \mathbb{P}_{\varphi(n) \sum_{j=1}^n Z_j}. \tag{4}$$

or in equivalent form

$$Z \stackrel{D}{=} \varphi(v_p) \sum_{j=1}^{v_p} Z_j, \tag{5}$$

where $v_p \sim Geo(p), p \in (0, 1)$, independent of all $Z_j, j \geq 1$ for each $p \in (0, 1)$. Here and from now on, the notation $\stackrel{D}{=}$ denotes identity in distribution.

Note that the condition $v_p \xrightarrow{P} +\infty$ as $p \searrow 0^+$ is used for Definition 1.1 in (4) and (5) whose proof will be confirmed by Proposition 4.1 in Appendix. Moreover, condition $\varphi(v_p) \stackrel{a.s.}{=} o(1)$ (equality in almost sure) when $p \searrow 0^+$ is assumed in this paper. It is worth pointing out that the concept of φ -decomposability, defined by Butzer and Schulz in [5] can be extended to concept of φ -Geometrically decomposable random variables since the range of the geometric random variable $v_p, p \in (0, 1)$ is a subset of \mathbb{N} .

The main purpose of this paper is to study weak limit theorems together with convergence rates for randomly normalized geometric random sums $\varphi(v_p) \sum_{j=1}^{v_p} X_j$ when $p \searrow 0^+$. In this article the considered limiting random variables are assumed to be φ -geometrically decomposable (Theorem 3.4) or the standard normal distributed random variable (Theorem 3.6) and the symmetric stable distributed random variables with $1 < \alpha < 2$ and $\sigma = 1$ (Theorem 3.8). Using Zolotarev’s probability metric [33], a general theorem on rate of convergence for normalized geometric random sums $\varphi(v_p) \sum_{j=1}^{v_p} X_j$ is established (Theorem 3.4).

Particularly, when $\varphi(v_p) = v_p^{-1/2}$, the rate of convergence for distribution of normalized geometric random sum $v_p^{-1/2} \sum_{j=1}^{v_p} X_j$ to standard normal distribution will be obtained (Theorem 3.6). Furthermore, for $1 < \alpha < 2$

and $\varphi(v_p) = v_p^{-1/\alpha}$, the rate of convergence for distribution of normalized geometric random sums $v_p^{-1/\alpha} \sum_{j=1}^{v_p} X_j$ towards the stable laws are also established (Theorem 3.8). Moreover, weak limit theorems for normalized geometric random sums are shown directly from Theorems 3.4, 3.6 and 3.8 through corresponding remarks.

It is worth pointing out that, Zolotarev’s probability metric used in our paper is an ideal metric (see for instance [32] and [24]), so it is easy to estimate the approximations concerning with geometric random sums of i.i.d. random variables. Moreover, this metric can be compared with well-known metrics like Kolmogorov metric, total variation metric, Lévy-Prokhorov metric and probability metric based on Trotter’s operator, etc. (see [3], [30], [31], [32], [10] and [11] for more details).

The article is organized as follows. The definition of Zolotarev’s probability metric and some related auxiliary results will be recalled in Section 2. Section 3 is devoted to main results of our paper with detailed proofs. Throughout this paper, we denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers, by $\mathbb{R} = (-\infty, +\infty)$ the set of real numbers, the symbol $\stackrel{a.s.}{=}$ denotes that the identity is almost sure. The notations \xrightarrow{D} and \xrightarrow{P} denote convergence in distribution and convergence in probability, respectively.

2. Zolotarev’s probability metric

Before stating the main results we first recall the definition of Zolotarev’s probability metric and provide some auxiliary results which will be used in this paper. For a deeper discussion of Zolotarev’s probability metric and its applications we refer the reader to [3], [30], [31], [32], [33], and [12]. We denote by \mathfrak{X} the set of all random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Definition 2.1 ([30]). *The mapping $d : \mathfrak{X} \times \mathfrak{X} \rightarrow [0, +\infty]$ (infinite values of d are accepted) is called a probability metric, denoted by $d(X, Y)$, if for all random variables $X, Y, Z \in \mathfrak{X}$, the following statements hold:*

1. $\mathbb{P}(X = Y) = 1 \implies d(X, Y) = 0$;
2. $d(X, Y) = d(Y, X)$;
3. $d(X, Y) \leq d(X, Z) + d(Z, Y)$.

