Filomat 33:10 (2019), 3061–3072
https://doi.org/10.2298/FIL1910061Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The New Revisitation of Core EP Inverse of Matrices

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Abstract. In this paper, we study some properties of *core-EP inverse* of square matrices. Firstly, we extend the obtained theorem proved by K.M. Prasad and K.S. Mohana. Then some properties of core-EP inverse have been given, through applying the conditions $(AX)^* = AX$, $XA^{k+1} = A^k$ and $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$. Secondly, we get some characterizations of *core-EP inverse* by employing the conditions $AX = P_{A^k}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^{k+1})^*A)}$. Finally, we get some properties of *core-EP inverse* by utilizing the condition $A^{k+1}X = A^kP_{A^k}$.

1. Introduction

Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices. For $A \in \mathbb{C}^{m\times n}$, the symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* and $r(A)$ denote the *range space*, *null space*, *conjugate transpose* and *rank* of A, respectively. Moreover, the identity matrix of order n is denoted by I_n.

Let $A \in \mathbb{C}^{m\times n}$. The unique matrix $X \in \mathbb{C}^{n\times m}$, which satisfying the following conditions:

$$(i)\ AXA = A, \quad (ii)\ XAX = X, \quad (iii)\ (AX)^* = AX, \quad (iv)\ (XA)^* = XA,$$

is called the *Moore-Penrose* inverse of A and written by $A^\dagger$ [1] . If a matrix $X \in \mathbb{C}^{n\times m}$ only satisfies the equality $AXA = A$, then X is an inner inverse of A and we denote it by A^- [1] . Moreover, we denote the class of all inner inverse of A by $A^{(1)}$ [1] . A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies $XAX = X$ is an outer inverse of A and we denote it by $A^{(2)}$ [1] . Furthermore, let P and L be two complementary subspaces in $\mathbb{C}^n$. If the matrix X satisfies the following conditions:

$$(i)\ X \in A^{(2)}, \quad (ii)\ \mathcal{R}(X) = \mathcal{R}(P), \quad (iii)\ \mathcal{N}(X) = \mathcal{N}(L),$$

then X is denoted by $A^{(2)}_{P,L}$ [9] .

Here, we mainly consider the square matrices. The smallest nonnegative integer k, which satisfies $r(A^k) = r(A^{k+1})$, is called the index of A and we denote it as $Ind(A)$. Furthermore, the set of all index 1 matrices also known as core matrices is denoted by $\mathbb{C}^{CM}_n$. The matrix A satisfying $\mathcal{R}(A^*) = \mathcal{R}(A)$, is called *EP-matrix* and set of *EP-matrices* in $\mathbb{C}^{n\times n}$ is $\mathbb{C}^{EP}_n$. The matrix A satisfying $A^2 = A$, is called idempotent matrix and it is denoted by $\mathbb{C}^P_n$. For $A \in \mathbb{C}^{n\times n}$, if a matrix $X \in \mathbb{C}^{n\times n}$ satisfies the following three conditions:

$$(i)\ XAX = X, \quad (ii)\ XA = AX, \quad (iii)\ XA^{k+1} = A^k, \text{for some positive integer } k,$$

2010 *Mathematics Subject Classification*. 15A09
Keywords. core-EP inverse, core-EP decomposition of matrices, core inverse, the orthogonal projection of matrices.
Received: 29 October 2018; Accepted: 03 January 2019
Communicated by Dragana Cvetković-Ilić
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then X is called the *Drazin inverse* of A and written as A^D [1] . Moreover, when $Ind(A) \leq 1$, the matrix X is known as the *group inverse* and noticed by $A^\sharp$ [1] . More on Drazin inverses and generalized Drazin inverses see in [10, 17]. For two complementary subspaces $L, M \in \mathbb{C}^n$, that satisfy $L \bigoplus M = \mathbb{C}^n$. $P_{L,M}$ is said to be the oblique projector onto L along M. Additionally, if M is the subspace orthogonal to L, we denote the orthogonal projection onto L by $P_{L,M}$. For $A \in \mathbb{C}^{n\times n}$, P_A is said to be the orthogonal projection onto $\mathcal{R}(A)$, i.e. $P_A = AA^\dagger$.

