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On the Schwarz Lemma at the Upper Half Plane

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Abstract. In this paper, we give a simple proof for the boundary Schwarz lemma at the upper half plane. Considering that $f(z)$ is a holomorphic function defined on the upper half plane, we derive inequalities for the modulus of derivative of $f(z)$, $\left|f'(0)\right|$, by assuming that the $f(z)$ function is also holomorphic at the boundary point $z = 0$ on the real axis with $f(0) = \Re f(i)$.

1. Introduction

The most classical version of the Schwarz Lemma examines the behavior of a bounded, holomorphic function mapping the origin to origin in the unit disc $E = \{w : |w| < 1\}$. It is possible to see its effectiveness in the proofs of many important theorems. The Schwarz Lemma which has quite wide application area and is the direct application of the maximum modulus principle is given in the most basic form as follow [5]:

Let E be the unit disc in the complex plane $\mathbb{C}$. Let $f : E \to E$ be a holomorphic function with $f(0) = 0$. Under these conditions, $\left|f(w)\right| \leq |w|$ for all $w \in E$ and $\left|f'(0)\right| \leq 1$. In addition, if the equality $\left|f(w)\right| = |w|$ holds for any $w \neq 0$, or $\left|f'(0)\right| = 1$, then f is a rotation; that is $f(w) = we^{i\theta}$, θ real. The Schwarz lemma is one of the most important results in the classical complex analysis,which has become a crucial theme in many branches of mathematical research for over a hundred years [13, 15]. Also, in [14], they gave simple proofs of various versions of the Schwarz lemma for real-valued harmonic functions and for holomorphic (more generally harmonic quasiregular, shortly (HQR) mappings with the strip codomain.

Let E be the unit disc and $S = \{z \in C : \Im z > 0\}$ the upper half plane in $\mathbb{C}$. For $i \in E$, $\frac{w-i}{1+iw}$ defines a conformal self-map of E carrying i to 0. Similary, for any $i \in S$,

$$w \to \frac{z-i}{z+i}$$

is conformal map of S onto E, i to 0. It follows in particular the S and E are conformal equivalent.

Consider the function

$$f(w) = \frac{f(z) - f(i)}{f(z) - \overline{f(i)}}, \quad w = \frac{z-i}{z+i},$$

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where $f(z) = f(i) + c_1 (z-i) + c_2 (z-i)^2 + ...$

Here, $f(w)$ is a holomorphic function in E, $f(0) = 0$ and $\left|f(w)\right| < 1$ for $w \in E$. Applying the Schwarz lemma for the function $f(w)$, we obtain

$$f'(w) = \frac{\frac{2i}{(1-w)^2} f'\left(i\frac{1+w}{1-w}\right)\left(f(i) - \overline{f(i)}\right)}{\left(f\left(i\frac{1+w}{1-w}\right) - \overline{f(i)}\right)^2},$$

$$\begin{aligned} f'(0) &= \frac{2if'(i)\left(f(i) - \overline{f(i)}\right)}{\left(f(i) - \overline{f(i)}\right)^2} = \frac{2if'(i)}{f(i) - \overline{f(i)}} \\ &= \frac{f'(i)}{\frac{f(i)-\overline{f(i)}}{2i}} = \frac{f'(i)}{\Im f(i)} \end{aligned}$$

and

$$\left|f'(i)\right| \leq \Im f(i).$$

The result is sharp and the extremal function is

$$f(z) = z\Im f(i) + \Re f(i).$$

In this case, it is get the following lemma.

Lemma 1.1. *Let $f : S \to S$ be holomorphic function. Then*

$$\left|f'(i)\right| \leq \Im f(i). \tag{1.1}$$

The inequality (1.1) *is sharp with equality for the function*

$$f(z) = z\Im f(i) + \Re f(i).$$

Consider the product

$$B_0(w) = \prod_{k=1}^{n} \frac{w - w_k}{1 - \overline{w_k} w}.$$

The function $B_0(w)$ is called a finite Blaschke product, where $w_1, w_2, ..., w_n \in E$.

Let

$$\Omega(w) = \frac{f(w)}{B_0(w)} = \frac{f(w)}{\prod\limits_{k=1}^{n} \frac{w-w_k}{1-\overline{w_k}w}},\ w_k = \frac{z_k - i}{z_k + i}.$$

Here, $z_1, z_2, ..., z_k$ are points in the upper haşf plane with $f(z_k) = f(i)$ and $w_1, w_2, ..., w_n$ are zeros $f(w)$. In addition, $\Omega(w)$ is holomorphic function in E, $\Omega(0) = 0$ and $|\Omega(w)| < 1$ for $z \in E$. Therefore, $\Omega(w)$ satisfy the conditions of the Schwarz lemma. Thus, from the Schwarz lemma, we obtain

$$\begin{aligned} \Omega(w) &= \frac{f(z) - f(i)}{f(z) - \overline{f(i)}} \frac{1}{\prod\limits_{k=1}^{n} \frac{w-w_k}{1-\overline{w_k}w}} \\ &= \frac{c_1\left(i\frac{2w}{1-w}\right) + c_2\left(i\frac{2w}{1-w}\right)^2 + ...}{2i\Im f(i) + c_1\left(i\frac{2w}{1-w}\right) + c_2\left(i\frac{2w}{1-w}\right)^2 + ...} \frac{1}{\prod\limits_{k=1}^{n} \frac{w-w_k}{1-\overline{w_k}w}}, \end{aligned}$$