Definition 2.2 ([30]). *A metric d is called simple if its values are determined by a pair of marginal distributions P_X and P_Y .*

It should be noted that, for a simple metric the following forms are equivalent

$$d(X, Y) = d(P_X, P_Y) = d(F_X, F_Y).$$

According to Zolotarev [30, p. 418], if d is a simple metric, then

$$d(X, Y) = 0 \iff X \stackrel{D}{=} Y.$$

Definition 2.3 ([30]). A simple metric d is called an ideal probability metric of order $s \geq 0$ on \mathfrak{X} , if for all $X, Y, Z \in \mathfrak{X}$, the following statements hold:

1. Regularity:

$$d(X + Z, Y + Z) \leq d(X, Y),$$

for Z independent of X and Y .

2. Homogeneity of order s :

$$d(cX, cY) = |c|^s d(X, Y),$$

for any $c \neq 0$.

We denote by $C(\mathbb{R})$ the set of all real-valued, bounded, uniformly continuous functions defined on \mathbb{R} , with norm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$. Furthermore, for $r \in \mathbb{N}$, $\beta \in (0, 1]$ and $s = r + \beta$, let us set

$$C^r(\mathbb{R}) = \{f \in C(\mathbb{R}) : f^{(k)} \in C(\mathbb{R}), 1 \leq k \leq r\},$$

and

$$\mathfrak{D}_s = \{f \in C^r(\mathbb{R}) : |f^{(r)}(x) - f^{(r)}(y)| \leq |x - y|^\beta\}.$$

Definition 2.4 ([30]). Let $X, Y \in \mathfrak{X}$. Zolotarev's probability metric on \mathfrak{X} between two random variables X and Y , denoted by $d_Z(X, Y)$, is defined by

$$d_Z(X, Y) = \sup_{f \in \mathfrak{D}_s} \left| \mathbb{E}[f(X) - f(Y)] \right|.$$

Remark 2.5 ([30]).

1. Zolotarev's probability metric $d_Z(X, Y)$ on \mathfrak{X} is an ideal metric of order s , i.e., for any $c \neq 0$, and for $X, Y, Z \in \mathfrak{X}$,

$$d_Z(X + Z, Y + Z) \leq d_Z(X, Y),$$

for Z independent of X and Y , and

$$d_Z(cX, cY) = |c|^s d_Z(X, Y).$$

2. Let $d_Z(X_n, \bar{X}) \rightarrow 0$ as $n \rightarrow \infty$. Then $X_n \xrightarrow{D} \bar{X}$ as $n \rightarrow \infty$. (see for instance [30], p. 424).

The following lemmas are useful in next section.

Lemma 2.6. Let $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ be two sequences of independent random variables that are identically distributed within each sequence. Then,

$$d_Z \left(\sum_{j=1}^n X_j, \sum_{j=1}^n Y_j \right) \leq n \cdot d_Z(X_1, Y_1). \tag{6}$$

Proof. The proof is immediate from [33, Corollary 1.4.1, page 36]. \square

Lemma 2.7. Let $X, Y \in \mathfrak{X}$ with $\mathbb{E}|X|^s < +\infty$ and $\mathbb{E}|Y|^s < +\infty$, for $s = r + \beta, \beta \in (0, 1]$ and $r \in \mathbb{N}$. Then

$$d_Z(X, Y) \leq \sum_{k=1}^r \frac{M_k}{k!} \left| \mathbb{E}(X^k) - \mathbb{E}(Y^k) \right| + \frac{\theta^\beta}{r!} (\mathbb{E}|X|^s + \mathbb{E}|Y|^s),$$

where $M_k = \sup_{f \in \mathfrak{D}_s} |f^{(k)}(0)|$.