Baksalary and Trenker [6] introduced the *core inverse* on the set $\mathbb{C}_n^{CM}$: For $A \in \mathbb{C}_n^{CM}$, the core inverse of A is defined to be the unique matrix X such that

$$AX = P_A \text{ and } \mathcal{R}(X) \subseteq \mathcal{R}(A),$$

and written as $A^{\textcircled{\#}}$. Moreover, three kinds of generalizations of the core inverses [3,19,22] were given for $n\times n$ complex matrices, called *core-EP inverse*, *BT-inverse*, and *DMP-inverse*, respectively. In order to introduce these inverses, we assume that $A \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. Firstly, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X \text{ and } \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

is called the core-EP inverse of A and noticed by $A^{\textcircled{\dagger}}$ [22]. It is clear that $A^{\textcircled{\dagger}} \in \mathbb{C}_n^{EP}$. Secondly, for $A \in \mathbb{C}^{n\times n}$, the *DMP-inverse* of A, written by $A^{D,\dagger}$ [19], is defined as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X, \quad XA = A^DA \text{ and } A^kX = A^kA^\dagger.$$

Finally, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$X = (AP_A)^\dagger,$$

is called the BT-inverse of A and noticed by $A^\diamond$ [3].

In 2017, Drazin introduced (B, C)-inverse on the ring. And then Benitez and Boasso et. al [7] researched (B, C)-inverse on the set $\mathbb{C}^{m\times n}$.

Definition 1.1. [7] *Let $A \in \mathbb{C}^{m\times n}$ and $B, C \in \mathbb{C}^{n\times m}$. The matrix A is said to be (B, C)-invertible, if there exist a matrix $X \in \mathbb{C}^{n\times m}$, satisfying the following conditions:*

$$XAB = B, \quad CAX = C, \quad \mathcal{N}(X) = \mathcal{N}(C) \text{ and } \mathcal{R}(X) = \mathcal{R}(B).$$

Furthermore, the matrix X is called (B, C)-inverse of A. And the matrix X is unique.

In [24] , Wang introduced core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ as follows:

Lemma 1.2. [24] *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then A can be written as the sum of matrices A_1 and A_2, i.e. $A = A_1 + A_2$, where*
(i) $A_1 \in \mathbb{C}_n^{CM}$;
(ii) $A_2^k = 0$;
(iii) $A_1^*A_2 = A_2A_1 = 0$.
Moreover, A_1 is core partial and A_2 is nilpotent partial. Then we notice k is nilpotent index, moreover the nilpotent index of A_2 is equal to index of the matrix A.

Moreover, Wang [24] researched characterization of core-EP decomposition by using Schur lemma.

Lemma 1.3. [24] *Let the core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ be as in Lemma 1.2. Then there exists the unitary matrix U such that*

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*, \tag{1}$$

where T is non-singular and N is nilpotent. Moreover, A_1 *and* A_2 *can be represented by*

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \text{ and } A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{2}$$

and A^k *can be given by*

$$A^k = U\begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}U^*, \tag{3}$$

where $\widetilde{S} = \sum_{i=0}^{k-1} T^i S N^{k-i}$.

Furthermore, in [24] , some characterizations of *core-EP inverse* were introduced.

Lemma 1.4. [24] *Let* $A \in \mathbb{C}^{n\times n}$ *with* $Ind(A) = k$*, and let the core-EP decomposition of A be as in Lemma 1.3. Then*

$$A^{\textcircled{\dagger}} = A_1^{\textcircled{\#}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

In [24] , the author obtained the relationship between *core-EP inverse* and *core inverse* by using the core-EP decomposition. In [12] , some representations for *core-EP inverse* have been given.

In this paper we are concerned with some properties of *core-EP inverse* of square matrices by using core-EP decomposition. In Section 2, some necessary and sufficient conditions for *core-EP inverse* will be given by using the conditions $(AX)^* = AX$, $XA^{k+1} = A^k$ and $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$. In Section 3, we derived necessary and sufficient condition for *core-EP inverse* by using the conditions $AX = P_{A^k}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^{k+1})^*A)}$. In Section 4, we get some properties of *core-EP inverse* by utilizing the condition $A^{k+1}X = A^kP_{A^k}$. Also, we devoted a new representation of the core-EP inverse. Then by the definition of (B, C)-inverse, we get a result that the *core-EP inverse* is a specific (B, C)-inverse.

2. Some revisitations about *core-EP inverse*

In [22] , Prasad and Mohana defined *core-EP inverse* for square matrices and presented some properties. In the following lemma, we provide one of its properties:

Lemma 2.1. [22] *Let* A, $X \in \mathbb{C}^{n\times n}$ *and* $Ind(A) = k$*. Then X is core-EP inverse of A if and only if X satisfies the following four conditions:*

$$XA^{k+1} = A^k, \quad XAX = X, \quad (AX)^* = AX, \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k).$$

Whereas, we will see that the condition $XAX = X$ is superfluous for Lemma 2.1. Therefore, we have the following theorem.