$$\frac{\Omega(w)}{w} = \frac{c_1\left(\frac{2i}{1-w}\right) + c_2\left(\frac{2i}{1-w}\right)^2 w + ...}{2i\Im f(i) + c_1\left(i\frac{2w}{1-w}\right) + c_2\left(i\frac{2w}{1-w}\right)^2 + ...} \frac{1}{\prod_{k=1}^{n} \frac{w-w_k}{1-\overline{w_k}w}},$$

$$|\Omega'(0)| = \frac{|c_1|}{\Im f(i) \prod_{k=1}^{n} |w_k|} \leq 1$$

and

$$|c_1| \leq \Im f(i) \prod_{k=1}^{n} |w_k|.$$

The result is sharp and the extremal function is

$$f(z) = \frac{f(i) - \overline{f(i)}\frac{z-i}{z+i} \prod_{k=1}^{n} \frac{\frac{z-i}{z+i} - \frac{z_k-i}{z_k+i}}{1-\overline{\frac{z_k-i}{z_k+i}}\frac{z-i}{z+i}}}{1 - \frac{z-i}{z+i} \prod_{k=1}^{n} \frac{\frac{z-i}{z+i} - \frac{z_k-i}{z_k+i}}{1-\overline{\frac{z_k-i}{z_k+i}}\frac{z-i}{z+i}}},$$

where $z_1, z_2, ..., z_k$ are positive real numbers.

In this case, it is get the following lemma.

Lemma 1.2. *Let $f : S \to S$ be holomorphic function. Assume that $z_1, z_2, ..., z_k$ are points in the upper half plane with $f(z_k) = f(i)$. Then we have the inequality*

$$|c_1| \leq \Im f(i) \prod_{k=1}^{n} |w_k|.$$

The result is sharp and the extremal function is

$$f(z) = \frac{f(i) - \overline{f(i)}\frac{z-i}{z+i} \prod_{k=1}^{n} \frac{\frac{z-i}{z+i} - \frac{z_k-i}{z_k+i}}{1-\overline{\frac{z_k-i}{z_k+i}}\frac{z-i}{z+i}}}{1 - \frac{z-i}{z+i} \prod_{k=1}^{n} \frac{\frac{z-i}{z+i} - \frac{z_k-i}{z_k+i}}{1-\overline{\frac{z_k-i}{z_k+i}}\frac{z-i}{z+i}}},$$

where $z_1, z_2, ..., z_k$ are positive real numbers.

A significant result of the Schwarz lemma is given by Osserman as follows [16]:

Let $f : E \to E$ be an analytic function with $f(0) = 0$. Assume that there is a $b \in \partial E$ so that f extends continuously to b, $\left|f(b)\right| = 1$ and $f'(b)$ exists. Then

$$\left|f'(b)\right| \geq \frac{2}{1 + \left|f'(0)\right|}. \tag{1.2}$$

Thus, by the classical Schwarz lemma, it follows that

$$\left|f'(b)\right| \geq 1. \tag{1.3}$$

In addition, for $b = 1$ in the inequality (2), equality occurs for the function $f(z) = w\frac{w+\gamma}{1+\gamma w}$, $\gamma \in [0, 1]$. Also, $\left|f'(c)\right| > 1$ unless $f(z) = we^{i\theta}$, θ real.

Chelst, Osserman, Burns and Krantz ([2, 8, 16]) studied the Schwarz lemma at the boundary of the unit disk, respectively. The similar types of results which are related with the subject of the paper can be found in ([9–11]). In addition, the concerning results in more general aspects is discussed by M. Mateljević in [12] where was announced on ResearchGate. In recent years, the Schwarz lemma at the boundary for holomorphic mappings (see,[1], [3], [4], [6], [7],[16] [17], [18], [19], [20], [21], and references therein). Some of them are about the estimates from below for the modulus of the derivative of the function at the boundary points which satisfy the condition $\left|f(b)\right| = 1$. Also, M. Jeong [6] got some inequalities at a boundary point for a different form of holomorphic functions and showed the sharpness of these inequalities. Also, M. Jeong found a necessary and sufficient condition for a holomorphic map to have fixed points only on the boundary of the unit disc and compared its derivatives at fixed points to get some relations among them [7]. For historical background about the Schwarz Lemma and its applications on the boundary of the unit disc, we refer to (see [1], [22]).

2. Main Results

In this section, we give a simple proof for the boundary Schwarz lemma at the upper half plane. Considering that $f(z)$ is a holomorphic function defined on the upper half plane, we derive inequalities for the modulus of derivative of $f(z)$, $\left|f'(0)\right|$, by assuming that the $f(z)$ function is also holomorphic at the boundary point $z = 0$ on the real axis with $f(0) = \Re f(i)$.