Proof. According to Taylor series expansion for a function $f \in \mathfrak{D}_s$ and $x \in \mathbb{R}$ with Lagrange remainder (see for instance [27], page 110), we have

$$\begin{aligned} f(x) &= f(0) + \sum_{k=1}^{r-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(r)}(\theta x)}{r!} x^r \\ &= f(0) + \sum_{k=1}^{r-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(r)}(0)}{r!} x^r + \frac{f^{(r)}(\theta x)}{r!} x^r - \frac{f^{(r)}(0)}{r!} x^r \\ &= f(0) + \sum_{k=1}^r \frac{f^{(k)}(0)}{k!} x^k + \frac{x^r}{r!} [f^{(r)}(\theta x) - f^{(r)}(0)], \end{aligned}$$

where $0 < \theta < 1$.

Thus, for any $x, y \in \mathbb{R}$, it follows that

$$\begin{aligned} f(x) - f(y) &= \sum_{k=1}^r \frac{f^{(k)}(0)}{k!} (x^k - y^k) + \frac{x^r}{r!} [f^{(r)}(\theta x) - f^{(r)}(0)] - \frac{y^r}{r!} [f^{(r)}(\theta y) - f^{(r)}(0)] \\ &\leq \sum_{k=1}^r \frac{f^{(k)}(0)}{k!} (x^k - y^k) + \frac{|x|^r}{r!} |f^{(r)}(\theta x) - f^{(r)}(0)| + \frac{|y|^r}{r!} |f^{(r)}(\theta y) - f^{(r)}(0)| \\ &\leq \sum_{k=1}^r \frac{f^{(k)}(0)}{k!} (x^k - y^k) + \frac{|x|^r}{r!} |\theta x|^\beta + \frac{|y|^r}{r!} |\theta y|^\beta \\ &= \sum_{k=1}^r \frac{f^{(k)}(0)}{k!} (x^k - y^k) + \frac{\theta^\beta}{r!} (|x|^s + |y|^s). \end{aligned}$$

Therefore,

$$\begin{aligned} d_Z(X, Y) &= \sup_{f \in \mathfrak{D}_s} \left| \mathbb{E}[f(X) - f(Y)] \right| \\ &\leq \sup_{f \in \mathfrak{D}_s} \sum_{k=1}^r \frac{|f^{(k)}(0)|}{k!} \left| \mathbb{E}(X^k) - \mathbb{E}(Y^k) \right| + \frac{\theta^\beta}{r!} (\mathbb{E}|X|^s + \mathbb{E}|Y|^s) \\ &= \sum_{k=1}^r \frac{M_k}{k!} \left| \mathbb{E}(X^k) - \mathbb{E}(Y^k) \right| + \frac{\theta^\beta}{r!} (\mathbb{E}|X|^s + \mathbb{E}|Y|^s), \end{aligned}$$

where $M_k = \sup_{f \in \mathfrak{D}_s} |f^{(k)}(0)|$.

Based on boundedness of $\mathbb{E}|X|^s$ and $\mathbb{E}|Y|^s$, since $f \in \mathfrak{D}_s$, it follows that $d_Z(X, Y)$ is finite. The proof is straightforward. \square

Lemma 2.8. Let $X, Y \in \mathfrak{X}$ with $\mathbb{E}|X| < +\infty$ and $\mathbb{E}|Y| < +\infty$. Then,

$$d_Z(X, Y) \leq \sup_{f \in \mathfrak{D}_2} \|f'\| \cdot (\mathbb{E}|X| + \mathbb{E}|Y|),$$

where $\|f'\| = \sup_{w \in \mathbb{R}} |f'(w)|$.