Theorem 2.2. *Let* A, $X \in \mathbb{C}^{n\times n}$ *and* $Ind(A) = k$*. Then X is core-EP inverse of A if and only if X satisfies the conditions:*

$$XA^{k+1} = A^k \quad (AX)^* = AX, \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k).$$

Proof. Suppose that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$ and $XA^{k+1} = A^k$. We have $X = A^kT$, for some $T \in \mathbb{C}^{n\times n}$. Hence,

$$XAX = XA^{k+1}T = A^kT = X.$$

□

According to Theorem 2.2, we can find that X satisfies $XA^{k+1} = A^k$ and $(AX)^* = AX$, where X is *core-EP inverse* of A. Therefore, in the following theorem, we get some necessary and sufficient conditions about *core-EP inverse* by $(AX)^* = AX$ and $XA^{k+1} = A^k$. To prove the theorem, we need the following lemma.

Lemma 2.3. *Let $A,\ X \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. Suppose that the core-EP decomposition of A is given by $A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*$. If $XA^{k+1} = A^k$, then X can be written as $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$, where X_2 and X_4 are arbitrary.*

Proof. We assume that A can be written as (1). Then A^k can be written by (3). Assume that $X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*$. If X satisfies $XA^{k+1} = A^k$, we get

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} T^{k+1} & T\widetilde{S} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} X_1T^{k+1} & X_1T\widetilde{S} \\ X_3T^{k+1} & X_3T\widetilde{S} \end{bmatrix} = \begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}.$$

Therefore, we can obtain the following equalities

$$X_1T^{k+1} = T^k, \tag{4}$$

$$X_1T\widetilde{S} = \widetilde{S}, \tag{5}$$

$$X_3T^{k+1} = 0, \tag{6}$$

$$X_3T\widetilde{S} = 0. \tag{7}$$

It can be easily seen that $X_1 = T^{-1}$ and $X_3 = 0$. Therefore, X can be written as $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$. □

Theorem 2.4. *Let $A,\ X \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. If $A = A_1 + A_2$ is the core-EP decomposition of A(from Lemma 1.2), where A_1 is core partial of A and A_2 is nilpotent partial of A. Then the following conditions are equivalent:*

(i) $X = A^{\oplus}$;
(ii) $(AX)^* = AX$, $XA^{k+1} = A^k$ and $r(A^k) = r(X)$;
(iii) $(AX)^* = AX$, $XA^{k+1} = A^k$ and $A_1X^2 = X$;
(iv) $(AX)^* = AX$, $XA^{k+1} = A^k$ and $A^sX^{s+1} = X$, for some positive integer s;
(v) $(AX)^* = AX$, $XA^{k+1} = A^k$ and $XA_1X = X$.

Proof. The proofs of (i)⇒(ii), (i)⇒(iii), (i)⇒(iv) and (i)⇒(v) are a direct consequences of Lemma 1.4.

(ii)⇒(i) If $XA^{k+1} = A^k$ then $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$. Since $r(A^k) = r(X)$, we get $\mathcal{R}(A^k) = \mathcal{R}(X)$. Hence due to Theorem 2.2, we have $X = A^{\oplus}$.

(iii)⇒(i) Suppose that $X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*$. Let A be of the form (1). Moreover according to Lemma 1.3, it follows that $A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*$ and $A^k = U\begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}U^*$. From Lemma 2.3, we set that X can be given by $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$. By $A_1X^2 = X$, we obtain

$$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix} = \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}, \tag{8}$$

$$\begin{bmatrix} T^{-1} & X_2 + TX_2X_4 + SX_4^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}. \tag{9}$$

So we get $X_4 = 0$. From $(AX)^* = AX$ we get $X_2 = 0$. Hence $X = A^{\oplus}$.

(iv)⇒(i) By $XA^{k+1} = A^k$, we obtain $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$. Also, by $A^sX^{s+1} = X$, we have that

$$\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}^s \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}^{s+1} = \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} T^s & \widetilde{S} \\ 0 & N^s \end{bmatrix}\begin{bmatrix} T^{-(s+1)} & \widetilde{M} \\ 0 & X_4^{s+1} \end{bmatrix} = \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}, \tag{11}$$

where $\widetilde{S} = \sum_{i=0}^{s-1} T^iSN^{s-i}$ and $\widetilde{M} = \sum_{i=0}^{s} T^{-i}X_2X_4^{s-i}$. So we get

$$\begin{bmatrix} T^{-1} & T^s\widetilde{M} + \widetilde{S}X_4^{s+1} \\ 0 & N^sX_4^{s+1} \end{bmatrix} = \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}$$

Hence $N^sX_4^{s+1} = X_4$. If $s \geq k$, then it is easy to check that $X_4 = 0$. If $s < k$, then $X_4 = N^sX_4^{s+1} = N^{2s}X_4^{2s+1} = \cdots = N^{ks}X_4^{ks+1}$. Due to $N^k = 0$, we obtain $X_4 = 0$. Thus, we have $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & 0 \end{bmatrix}U^*$. Then, from $(AX)^* = AX$ we get $X_2 = 0$. Therefore, $X = A^{\oplus}$.