Theorem 2.1. *Let $f : S \to S$ be holomorphic function and it is also holomorphic function at the point $z = 0$ of the real axis with $f(0) = \Re f(i)$. Then*

$$\left|f'(0)\right| \geq \Im f(i). \tag{2.1}$$

The inequality (2.1) *is sharp with extremal function*

$$f(z) = z\Im f(i) + \Re f(i).$$

Proof. Let us consider the following function

$$f(w) = \frac{f(z) - f(i)}{f(z) - \overline{f(i)}}, \quad w = \frac{z - i}{z + i}.$$

Then $f(w)$ is holomorphic function in the unit disc E, $f(0) = 0$ and we take $\left|f(w)\right| < 1$ for $|w| < 1$. Also, we have $\left|f(-1)\right| = 1$ for $-1 = b \in \partial E$. That is, since

$$f(0) = \Re f(i) = \frac{f(i) + \overline{f(i)}}{2},$$

we take

$$f(w) = \frac{f(i\frac{1+w}{1-w}) - f(i)}{f(i\frac{1+w}{1-w}) - \overline{f(i)}},$$

$$f(-1) = \frac{f(0) - f(i)}{f(0) - \overline{f(i)}}$$

and

$$\left|f(-1)\right| = 1.$$

Therefore, from (1.3), we obtain

$$f'(w)=\frac{\frac{2i}{(1-w)^2}f'\left(i\frac{1+w}{1-w}\right)\left(f(i)-\overline{f(i)}\right)}{\left(f\left(i\frac{1+w}{1-w}\right)-\overline{f(i)}\right)^2},$$

$$\begin{aligned}1 \leq \left|f'(-1)\right| &= \left|\frac{\frac{1}{2}if'(0)\left(f(i)-\overline{f(i)}\right)}{\left(f(0)-\overline{f(i)}\right)^2}\right| \\ &= \left|\frac{\frac{1}{2}if'(0)\left(f(i)-\overline{f(i)}\right)}{\left(\frac{f(i)-\overline{f(i)}}{2}-\overline{f(i)}\right)^2}\right| \\ &= \frac{\left|f'(0)\right|}{\left|\frac{f(i)-\overline{f(i)}}{2}\right|}=\frac{\left|f'(0)\right|}{\Im f(i)},\end{aligned}$$

and

$$\left|f'(0)\right| \geq \Im f(i).$$

Now we shall show that the inequality (2.1) is sharp. Let

$$f(z)=z\Im f(i)+\Re f(i).$$

Then

$$f'(z)=\Im f(i)$$

and

$$\left|f'(0)\right|=\Im f(i).$$

□

The inequality (2.1) can be strengthened as below by taking into account $c_1=f'(i)$ which is first coeffcient in the expansion of the function $f(z)$.

Theorem 2.2. *Under the hypothesis of the Theorem 2.1. Then*

$$\left|f'(0)\right| \geq \frac{2\left[\Im f(i)\right]^2}{\Im f(i)+\left|f'(i)\right|}. \tag{2.2}$$

The inequality (2.2) *is sharp with extremal function*

$$f(z)=\frac{\left(1-a\frac{z-i}{z+i}\right)f(i)+\left(\left(\frac{z-i}{z+i}\right)^2-a\frac{z-i}{z+i}\right)\overline{f(i)}}{1-2a\frac{z-i}{z+i}+\left(\frac{z-i}{z+i}\right)^2},$$

where $a=\frac{\left|f''(i)\right|}{\Im f(i)}$ *is an arbitrary number from* $[0,1]$ (*see* (1.1)).

Proof. Let $f(w)$ be the same as in the proof of Theorem 2.1. Therefore, from (1.2),

$$\frac{2}{1+\left|f'(0)\right|} \leq \left|f'(-1)\right| = \frac{\left|f'(0)\right|}{\Im f(i)}.$$

Since

$$f'(w) = \frac{\frac{2i}{(1-w)^2} f'\left(i\frac{1+w}{1-w}\right)\left(f(i) - \overline{f(i)}\right)}{\left(f\left(i\frac{1+w}{1-w}\right) - \overline{f(i)}\right)^2}$$

and

$$f'(0) = \frac{f'(i)}{\Im f(i)}$$

it is clear that

$$\left|f'(0)\right| = \frac{\left|f'(i)\right|}{\Im f(i)}.$$

Then

$$\frac{2}{1+\frac{\left|f'(i)\right|}{\Im f(i)}} \leq \frac{\left|f'(0)\right|}{\Im f(i)}$$

and

$$\left|f'(0)\right| \geq \frac{2\left[\Im f(i)\right]^2}{\Im f(i) + \left|f'(i)\right|}.$$

The last inequality shows that the inequality intended is obtained.