Proof. For any $f \in C^1(\mathbb{R})$, on account of Mean Value Theorem ([27], page 107), for z is between x and y , we have

$$f(x) - f(y) = (x - y)f'(z).$$

Since $f \in C^1(\mathbb{R})$ and for all $z \in \mathbb{R}$, one has

$$|f'(z)| \leq \sup_{w \in \mathbb{R}} |f'(w)| = \|f'\|.$$

Then, we infer that, for $f \in C^1(\mathbb{R})$,

$$f(x) - f(y) \leq |x - y| |f'(z)| \leq \|f'\|(|x| + |y|).$$

Therefore,

$$d_Z(X, Y) = \sup_{f \in \mathcal{D}_2} \left| \mathbb{E}[f(X) - f(Y)] \right| \leq \sup_{f \in \mathcal{D}_2} \|f'\| (\mathbb{E}|X| + \mathbb{E}|Y|).$$

The proof is immediate. \square

3. Main results

The following proposition is one of the important properties of geometric random variable which will be used in the sequel.

Proposition 3.1. *Let v_p be a geometric random variable with parameter $p \in (0, 1)$. Then, for any $\gamma \in (0, 1)$,*

$$\mathbb{E}(v_p^{-\gamma}) \leq \frac{p^\gamma}{1-p} \Gamma(1-\gamma),$$

where $\Gamma(x), x > 0$ is Gamma function (see for instance [27], Definition 8.17, page 192).

Proof. It is plain that

$$\mathbb{E}(v_p^{-\gamma}) = \sum_{k=1}^{\infty} \mathbb{P}(v_p = k) k^{-\gamma} = \left(\frac{p}{1-p} \right) \sum_{k=1}^{\infty} (1-p)^k k^{-\gamma}.$$

Let g be a decreasing, positive and continuous function on $(0, +\infty)$. Then, we shall begin with showing that

$$\sum_{k=1}^{\infty} g(k) \leq \int_0^{\infty} g(x) dx.$$

Consider

$$g(x) = (1-p)^x x^{-\gamma},$$

a decreasing, positive and continuous function on $(0, +\infty)$.

Taking $(1-p)^x = e^{-t}$, yields

$$\begin{aligned} \sum_{k=1}^{\infty} (1-p)^k k^{-\gamma} &\leq \int_0^{\infty} (1-p)^x x^{-\gamma} dx = \left(\frac{1}{\ln(1/(1-p))} \right)^{1-\gamma} \int_0^{\infty} e^{-t} t^{-\gamma} dt \\ &= \left(\frac{1}{\ln(1/(1-p))} \right)^{1-\gamma} \Gamma(1-\gamma), \end{aligned}$$

where $\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$ is Gamma function.

Thus,

$$\mathbb{E}(v_p^{-\gamma}) \leq \left(\frac{p}{1-p}\right) \left(\frac{1}{\ln \frac{1}{1-p}}\right)^{1-\gamma} \Gamma(1-\gamma).$$

Since $\ln\left(\frac{1}{1-p}\right) > p$ for all $p \in (0, 1)$, we get

$$\mathbb{E}(v_p^{-\gamma}) \leq \left(\frac{p}{1-p}\right) \left(\frac{1}{p}\right)^{1-\gamma} \Gamma(1-\gamma) = \left(\frac{p^\gamma}{1-p}\right) \Gamma(1-\gamma).$$

The proof is straight-forward. \square

Remark 3.2. It is clear that with $\gamma = 1/2$, since $\Gamma(1/2) = \sqrt{\pi}$, we have

$$\mathbb{E}(v_p^{-1/2}) \leq \frac{p^{1/2} \sqrt{\pi}}{1-p}.$$

We follow the notation used in [25]. A random variable Y is said to be *symmetric stable distributed random variable* with parameters $\alpha \in (0, 2]$ and $\sigma > 0$, denoted by $Y \sim S(\alpha, \sigma)$, if its characteristic function is defined as follows

$$f_Y(t) = \exp\{-\sigma^\alpha |t|^\alpha\}, \quad t \in \mathbb{R}. \tag{7}$$

Proposition 3.3. Let $Y \sim S(\alpha, \sigma)$ with $\alpha \in (0, 2]$ and $\sigma > 0$. Then Y is a φ -geometrically decomposable random variable.