The proof of (v)⇒(i) is the same as (iii)⇒(i). □

The following example shows that we can't lead to $X = A^{\oplus}$ by substituting $XAX = X$ for $XA_1X = X$ in Theorem 2.4(v).

Example 2.5. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{and } X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to see that $Ind(A) = 2$. *Also, we can verify that* X *satisfies* $(AX)^* = AX$, $XA^3 = A^2$ *and* $XAX = X$. *However, we can easily check that* $\mathcal{R}(X) \neq \mathcal{R}(X^*) \neq \mathcal{R}(A^k)$. *Therefore,* $X \neq A^{\oplus}$.

In [22] , the author introduced the definition of *core-EP inverse* which satisfies $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$ and $XAX = X$. In the following theorem, we show the other properties of *core-EP inverse* by using the condition $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$. To prove the theorem, we need the following lemma.

Lemma 2.6. *Let* $A,\ X \in \mathbb{C}^{n\times n}$ *and* $Ind(A) = k$. *Let the core-EP decomposition of* A *be given by* $A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*$. *If* $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, *then* X *can be expressed as* $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$, *where* X_1 *is invertible.*

Proof. By Lemma 1.3, we have that A^k can be written by (3). Let X be given by $X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*$. Due to $\mathcal{R}(A^k) = \mathcal{R}(X)$, then there exists a matrix Y satisfying $X = A^kY$. So we divide the matrix Y into four blocks as $Y = U\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^*$. From $X = A^kY$, we get

$$U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^* = U\begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}U^*,$$

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} T^kY_1 + \widetilde{S}Y_3 & T^kY_2 + \widetilde{S}Y_4 \\ 0 & 0 \end{bmatrix}.$$

Thus, $X_3 = 0$ and $X_4 = 0$. Similarly, according to $\mathcal{R}(X^*) = \mathcal{R}(A^k)$, it follows that $X_2 = 0$. Thus the matrix X can be written as $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$, where X_1 is invertible. □

Theorem 2.7. *Let $A,\ X \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. Then the following are equivalent:*
(i) $X = A^{\textcircled{\dagger}}$;
(ii) $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$ and $XA^{k+1} = A^k$;
(iii) $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$ and $A^sX^{s+1} = X$, for some positive integer s;
(iv) $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$ and $XA \in \mathbb{C}_n^P$;
(v) $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$ and $AX \in \mathbb{C}_n^P$;
(vi) $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$ and $AXA^k = A^k$.

Proof. The proofs of (i)⇒(ii), (i)⇒(iii), (i)⇒(iv), (i)⇒(v) and (i)⇒(vi) are trivial by Lemma 1.4.

(ii)⇒(i), (iii)⇒(i), and (vi)⇒(i). We suppose that A is written as in (1). Then $A^k = U\begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}U^*$. Assume that $X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*$. By Lemma 2.6, we have $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, which yields $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$, where X_1 is invertible. Then following the other conditions, we can obtain $X_1 = T^{-1}$ by a direct calculation. Hence we have $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\textcircled{\dagger}}$.

(iv)⇒(i) By Lemma 2.6, we have $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$, where X_1 is invertible. Due to $XA \in \mathbb{C}_n^P$, we get

$$\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix},$$

$$\begin{bmatrix} X_1TX_1T & X_1TX_1S \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1T & X_1S \\ 0 & 0 \end{bmatrix}.$$

Thus we have $X_1TX_1T = X_1T$. Since X_1 and T are invertible, we conclude $X_1T = I$. Then we have $X_1 = T^{-1}$. Hence we have $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\textcircled{\dagger}}$.