Now we shall show that the inequality (2.2) is sharp. Let

$$f(z) = \frac{\left(1 - a\frac{z-i}{z+i}\right) f(i) + \left(\left(\frac{z-i}{z+i}\right)^2 - a\frac{z-i}{z+i}\right)\overline{f(i)}}{1 - 2a\frac{z-i}{z+i} + \left(\frac{z-i}{z+i}\right)^2}.$$

From the last equation, we have

$$f\left(i\frac{1+w}{1-w}\right) = \frac{f(i) + \frac{w^2-aw}{1-aw}\overline{f(i)}}{1 + \frac{w^2-aw}{1-aw}}, \quad w = \frac{z-i}{z+i}.$$

Then

$$\begin{aligned}\frac{2i}{(1-w)^2} f'\left(i\frac{1+w}{1-w}\right) &= \frac{\overline{f(i)}\frac{2w-aw^2-a}{(1-aw)^2}\left(1 + \frac{w^2-aw}{1-aw}\right)}{\left(1 + \frac{w^2-aw}{1-aw}\right)^2} \\ &\quad - \frac{\frac{2w-aw^2-a}{(1-aw)^2}\left(f(i) + \frac{w^2-aw}{1-aw}\overline{f(i)}\right)}{\left(1 + \frac{w^2-aw}{1-aw}\right)^2}\end{aligned}$$

and for $w = -1$,

$$f'(0) = \frac{f(i) - \overline{f(i)}}{2i}\frac{2}{1+a} = \frac{2}{1+a}\Im f(i).$$

Since $a = \frac{\left|f''(i)\right|}{\Im f(i)}$, (2.2) *is satisfied with equality.* □

The inequality (2.2) can be strengthened as below by taking into account $c_2 = \frac{f''(i)}{2!}$ which is second coefficient in the expansion of the function $f(z) = f(i) + c_1(z-i) + c_2(z-i)^2 + ...$

Theorem 2.3. *Let $f : S \to S$ be holomorphic function and it is also holomorphic function at the point $z = 0$ of the real axis with $f(0) = \Re f(i)$. Then*

$$\left|f'(0)\right| \geq \Im f(i)\left(1 + \frac{2\left(\Im f(i) - |c_1|\right)^2}{\left(\Im f(i)\right)^2 - \left|c_1^2\right| + \left|2ic_2\Im f(i) - c_1^2\right|}\right). \tag{2.3}$$

The equality in (2.3) *occurs for the function*

$$f(z) = \frac{f(i) + \left(\frac{z-i}{z+i}\right)^2 \overline{f(i)}}{1 + \left(\frac{z-i}{z+i}\right)^2}.$$

Proof. Let $B(w) = w$, $w \in E$ and $f(w)$ be as in the proof of Theorem 2.1. $B(w)$ is a holomorphic in E and $|B(w)| < 1$ for $|w| < 1$. Maximum principle implies that for each $w \in E$, we have $\left|f(w)\right| < |B(w)|$. Therefore,

$$\phi(w) = \frac{f(w)}{B(w)}$$

is a holomorphic in E and $\left|\phi(w)\right| < 1$ for $|w| < 1$. In particular, we take

$$\begin{aligned}
\phi(w) &= \frac{f(w)}{B(w)} = \frac{f(i\frac{1+w}{1-w}) - f(i)}{\left(f(i\frac{1+w}{1-w}) - \overline{f(i)}\right)w} \\
&= \frac{f(i) + c_1\left(i\frac{1+w}{1-w} - i\right) + c_2\left(i\frac{1+w}{1-w} - i\right)^2 + ... - f(i)}{\left(f(i) + c_1\left(i\frac{1+w}{1-w} - i\right) + c_2\left(i\frac{1+w}{1-w} - i\right)^2 + ... - \overline{f(i)}\right)w} \\
&= \frac{c_1\left(i\frac{1+w}{1-w} - i\right) + c_2\left(i\frac{1+w}{1-w} - i\right)^2 + ...}{\left(f(i) - \overline{f(i)} + c_1\left(i\frac{1+w}{1-w} - i\right) + c_2\left(i\frac{1+w}{1-w} - i\right)^2 + ...\right)w} \\
&= \frac{c_1\left(i\frac{2w}{1-w}\right) + c_2\left(i\frac{2w}{1-w}\right)^2 + c_3\left(i\frac{2w}{1-w}\right)^3 + ...}{\left(2i\Im f(i) + c_1\left(i\frac{2w}{1-w}\right) + c_2\left(i\frac{2w}{1-w}\right)^2 + c_3\left(i\frac{2w}{1-w}\right)^3 + ...\right)w} \\
&= \frac{c_1\left(i\frac{2}{1-w}\right) + c_2\left(i\frac{2}{1-w}\right)^2 w + c_3\left(i\frac{2}{1-w}\right)^3 w^2 + ...}{\left(2i\Im f(i) + c_1\left(i\frac{2w}{1-w}\right) + c_2\left(i\frac{2w}{1-w}\right)^2 + c_3\left(i\frac{2w}{1-w}\right)^3 + ...\right)},
\end{aligned}$$

$$\left|\phi(0)\right| = \frac{|c_1|}{\Im f(i)} \leq 1 \tag{2.4}$$

and

$$\left|\phi'(0)\right| = \frac{\left|2ic_2\Im f(i) - c_1^2\right|}{\left(\Im f(i)\right)^2}.$$

In addition, it can be seen that

$$\frac{bf'(b)}{f(b)} = \left|f'(b)\right| \geq |B'(b)| = \frac{bB'(b)}{B(b)},$$

where $b = -1 \in \partial E$.