Proof. Let Y_1, Y_2, \dots be a sequence of i.i.d. random variables copied from Y . The characteristic function of sum $\sum_{j=1}^n Y_j$ is defined by

$$f_{\sum_{j=1}^n Y_j}(t) = \mathbb{E}\left(e^{it(Y_1+Y_2+\dots+Y_n)}\right) = \left[\mathbb{E}\left(e^{itY}\right)\right]^n = \exp\{-n\sigma^\alpha |t|^\alpha\}, \quad \text{for } t \in \mathbb{R}.$$

Thus, characteristic function of the sum $n^{-1/\alpha} \sum_{j=1}^n Y_j$ will be given by

$$f_{n^{-1/\alpha} \sum_{j=1}^n Y_j}(t) = \mathbb{E}\left[\exp\left\{itn^{-1/\alpha} \sum_{j=1}^n Y_j\right\}\right] = f_{\sum_{j=1}^n Y_j}(n^{-1/\alpha}t) = \exp\{-\sigma^\alpha |t|^\alpha\}, \quad \text{for } t \in \mathbb{R}.$$

Let us consider the normalized function $\varphi(v_p) = v_p^{-1/\alpha}$. Then the characteristic function of the normalized geometric sums $v_p^{-1/\alpha} \sum_{j=1}^{v_p} Y_j$ is given by

$$\begin{aligned} f_{v_p^{-1/\alpha} \sum_{j=1}^{v_p} Y_j}(t) &= \mathbb{E}\left[\exp\left\{itv_p^{-1/\alpha} \sum_{j=1}^{v_p} Y_j\right\}\right] \\ &= \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{P}(v_p = n) \exp\left\{itn^{-1/\alpha} \sum_{j=1}^n Y_j\right\}\right] \\ &= \sum_{n=1}^{\infty} \mathbb{P}(v_p = n) \mathbb{E}\left[\exp\left\{itn^{-1/\alpha} \sum_{j=1}^n Y_j\right\}\right] \\ &= \sum_{n=1}^{\infty} \mathbb{P}(v_p = n) \exp\{-\sigma^\alpha |t|^\alpha\} = f_Y(t), \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

This finishes the proof. \square

The general theorem on rate of convergence for normalized geometric random sums $\varphi(v_p) \sum_{j=1}^{v_p} X_j$ will be established as follows

Theorem 3.4. *Let X, X_1, X_2, \dots be a sequence of i.i.d. random variables with $\mathbb{E}|X|^s < +\infty$ for $1 \leq s \leq r, r \in \mathbb{N}$. Let v_p be a geometric random variable with mean $1/p, p \in (0, 1)$, independent of all X_j for $j \geq 1$. Assume that Y is a φ -geometrically decomposable random variable with $\mathbb{E}|Y|^s < +\infty$ for $s = r + \beta, \beta \in (0, 1]$ and $r \in \mathbb{N}$. Then,*

$$d_Z\left(\varphi(v_p) \sum_{j=1}^{v_p} X_j, Y\right) \leq \mathbb{E}\left\{[\varphi(v_p)]^{v_p}\right\} \left\{ \sum_{k=1}^r \frac{M_k}{k!} |\mathbb{E}X^k - \mathbb{E}Y^k| + \frac{\theta^\beta}{r!} (\mathbb{E}|X|^s + \mathbb{E}|Y|^s) \right\},$$

where $s = r + \beta, \beta \in (0, 1], r \in \mathbb{N}$ and $M_k = \sup_{f \in \mathfrak{D}_s} |f^{(k)}(0)|$.