The proof of (v)⇒(i) is similar to the proof of (iv)⇒(i). □

Remark 2.8. *We obtain $X = A^{\textcircled{\dagger}}$ by substituting A_1 for A in Theorem 2.7 (ii)-(vi).*

3. Some properties of *core-EP inverse* under the condition $AX = P_{A^k}$

In this section, we mainly show several characterizations of *core-EP inverse* by applying some properties in [12]. Ferreyra, Levis, et. al [12] have presented some new characterizations about the *core-EP inverse* of a square matrix. In the following lemma, the important conclusion from [12] is given:

Lemma 3.1. [12] *Let $A, X \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. Then X is core-EP inverse of A if and only if $AX = P_{A^k}$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$.*

From this lemma we can obtain $AX = P_{A^k}$, if the matrix X is *core-EP inverse* of the square matrix A. Therefore, in the following theorem, we get some necessary and sufficient conditions about *core-EP inverse* by taking $AX = P_{A^k}$ into consideration.

Theorem 3.2. *Let $A,\ X \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. And $A = A_1 + A_2$ is the core-EP decomposition of A(from Lemma 1.1), where A_1 is core partial of A and A_2 is nilpotent partial of A. Then the following conditions are equivalent:*
(i) $X = A^{\textcircled{\dagger}}$;
(ii) $AX = P_{A^k}$ and $AX^2 = X$;
(iii) $AX = P_{A^k}$ and $A_1X^2 = X$;
(iv) $AX = P_{A^k}$, $X \in \mathbb{C}_n^{EP}$ and $XAX = X$.

Proof. The proofs of (i)⇒(ii) and (i)⇒(iii) are trivial by applying Lemma 1.4.

(i)⇒(iv) By Lemma 3.1, we have $AX = P_{A^k}$. And due to the definition of core-EP inverse, we can conclude $\mathcal{R}(X) = \mathcal{R}(X^*)$ and $XAX = X$. Also, since $\mathcal{R}(X) = \mathcal{R}(X^*)$, we get $X \in \mathbb{C}_n^{EP}$.

(ii)⇒(i) If $AX = P_{A^k}$, then by $AX^2 = X$ we conclude $P_{A^k}X = X$. Also, we have $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. Hence from Lemma 3.1 we have $X = A^{\oplus}$.

(iii)⇒(i) Assume that $X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*$. Let A be as in (1). Then A_1 can be written as the equality (2) and write $A^k = U\begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}U^*$. Due to Lemma 3.1, we have

$$P_{A^k} = AA^{\oplus} = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Therefore, we have $P_{A^k} = U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*$. From $A_1X^2 = X$, we obtain

$$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

By a direct calculation, we get

$$\begin{bmatrix} TX_1^2 + SX_3X_1 + TX_2X_3 + SX_4X_3 & TX_1X_2 + SX_3X_2 + TX_2X_4 + SX_4^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix},$$

which yield

$$TX_1^2 + SX_3X_1 + TX_2X_3 + SX_4X_3 = X_1, \tag{12}$$

$$TX_1X_2 + SX_3X_2 + TX_2X_4 + SX_4^2 = X_2, \tag{13}$$

$$X_3 = 0, \tag{14}$$

$$X_4 = 0. \tag{15}$$

Therefore, $X_3 = 0$ and $X_4 = 0$. So the matrix X is of the form

$$X = U\begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}U^*.$$

Since $AX = P_{A^k}$, we have

$$\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

So we get $X_1 = T^{-1}$ and $X_2 = 0$. Above all, we conclude $X = A^{\oplus}$.

(iv)⇒(i) According to $AX = P_{A^k}$, we get $XAX = X$, which yields $XP_{A^k} = X$. So we have $\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ and $\begin{bmatrix} X_1 & 0 \\ X_3 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$. Therefore, we obtain $X_2 = 0$ and $X_4 = 0$. According to $X \in \mathbb{C}_n^{EP}$, we have $X_3 = 0$. Then since $AX = P_{A^k}$, it follows that $X = A^{\oplus}$. □

In Theorem 3.2 (iii), $X \in \mathbb{C}_n^{EP}$ is necessary to check $X = A^{\oplus}$. In the following example, we will demonstrate it.

Example 3.3. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{and } X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can easily check that $Ind(A) = 2$. *Also,* X *satisfies* $AX = P_{A^2}$ *and* $XAX = X$. *However, we have* $\mathcal{R}(X) \neq \mathcal{R}(X^*)$ *by a direct calculation. So* $X \neq A^{\oplus}$.

Furthermore, in [12], the authors have provided other properties of *core-EP inverse*. In the following lemma, one of properties in [12] will be recalled.

Lemma 3.4. [12] *Let* $A \in \mathbb{C}^{n\times n}$ *and* $Ind(A) = k$. *Then,*
(i) $AA^{\oplus} = P_{A^k}$;
(ii) $A^{\oplus}A = P_{\mathcal{R}(A^k),\mathcal{N}((A^{k+1})^*A)}$.