The quotient function

$$\Theta(w) = \frac{\phi(w) - \phi(0)}{1 - \overline{\phi(0)}\phi(w)}$$

satisfies the hypothesis of the Schwarz lemma on the boundary, from where we obtain estimate

$$\begin{aligned}
\frac{2}{1 + |\Theta'(0)|} &\leq |\Theta'(-1)| = \frac{1 - \left|\phi(0)\right|^2}{\left|1 - \overline{\phi(0)}\phi(w)\right|^2} \left|\phi'(-1)\right| \\
&\leq \frac{1 + \left|\phi(0)\right|}{1 - \left|\phi(0)\right|} \left|\frac{f'(-1)}{B(-1)} - \frac{f(-1)B'(-1)}{B^2(-1)}\right| \\
&= \frac{1 + \left|\phi(0)\right|}{1 - \left|\phi(0)\right|} \left\{\left|f'(-1)\right| - |B'(-1)|\right\} \\
&= \frac{1 + \left|\phi(0)\right|}{1 - \left|\phi(0)\right|} \left\{\left|f'(-1)\right| - 1\right\}.
\end{aligned}$$

Since

$$\Theta'(w) = \frac{1 - \left|\phi(0)\right|^2}{\left(1 - \overline{\phi(0)}\phi(w)\right)^2}\phi'(w),$$

$$|\Theta'(0)| = \frac{\left|\phi'(0)\right|}{1 - \left|\phi(0)\right|^2} = \frac{\frac{\left|2ic_2\Im f(i) - c_1^2\right|}{\left(\Im f(i)\right)^2}}{1 - \left(\frac{|c_1|}{\Im f(i)}\right)^2} = \frac{\left|2ic_2\Im f(i) - c_1^2\right|}{\left(\Im f(i)\right)^2 - |c_1|^2}$$

and

$$\left|f'(-1)\right| = \frac{\left|f'(0)\right|}{\Im f(i)},$$

we obtain

$$\frac{2}{1 + \frac{\left|2ic_2\Im f(i) - c_1^2\right|}{\left(\Im f(i)\right)^2 - |c_1|^2}} \leq \frac{\Im f(i) + |c_1|}{\Im f(i) - |c_1|} \left\{\frac{\left|f'(0)\right|}{\Im f(i)} - 1\right\},$$

$$\frac{2\left(\left(\Im f(i)\right)^2 - |c_1|^2\right)}{\left(\Im f(i)\right)^2 - \left|c_1^2\right| + \left|2ic_2\Im f(i) - c_1^2\right|} \frac{\Im f(i) - |c_1|}{\Im f(i) + |c_1|} \leq \left\{\frac{\left|f'(0)\right|}{\Im f(i)} - 1\right\},$$

$$\frac{2\left(\Im f(i) - |c_1|\right)^2}{\left(\Im f(i)\right)^2 - \left|c_1^2\right| + \left|2ic_2\Im f(i) - c_1^2\right|} \leq \left\{\frac{\left|f'(0)\right|}{\Im f(i)} - 1\right\}$$

and

$$\left|f'(0)\right| \geq \Im f(i)\left(1 + \frac{2\left(\Im f(i) - |c_1|\right)^2}{\left(\Im f(i)\right)^2 - \left|c_1^2\right| + \left|2ic_2\Im f(i) - c_1^2\right|}\right).$$

Therefore, we get the inequality (2.3).

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z)=\frac{f(i)+\left(\frac{z-i}{z+i}\right)^2\overline{f(i)}}{1+\left(\frac{z-i}{z+i}\right)^2}.$$

Then

$$\left|f'(0)\right|=2\Im f(i).$$

Also, since $c_1=0$ and $c_2=\frac{i}{2}\Im f(i)$, we take

$$\Im f(i)\left(1+\frac{2\left(\Im f(i)-|c_1|\right)^2}{\left(\Im f(i)\right)^2-\left|c_1^2\right|+\left|2ic_2\Im f(i)-c_1^2\right|}\right)=2\Im f(i).$$

□

If $f(z)-f(i)$ has no points different from $z=i$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.4. *Let $f:S\to S$ be holomorphic function. $f(z)-f(i)$ has no points in S except $z=i$ and $c_1>0$ is also holomorphic function at the point $z=0$ of the real axis with $f(0)=\Re f(i)$. Then*

$$\left|f'(0)\right|\geq\Im f(i)\left(1-\frac{2\Im f(i)\,|c_1|\ln^2\left(\frac{|c_1|}{\Im f(i)}\right)}{2\Im f(i)\,|c_1|\ln\left(\frac{|c_1|}{\Im f(i)}\right)-\left|2ic_2\Im f(i)-c_1^2\right|}\right).\tag{2.5}$$

In addition, the result is sharp and the extremal function is

$$f(z)=z\Im f(i)+\Re f(i).$$

Proof. Let $c_1>0$, $f(w)$, $B(w)$ and $\phi(w)$ be as in the proof of Theorem 2.3 and the function $f(z)-f(i)$ has no points in S except $z=i$. Thus, the function $f(w)$ has no more zero than zero. Having in mind inequality (2.4), we denote by $\ln\varphi(z)$ the analytic branch of the logarithm normalized by the condition

$$\ln\phi(0)=\ln\left(\frac{|c_1|}{\Im f(i)}\right)<0,\ \frac{|c_1|}{\Im f(i)}\leq 1.$$