Proof. Since Y is a φ -geometrically decomposable random variable, so there exist the i.i.d. random variables Y_1, Y_1, \dots are copied from Y , such that

$$Y \stackrel{D}{=} \varphi(v_p) \sum_{j=1}^{v_p} Y_j.$$

Based on ideality of Zolotarev’s probability metric of order s , according to Lemma 2.6 and Lemma 2.7, we have

$$\begin{aligned} d_Z\left(\varphi(v_p) \sum_{j=1}^{v_p} X_j, Y\right) &= d_Z\left(\varphi(v_p) \sum_{j=1}^{v_p} X_j, \varphi(v_p) \sum_{j=1}^{v_p} Y_j\right) \\ &= \sum_{n=1}^{\infty} \left\{ \mathbb{P}(v_p = n) d_Z\left(\varphi(n) \sum_{j=1}^n X_j, \varphi(n) \sum_{j=1}^n Y_j\right) \right\} \\ &\leq \sum_{n=1}^{\infty} \left\{ \mathbb{P}(v_p = n) [\varphi(n)]^s n d_Z(X_1, Y_1) \right\} = \mathbb{E}\left\{[\varphi(v_p)]^{v_p}\right\} d_Z(X, Y) \\ &\leq \mathbb{E}\left\{[\varphi(v_p)]^{v_p}\right\} \left\{ \sum_{k=1}^r \frac{M_k}{k!} |\mathbb{E}X^k - \mathbb{E}Y^k| + \frac{\theta^\beta}{r!} (\mathbb{E}|X|^s + \mathbb{E}|Y|^s) \right\}, \end{aligned}$$

where $M_k = \sup_{f \in \mathfrak{D}_s} |f^{(k)}(0)|$. The proof is complete. \square

On account of Theorem 3.4, the limit theorem for normalized geometric random sums will be formulated as follows

Remark 3.5. *Suppose that*

$$\mathbb{E}\left\{[\varphi(v_p)]^{v_p}\right\} = o(1) \quad \text{as } p \searrow 0^+.$$

Then

$$\varphi(v_p) \sum_{j=1}^{v_p} X_j \xrightarrow{D} Y, \quad \text{as } p \searrow 0^+.$$

From now on, let X^* be a standard normal distributed random variable, denoted by $X^* \sim N(0, 1)$, with the characteristic function

$$f_{X^*}(t) = \exp\left\{-\frac{t^2}{2}\right\}, \quad t \in \mathbb{R}.$$

It is easily seen that, $\mathbb{E}(X^*) = 0, \mathbb{E}(X^{*2}) = 1$ and $\mathbb{E}(|X^*|^3) = 4/\sqrt{2\pi}$. Moreover, when $\alpha = 2$ and $\sigma = 1/\sqrt{2}$, the symmetric stable laws reduce to the standard normal distribution. Hence, according to Proposition 3.3, the X^* is a φ -geometrically decomposable random variable.

Theorem 3.6. *Let X, X_1, X_2, \dots be a sequence of i.i.d. random variables with moments $\mathbb{E}X = 0, \mathbb{E}X^2 = 1$ and $\mathbb{E}|X|^3 = \rho < +\infty$. Let v_p be a geometric random variable with mean $p^{-1}, p \in (0, 1)$, independent of X and $X_j, j \geq 1$. Then,*

$$d_Z\left(v_p^{-1/2} \sum_{j=1}^{v_p} X_j, X^*\right) \leq \frac{p^{1/2} \sqrt{\pi}}{1-p} \left(\frac{\rho}{2} + \frac{2}{\sqrt{2\pi}}\right).$$

Proof. Applying to Theorem 3.4 for normalized function $\varphi(v_p) = v_p^{-1/2}, r = 2, \beta = 1$ and $s = r + \beta = 3$, we obtain

$$\begin{aligned} d_Z\left(v_p^{-1/2} \sum_{j=1}^{v_p} X_j, X^*\right) &\leq \mathbb{E}\left[\left(v_p^{-1/2}\right)^3 v_p\right] \left\{\frac{\theta}{2!}(\mathbb{E}|X|^3 + \mathbb{E}|X^*|^3)\right\} \\ &\leq \mathbb{E}\left(v_p^{-1/2}\right) \left(\frac{\rho}{2} + \frac{2}{\sqrt{2\pi}}\right). \end{aligned}$$