Moreover, combining the equality (ii) in Lemma 3.3 with the condition $\mathcal{N}((A^k)^*A) = \mathcal{N}((A^{k+1})^*A)$ lead to $A^{\oplus}A = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$. It is clear that $AX = P_{A^k}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ by $X = A^{\oplus}$. However, the conditions $AX = P_{A^k}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ can't deduce that $X = A^{\oplus}$.

Example 3.5. *Let*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{and } X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can easy to check that $Ind(A) = 2$. *Also,* X *satisfies* $AX = P_{A^2}$ *and* $XA = P_{\mathcal{R}(A^2),\mathcal{N}((A^2)^*A)}$. *However, we have* $\mathcal{R}(X) \neq \mathcal{R}(X^*) \neq \mathcal{R}(A^2)$ *by a direct calculation. So* $X \neq A^{\oplus}$.

From Example 3.5, we see that the matrix X only satisfying $AX = P_{A^k}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ can not be *core-EP inverse* of A. Consequently, from the following theorem, we obtain various representions of *core-EP inverse* by $AX = P_{A^k}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$.

Theorem 3.6. *Let* $A,\ X \in \mathbb{C}^{n\times n}$ *and* $Ind(A) = k$. *If* $A = A_1 + A_2$ *is the core-EP decomposition of A(from Lemma 1.2), where* A_1 *is core partial of A and* A_2 *is nilpotent partial of A. Then the following conditions are equivalent:*
(i) $X = A^{\oplus}$;
(ii) $AX = P_{A^k}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ *and* $XAX = X$;
(iii) $AX = P_{A^k}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ *and* $r(X) = r(A^k)$;
(iv) $AX = P_{A^k}$, $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ *and* $XA_1X = X$;
(v) $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$, $X \in \mathbb{C}_n^{EP}$ *and* $XAX = X$.

Proof. The proofs of (i)⇒(ii), (i)⇒(iii) and (i)⇒(iv) can be showed by using Lemma 3.3 and Lemma 2.1.

(i)⇒(v) If (i) holds, by Lemma 3.4 we have $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$. By the definition of core-EP inverse, we have $\mathcal{R}(X) = \mathcal{R}(X^*)$ and $XAX = X$. Also, $X \in \mathbb{C}_n^{EP}$ by $\mathcal{R}(X) = \mathcal{R}(X^*)$.

(ii)⇒(i) Write $X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*$. Suppose that the core-EP decomposition of A be as in (1), then A_1 and A^k can be written as equalities (2) and (3). From $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$, we can obtain $XA^{k+1} = A^k$. From the following Lemma 2.3, the matrix X can be written as $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$. Then due to $AX = P_{A^k}$ and $XAX = X$, we have $XP_{A^k} = X$. So it can be easily checked that $X_2 = 0$ and $X_4 = 0$. Therefore, $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\oplus}$.

(iii)⇒(i) From (ii)⇒(i) and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$, the matrix X can be expressed as $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$. As $r(X) = r(A^k)$, we obtain $X_4 = 0$. For $AX = P_{A^k}$, we have the following equality

$$\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & X_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{16}$$

So $X_2 = 0$. Thus, $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\oplus}$.

(iv)⇒(i) From (ii)⇒(i) and combining $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$, the matrix X can be written as $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$. And thanks to $XA_1X = X$, we have

$$\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix} = \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix},$$

$$\begin{bmatrix} T^{-1} & X_2 + T^{-1}SX_4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix},$$

So we have $X_4 = 0$. Then following (i)⇒(iii), we have $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\oplus}$.

(v)⇒(i) Due to $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$, we have $XA^{k+1} = A^k$. Then X can be written as $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$ by Lemma 2.2. And due to $XAX = X$, we have $\mathcal{R}(X) \subseteq \mathcal{R}(XA)$. According to $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$, we have $\mathcal{R}(XA) = \mathcal{R}(A^k)$. It follows that $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. So we have $X_4 = 0$. By applying $X \in \mathbb{C}_n^{EP}$, we have $\mathcal{R}(X) = \mathcal{R}(X^*)$. So it is simple to show that $X_2 = 0$. And the matrix X can be written as $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*$. Then from Lemma 1.3, we can obtain $X = A^{\oplus}$. □

4. The other characterizations for *core-EP inverse*

In this section, we provide other properties of *core-EP inverse*. In fact, the following theorem will be useful to obtain the other revisitations of *core-EP inverse*.