The auxiliary function

$$\Phi(w)=\frac{\ln\phi(w)-\ln\phi(0)}{\ln\phi(w)+\ln\phi(0)}$$

is holomorphic in the unit disc E, $|\Phi(w)|<1$ for $w\in E$, $\Phi(0)=0$ and $|\Phi(-1)|=1$ for $-1\in\partial E$. So, we can apply (1.2) to the function $\Phi(w)$. Since

$$\Phi'(w)=\frac{2\ln\phi(0)}{\left(\ln\phi(w)+\ln\phi(0)\right)^2}\frac{\phi'(w)}{\phi(w)}$$

and

$$\Phi'(-1)=\frac{2\ln\phi(0)}{\left(\ln\phi(-1)+\ln\phi(0)\right)^2}\frac{\phi'(-1)}{\phi(-1)},$$

we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|} &\leq |\Phi'(-1)| = \frac{2\left|\ln\phi(0)\right|}{\left|\ln\phi(-1)+\ln\phi(0)\right|^2}\left|\frac{\phi'(-1)}{\phi(-1)}\right| \\ &= \frac{-2\ln\phi(0)}{\ln^2\phi(0)+\arg^2\phi(-1)}\left|\frac{f'(-1)}{B(-1)}-\frac{f(-1)B'(-1)}{B(-1)^2}\right| \\ &\leq \frac{-2\ln\phi(0)}{\ln^2\phi(0)+\arg^2\phi(-1)}\left\{\left|f'(-1)\right|-|B'(-1)|\right\} \\ &\leq \frac{-2\ln\phi(0)}{\ln^2\phi(0)}\left\{\frac{\left|f'(0)\right|}{\Im f(i)}-1\right\} \\ &= \frac{-2}{\ln\phi(0)}\left\{\frac{\left|f'(0)\right|}{\Im f(i)}-1\right\}.\end{aligned}$$

Since

$$\Phi'(0)=\frac{\phi'(0)}{2\phi(0)\ln\phi(0)},$$

$$|\Phi'(0)|=\frac{\frac{\left|2ic_2\Im f(i)-c_1^2\right|}{\left(\Im f(i)\right)^2}}{2\frac{|c_1|}{\Im f(i)}\left|\ln\left(\frac{|c_1|}{\Im f(i)}\right)\right|}=-\frac{\left|2ic_2\Im f(i)-c_1^2\right|}{2\Im f(i)\,|c_1|\ln\left(\frac{|c_1|}{\Im f(i)}\right)}$$

and therefore, we get

$$\frac{2}{1-\frac{\left|2ic_2\Im f(i)-c_1^2\right|}{2\Im f(i)|c_1|\ln\left(\frac{|c_1|}{\Im f(i)}\right)}}\leq\frac{-2}{\ln\left(\frac{|c_1|}{\Im f(i)}\right)}\left\{\frac{\left|f'(0)\right|}{\Im f(i)}-1\right\}.$$

By getting elementary arrangements, we obtain

$$-\frac{2\Im f(i)\,|c_1|\ln^2\left(\frac{|c_1|}{\Im f(i)}\right)}{2\Im f(i)\,|c_1|\ln\left(\frac{|c_1|}{\Im f(i)}\right)-\left|2ic_2\Im f(i)-c_1^2\right|}\leq\frac{\left|f'(0)\right|}{\Im f(i)}-1$$

and

$$\left|f'(0)\right|\geq\Im f(i)\left(1-\frac{2\Im f(i)\,|c_1|\ln^2\left(\frac{|c_1|}{\Im f(i)}\right)}{2\Im f(i)\,|c_1|\ln\left(\frac{|c_1|}{\Im f(i)}\right)-\left|2ic_2\Im f(i)-c_1^2\right|}\right).$$

Since $|c_1|=\Im f(i)$ is satisfied with equality. □

Theorem 2.5. *Let $f:S\to S$ be holomorphic function and it is also holomorphic function at the point $z=0$ of the real axis with $f(0)=\Re f(i)$. Assume that $i,z_1,z_2,...,z_k$ are points in the upper half plane with $f(z_k)=f(i)$. Then we have the inequality*

$$\left|f'(0)\right|\geq\Im f(i)\left(1+\sum_{k=1}^{n}\frac{\Im z_k}{|z_k|^2}\right. \tag{2.6}$$
$$\left.+\frac{2\left(\Im f(i)\prod_{k=1}^{n}\left|\frac{z_k-i}{z_k+i}\right|-|c_1|\right)^2}{\left(\Im f(i)\prod_{k=1}^{n}\left|\frac{z_k-i}{z_k+i}\right|\right)^2-|c_1|^2+\prod_{k=1}^{n}\left|\frac{z_k-i}{z_k+i}\right|\left|2i\Im f(i)c_2-c_1^2+\Im f(i)c_1\left(1+\sum_{k=1}^{n}\frac{4\Im z_k}{|z_k|^2-2i\Re z_k-1}\right)\right|}\right).$$