According to Remark 3.1, one has

$$\mathbb{E}\left(v_p^{-1/2}\right) \leq \frac{p^{1/2} \sqrt{\pi}}{1-p}.$$

Therefore,

$$d_Z\left(v_p^{-1/2} \sum_{j=1}^{v_p} X_j, X^*\right) \leq \frac{p^{1/2} \sqrt{\pi}}{1-p} \left(\frac{\rho}{2} + \frac{2}{\sqrt{2\pi}}\right).$$

The proof has completed. \square

It is worth pointing out that a type of Central limit theorem for normalized geometric random sums is formulated immediate from Theorem 3.6 as follows

Remark 3.7. *On account of Theorem 3.6, since $\frac{p^{1/2} \sqrt{\pi}}{1-p} = o(1)$ as $p \searrow 0^+$, it follows that*

$$d_Z\left(v_p^{-1/2} \sum_{j=1}^{v_p} X_j, X^*\right) \rightarrow 0 \quad \text{as } p \searrow 0^+$$

Hence

$$v_p^{-1/2} \sum_{j=1}^{v_p} X_j \xrightarrow{D} X^* \sim N(0, 1), \quad \text{as } p \searrow 0^+.$$

Theorem 3.8. Let X, X_1, X_2, \dots be a sequence of i.i.d. random variables with $\mathbb{E}|X| = \varrho < +\infty$. Let v_p be a geometric random variable with parameter $p \in (0, 1)$, independent of all $X_j, j \geq 1$. Assume that Y is a symmetric stable distributed random variable with $1 < \alpha < 2$ and $\sigma = 1$. Then,

$$d_Z\left(v_p^{-1/\alpha} \sum_{j=1}^{v_p} X_j, Y\right) \leq \frac{p^{\frac{2-\alpha}{\alpha}}}{1-p} \Gamma\left(\frac{2\alpha-2}{\alpha}\right) \sup_{f \in \mathfrak{D}_2} \|f'\| \left\{ \varrho + \frac{2}{\sqrt{\pi}} \Gamma\left(1 - \frac{1}{\alpha}\right) \right\}.$$

Proof. According to Proposition 3.3, since $Y \sim S(\alpha, 1)$, we have

$$Y \stackrel{D}{=} v_p^{-1/\alpha} \sum_{j=1}^{v_p} Y_j,$$

where Y, Y_1, Y_2, \dots are i.i.d. random variables and they are independent of v_p .

On account of ideality of Zolotarev’s probability metric of order s , from Lemma 2.6, it follows that

$$\begin{aligned} d_Z\left(v_p^{-1/\alpha} \sum_{j=1}^{v_p} X_j, Y\right) &= d_Z\left(v_p^{-1/\alpha} \sum_{j=1}^{v_p} X_j, v_p^{-1/\alpha} \sum_{j=1}^{v_p} Y_j\right) \\ &= \sum_{n=1}^{\infty} \left\{ \mathbb{P}(v_p = n) d_Z\left(n^{-1/\alpha} \sum_{j=1}^n X_j, n^{-1/\alpha} \sum_{j=1}^n Y_j\right) \right\} \\ &= \sum_{n=1}^{\infty} \left\{ \mathbb{P}(v_p = n) n^{-s/\alpha} d_Z\left(\sum_{j=1}^n X_j, \sum_{j=1}^n Y_j\right) \right\} \\ &\leq \sum_{n=1}^{\infty} \left\{ \mathbb{P}(v_p = n) n^{\frac{\alpha-s}{\alpha}} d_Z(X_1, Y_1) \right\} = \mathbb{E} \left\{ v_p^{\frac{\alpha-s}{\alpha}} \right\} d_Z(X, Y). \end{aligned}$$