Theorem 4.1. *Let $A,\ X \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. Then the following conditions are equivalent:*
(i) $X = A^{\oplus}$;
(ii) $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $XP_{A^k} = X$;
(iii) $XA^{k+1} = A^k$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $XP_{A^k} = X$;
(iv) $XA^{k+1} = A^k$ and $\mathcal{R}(X^*) = \mathcal{R}(A^k)$;
(v) $X = P_{A^k}X = XP_{A^k}$ and $P_{A^k} = XAP_{A^k}$;
(vi) $X = P_{A^k}X = XP_{A^k}$ and $P_{A^k} = P_{A^k}AX$.

Proof. The proof of (i)⇒(iv) is easy to check, by Lemma 1.4.

By Lemma 1.3, we have $P_{A^k} = U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*$, where $r(I) = r(A^k)$. Then we can check (i)⇒(ii), (i)⇒(iii), (i)⇒(v) and (i)⇒(vi), by using the definition of *core-EP inverse*.

(ii)⇒(i) A can be written as the equality (1). Also, A^k can be written as the equality (2). Partitioning of X as $X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*$ conformable for matrix with the partition of A. We know $P_{A^k} = U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*$. And since $XP_{A^k} = X$, we conclude the following equalities

$$U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{17}$$

$$\begin{bmatrix} X_1 & 0 \\ X_3 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \tag{18}$$

Hence $X_2 = 0$ and $X_4 = 0$. Then from Theorem 2.7 (ii)⇒(i), we have $X_3 = 0$ by $\mathcal{R}(X) = \mathcal{R}(A^k)$. So the matrix X can be written as $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$, where the matrix X_1 is invertible. Therefore, for $XAX = X$, we get $X_1 = T^{-1}$ by a simple calculation. Therefore, $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\oplus}$.

(iii)⇒(i) The proof is similar to (ii)⇒(i).

(iv)⇒(i) Following Lemma 2.3, the matrix X can be written as $X = U\begin{bmatrix} T^{-1} & X_2 \\ 0 & X_4 \end{bmatrix}U^*$. Then we have $X_2 = 0$ and $X_4 = 0$ by using $\mathcal{R}(X^*) = \mathcal{R}(A^k)$. Above all, the matrix X can be written as $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*$. So $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\oplus}$.

(v)⇒(i) As $P_{A^k} = U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*$, then we get $X_2 = 0$ and $X_4 = 0$ by $XP_{A^k} = X$. So we obtain $X = U\begin{bmatrix} X_1 & 0 \\ X_3 & 0 \end{bmatrix}U^*$. Thus from $X = P_{A^k}X$, we have $X_3 = 0$. Hence $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$. From $P_{A^k} = XAP_{A^k}$, we have

$$U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1T & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $X_1T = I$. Note that the matrix T is invertible, so we obtain $X_1 = T^{-1}$. Thus we get the conclusion that $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\oplus}$.

(vi)⇒(i) The proof is similar to (v)⇒(i). □

In the previous arguments, we always concern on the condition $XA^{k+1} = A^k$. Now, we will take the condition $A^{k+1}X = A^kP_{A^k}$ into consideration. According to Lemma 3.1, we can obtain $A^{k+1}X = A^kP_{A^k}$ by $X = A^{\oplus}$. Therefore, in the following theorem, we will present some characterizations in reference to *core-EP inverse* by using the condition $A^{k+1}X = A^kP_{A^k}$.

Theorem 4.2. *Let $A,\ X \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. And $A = A_1 + A_2$ is the core-EP decomposition of A(from Lemma 1.1), which A_1 is core partial of A and A_2 is nilpotent partial of A. Then the following conditions are equivalent:*
(i) $X = A^{\oplus}$;
(ii) $A^{k+1}X = A^kP_{A^k}$ and $\mathcal{R}(X) = \mathcal{R}(A^k)$;
(iii) $A^{k+1}X = A^kP_{A^k}$ and $P_{A^k}X = X$;
(iv) $A^{k+1}X = A^kP_{A^k}$ and $A_1X^2 = X$;
(v) $A^{k+1}X = A^kP_{A^k}$ and $AX^2 = X$.

Proof. (i)⇒(ii)-(v) are obvious by Lemma 1.4 and Lemma 3.1.

(ii)⇒(i) Let $A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*$. Then we have $A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*$ and $A^k = U\begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}U^*$. We suppose that $X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*$. For $A^{k+1}X = A^kP_{A^k}$, we have

$$U\begin{bmatrix} T^{k+1} & T\widetilde{S} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^* = U\begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

$$\begin{bmatrix} T^{k+1}X_1 + T\widetilde{S}X_3 & T^{k+1}X_2 + T\widetilde{S}X_4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T^k & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, we obtain the following equalities:

$$T^{k+1}X_1 + T\widetilde{S}X_3 = T^k, \tag{19}$$

$$T^{k+1}X_2 + T\widetilde{S}X_4 = 0. \tag{20}$$

Then from Lemma 2.6 and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we have $X_3 = 0$ and $X_4 = 0$. By equalities (19) and (20), we have $T^{k+1}X_1 = T^k$ and $T^{k+1}X_2 = 0$. Since the matrix T is invertible, we can get $X_1 = T^{-1}$ and $X_2 = 0$. So we have $X = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = A^{\oplus}$.