Moreover, the equality in (2.6) *occurs for the function*

$$f(z)=\frac{f(i)-\left(\frac{z-i}{z+i}\right)^2\overline{f(i)}\prod\limits_{k=1}^{n}\frac{\frac{z-i}{z+i}-\frac{z_k-i}{z_k+i}}{1-\overline{\frac{z_k-i}{z_k+i}}\frac{z-i}{z+i}}}{1-\left(\frac{z-i}{z+i}\right)^2\prod\limits_{k=1}^{n}\frac{\frac{z-i}{z+i}-\frac{z_k-i}{z_k+i}}{1-\overline{\frac{z_k-i}{z_k+i}}\frac{z-i}{z+i}}},$$

where $z_1, z_2, ..., z_n$ *are positive real numbers.*

Proof. Let

$$f(w)=\frac{f(i\frac{1+w}{1-w})-f(i)}{f(i\frac{1+w}{1-w})-\overline{f(i)}},\quad w=\frac{z-i}{z+i}.$$

Also, $w_1, w_2, ..., w_n$ be the zeros of the function $f(w)$ in E that are different from zero. The function

$$B_1(w)=w\prod_{k=1}^{n}\frac{w-w_k}{1-\overline{w_k}w}$$

is holomorphic in E and $|B_1(w)|<1$ for $z\in E$. From the maximum principle, for each $w\in E$, we have the inequality $\left|f(w)\right|\leq|B_1(w)|$. The composite function

$$\Lambda(w)=\frac{f(w)}{B_1(w)}$$

is holomorphic in E and $|\Lambda(w)|<1$ for $|w|<1$. In particular, we have

$$\begin{aligned}\Lambda(w) &= \frac{c_1\left(i\frac{2w}{1-w}\right)+c_2\left(i\frac{2w}{1-w}\right)^2+...}{2i\Im f(i)+c_1\left(i\frac{2w}{1-w}\right)+c_2\left(i\frac{2w}{1-w}\right)^2+...}\frac{1}{w\prod\limits_{k=1}^{n}\frac{w-w_k}{1-\overline{w_k}w}}\\ &= \frac{c_1\left(\frac{2i}{1-w}\right)+c_2\left(\frac{2i}{1-w}\right)^2w+...}{2i\Im f(i)+c_1\left(i\frac{2w}{1-w}\right)+c_2\left(i\frac{2w}{1-w}\right)^2+...}\frac{1}{\prod\limits_{k=1}^{n}\frac{w-w_k}{1-\overline{w_k}w}},\end{aligned}$$

$$|\Lambda(0)|=\frac{|c_1|}{\Im f(i)\prod\limits_{k=1}^{n}|w_k|}$$

and

$$|\Lambda'(0)|=\frac{\left|2i\Im f(i)c_2-c_1^2+\Im f(i)c_1\left(1+\sum\limits_{k=1}^{n}\frac{1-|w_k|^2}{w_k}\right)\right|}{(\Im f(i))^2\prod\limits_{k=1}^{n}|w_k|}.$$

Moreover, it can be easliy seen that for $b=-1$

$$\frac{bf'(b)}{f(b)}=\left|f'(b)\right|\geq\left|B_1'(b)\right|=\frac{bB_1'(b)}{B_1(b)}$$

and

$$\left|B_1'(-1)\right| = 1 + \sum_{k=1}^{n} \frac{1-|w_k|^2}{|1+w_k|^2}.$$

The auxiliary function

$$\Upsilon(w) = \frac{\Lambda(w) - \Lambda(0)}{1 - \overline{\Lambda(0)}\Lambda(w)}$$

is holomorphic in E and $|\Upsilon(w)| < 1$ for $|w| < 1$, $\Upsilon(0) = 0$ and $|\Upsilon(b)| = 1$ for $-1 = b \in \partial E$. From (1.2), we obtain

$$\begin{aligned} \frac{2}{1+|\Upsilon'(0)|} &\leq |\Upsilon'(-1)| = \frac{1-|\Lambda(0)|^2}{\left|1-\overline{\Lambda(0)}\Lambda(-1)\right|^2}\,|\Lambda'(-1)| \\ &\leq \frac{1+|\Lambda(0)|}{1-|\Lambda(0)|}\left\{\left|f'(-1)\right| - \left|B_1'(-1)\right|\right\}. \end{aligned}$$

Since

$$\Upsilon'(w) = \frac{1-|\Lambda(0)|^2}{\left(1-\overline{\Lambda(0)}\Lambda(w)\right)^2}\Lambda'(w)$$