According to Lemma 2.8 with $r = 1, \beta = 1$ and $s = 2$, we have

$$d_Z(X, Y) \leq \sup_{f \in \mathfrak{D}_2} \|f'\| \cdot (\mathbb{E}|X| + \mathbb{E}|Y|).$$

Since $Y \sim S(\alpha, 1)$ with $1 < \alpha < 2$, according to [19, Corollary 5, page 305], then

$$\mathbb{E}|Y| = \frac{2}{\sqrt{\pi}} \Gamma\left(1 - \frac{1}{\alpha}\right).$$

Moreover, based on hypothesis that $\mathbb{E}|X| = \varrho < +\infty$, we obtain

$$d_Z(X, Y) \leq \sup_{f \in \mathfrak{D}_2} \|f'\| \left\{ \varrho + \frac{2}{\sqrt{\pi}} \Gamma\left(1 - \frac{1}{\alpha}\right) \right\}.$$

Furthermore, according to Proposition 3.1 for $\gamma = \frac{2-\alpha}{\alpha}$, since $0 < \frac{2-\alpha}{\alpha} < 1$ with $1 < \alpha < 2$, it follows that

$$\mathbb{E} \left\{ v_p^{\frac{\alpha-2}{\alpha}} \right\} \leq \frac{p^{\frac{2-\alpha}{\alpha}}}{1-p} \Gamma\left(1 - \frac{2-\alpha}{\alpha}\right) = \frac{p^{\frac{2-\alpha}{\alpha}}}{1-p} \Gamma\left(\frac{2\alpha-2}{\alpha}\right).$$

Therefore,

$$d_Z\left(v_p^{-1/\alpha} \sum_{j=1}^{v_p} X_j, Y\right) \leq \frac{p^{\frac{2-\alpha}{\alpha}}}{1-p} \Gamma\left(\frac{2\alpha-2}{\alpha}\right) \sup_{f \in \mathfrak{D}_2} \|f'\| \left\{ \varrho + \frac{2}{\sqrt{\pi}} \Gamma\left(1 - \frac{1}{\alpha}\right) \right\}.$$

The proof is complete. \square

Remark 3.9. According to Remark 2.5, a weak limit theorem of geometric random sums may be concluded from Theorem 3.8 as follows

$$v_p^{-1/\alpha} \sum_{j=1}^{v_p} X_j \xrightarrow{D} Y \sim S(\alpha, 1) \quad \text{as } p \searrow 0.$$

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4. Appendix

Proposition 4.1. For any $p \in (0, 1)$, let v_p be a geometric random variable with mean $1/p$. Then,

$$v_p \xrightarrow{P} +\infty, \quad \text{as } p \searrow 0^+,$$

i. e. for any $M > 0$

$$\lim_{p \searrow 0^+} \mathbb{P}(v_p > M) = 1.$$

Proof. Since $v_p \sim \text{Geo}(p)$, it follows that for $M > 0$ is sufficiently large, we have

$$\begin{aligned} \mathbb{P}(v_p > M) &= 1 - \mathbb{P}(v_p \leq M) = 1 - \sum_{j=1}^M p(1-p)^{j-1} \\ &= 1 - \frac{p}{1-p} \sum_{j=1}^M (1-p)^j. \end{aligned}$$

It is clear that,

$$\sum_{j=1}^M (1-p)^j = \frac{(1-p)[1 - (1-p)^M]}{1 - (1-p)} = \frac{1-p}{p} [1 - (1-p)^M]$$

Then,

$$\mathbb{P}(v_p > M) = 1 - \frac{p}{1-p} \cdot \frac{1-p}{p} [1 - (1-p)^M] = (1-p)^M \rightarrow 1 \quad \text{as } p \searrow 0^+.$$

The proof is immediate. \square

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