(iii)⇒(i) By $A^{k+1}X = A^kP_{A^k}$, we get equalities (19) and (20). Then, by $P_{A^k}X = X$, we have

$$U\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^* = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*,$$

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix}.$$

Hence $X_3 = 0$ and $X_4 = 0$. Then we can obtain $X = A^{\oplus}$ by (ii)⇒(i).

(iv)⇒(i) From (ii)⇒(i) we have equalities (19) and (20). According to $A_1X^2 = X$, we have

$$U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^* = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*,$$

$$\begin{bmatrix} TX_1^2 + SX_3X_1 + TX_2X_4 + SX_4X_3 & TX_1X_2 + SX_3X_2 + TX_2X_4 + SX_4^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Hence we obtain $X_3 = 0$ and $X_4 = 0$. Then the following is same as (ii)⇒(i).

(v)⇒(ii) By $AX^2 = X$, we conclude $X = AX^2 = A^2X^3 = \cdots = A^kX^{k+1}$. Thus, we have $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, which implies $P_{A^k}X = X$. Hence (ii) holds. □

Then, in the following theorem, a simpler version of characterization about *core-EP inverse* by comparing with the matrix of Theorem 2.4 will be provided.

Theorem 4.3. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $Ind(A) = k$. *Then* $A^{\oplus} = A^{\{2\}}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}$.

Proof. Let X satisfies $X = A^{\{2\}}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$, then the theorem will be proved by showing that $X = A^{\oplus}$. From $X = A^{\{2\}}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$, we obtain $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $\mathcal{N}(X) = \mathcal{N}((A^k)^*)$. For any $A \in \mathbb{C}^{n\times n}$, we always have $\mathcal{N}(A) = \mathcal{R}(A^*)^{\perp}$. Therefore, we get $\mathcal{N}((A^k)^*) = \mathcal{R}(A^k)^{\perp}$ and $\mathcal{N}(X) = \mathcal{R}(X^*)^{\perp}$. Then, according to $\mathcal{N}(X) = \mathcal{N}((A^k)^*)$, we get $\mathcal{R}(A^k)^{\perp} = \mathcal{R}(X^*)^{\perp}$. Taking the complementary subspace of both sides, we have $\mathcal{R}(A^k) = \mathcal{R}(X^*)$. Therefore, we get $\mathcal{R}(A^k) = \mathcal{R}(X) = \mathcal{R}(X^*)$ and $XAX = X$. So $X = A^{\oplus}$. □

According to the definition of (B, C)-inverse and *core-EP inverse*, we can show that *core-EP inverse* is a specific (B, C)-inverse of A.

Theorem 4.4. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $Ind(A) = k$, *then core-EP inverse of A is a specific* (B, C)*-inverse of A, where* $B = A^k$ *and* $C = (A^*)^k$.

Proof. Suppose that the core-EP decomposition of A is $A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*$. Then A^k can be written as $A^k = U\begin{bmatrix} T^k & \widetilde{S} \\ 0 & 0 \end{bmatrix}U^*$. By applying Lemma 1.4, we get $A^{\oplus} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*$. Thus it can be easily verified that $A^{\oplus}A^{k+1} = A^k$ and $(A^*)^kAA^{\oplus} = (A^*)^k$. By the definition of *core-EP inverse* we have $\mathcal{R}(A^{\oplus}) = \mathcal{R}(A^k)$ and $\mathcal{N}(A^{\oplus}) = \mathcal{N}((A^*)^k)$. Therefore, according to the uniqueness of (B, C)-inverse, we obtain that core-EP inverse of A is a specific (B, C)-inverse of A, where $B = A^k$ and $C = (A^*)^k$. □

Remark 4.5. *Moreover, if the matrix $A \in \mathbb{C}^{n\times n}$ satisfies $Ind(A) \leq 1$, then $A^{\textcircled{\#}} = A^{\textcircled{\dagger}}$. All of the results obtained in this paper generalize the relevant ones in* [16] .

Acknowledgments. The authors would like to thank the referees and Professor Dragana Cvetković-Ilić for their helpful suggestions to the improvement of this paper.

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