and

$$\begin{aligned} |\Upsilon'(0)| &= \frac{|\Lambda'(0)|}{1-|\Lambda(0)|^2} = \frac{\frac{\left|2i\Im f(i)c_2 - c_1^2 + \Im f(i)c_1\left(1+\sum_{k=1}^{n}\frac{1-|w_k|^2}{w_k}\right)\right|}{\left(\Im f(i)\right)^2\prod_{k=1}^{n}|w_k|}}{1-\left(\frac{|c_1|}{\Im f(i)\prod_{k=1}^{n}|w_k|}\right)^2} \\ &= \prod_{k=1}^{n}|w_k|\,\frac{\left|2i\Im f(i)c_2 - c_1^2 + \Im f(i)c_1\left(1+\sum_{k=1}^{n}\frac{1-|w_k|^2}{w_k}\right)\right|}{\left(\Im f(i)\prod_{k=1}^{n}|w_k|\right)^2 - |c_1|^2}, \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{2}{1+\prod_{k=1}^{n}|w_k|\frac{\left|2i\Im f(i)c_2-c_1^2+\Im f(i)c_1\left(1+\sum_{k=1}^{n}\frac{1-|w_k|^2}{w_k}\right)\right|}{\left(\Im f(i)\prod_{k=1}^{n}|w_k|\right)^2-|c_1|^2}} \\ &\leq \frac{\Im f(i)\prod_{k=1}^{n}|w_k|+|c_1|}{f(i)\prod_{k=1}^{n}|w_k|-|c_1|}\left\{\frac{\left|f'(0)\right|}{\Im f(i)} - 1 - \sum_{k=1}^{n}\frac{1-|w_k|^2}{|1+w_k|^2}\right\}, \\ &\frac{2\left(\left(\Im f(i)\prod_{k=1}^{n}|w_k|\right)^2-|c_1|^2\right)}{\left(\Im f(i)\prod_{k=1}^{n}|w_k|\right)^2-|c_1|^2+\prod_{k=1}^{n}|w_k|\left|2i\Im f(i)c_2-c_1^2+\Im f(i)c_1\left(1+\sum_{k=1}^{n}\frac{1-|w_k|^2}{w_k}\right)\right|} \\ &\leq \frac{\Im f(i)\prod_{k=1}^{n}|w_k|+|c_1|}{f(i)\prod_{k=1}^{n}|w_k|-|c_1|}\left\{\frac{\left|f'(0)\right|}{\Im f(i)} - 1 - \sum_{k=1}^{n}\frac{1-|w_k|^2}{|1+w_k|^2}\right\} \end{aligned}$$

and

$$\left|f'(0)\right| \geq \Im f(i)\left(1 + \sum_{k=1}^{n}\frac{1-|w_k|^2}{|1+w_k|^2}\right.$$

$$+\frac{2\left(\Im f(i)\prod_{k=1}^{n}|w_k|-|c_1|\right)^2}{\left(\Im f(i)\prod_{k=1}^{n}|w_k|\right)^2-|c_1|^2+\prod_{k=1}^{n}|w_k|\left|2i\Im f(i)c_2-c_1^2+\Im f(i)c_1\left(1+\sum_{k=1}^{n}\frac{1-|w_k|^2}{w_k}\right)\right|}\Bigg).$$

Also, for $w_k=\frac{z_k-i}{z_k+i}$, we have

$$1-|w_k|^2=1-\left|\frac{z_k-i}{z_k+i}\right|^2=\frac{4\Im z_k}{|z_k+i|^2},$$

$$|1+w_k|^2=\left|1+\frac{z_k-i}{z_k+i}\right|^2=\frac{4|z_k|^2}{|z_k+i|^2},$$

$$\frac{1-|w_k|^2}{|1+w_k|^2}=\frac{\frac{4\Im z_k}{|z_k+i|^2}}{\frac{4|z_k|^2}{|z_k+i|^2}}=\frac{\Im z_k}{|z_k|^2}$$

and

$$\frac{1-|w_k|^2}{w_k}=\frac{\frac{4\Im z_k}{|z_k+i|^2}}{\frac{z_k-i}{z_k+i}}=\frac{4\Im z_k}{|z_k+i|^2}\frac{z_k+i}{z_k-i}=\frac{4\Im z_k}{|z_k|^2-2i\Re z_k-1}.$$

Therefore, we obtain inequality (2.6) with an obvious equality case. □

Motivated by the results of the work presented in [17], the following result has been obtained.

Theorem 2.6. *Let $f:S\to S$ be holomorphic function, $f(z_1)=f(i)$, $z_1\in S$ and it is also holomorphic function at the point $z=0$ of the real axis with $f(0)=\Re f(i)$. Then*

$$\left|f'(0)\right|\geq\Im f(i)\Bigg(1+\frac{\Im z_1}{|z_1|^2}+\frac{\Im f(i)|z_1-i|-\left|f'(i)\right||z_1+i|}{\Im f(i)|z_1-i|+\left|f'(i)\right||z_1+i|} \tag{2.7}$$

$$\times\left[1+\frac{(\Im f(i))^2\left(1-\frac{4\Im z_1}{|z_1+i|^2}\right)+|f'(z_1)|\Im z_1|f'(i)|-\Im f(i)|f'(z_1)|\Im z_1-\Im f(i)|f'(z_1)|}{(\Im f(i))^2\left(1-\frac{4\Im z_1}{|z_1+i|^2}\right)+|f'(z_1)|\Im z_1|f'(i)|+\Im f(i)|f'(z_1)|\Im z_1+\Im f(i)|f'(z_1)|}\frac{\Im z_1}{|z_1|^2}\right]\Bigg).$$

The inequality (2.6) *is sharp, with equality for each possible values $\left|f'(i)\right|$ and $\left|f'(z_1)\right|$.*

Proof. Let

$$q(w)=\frac{w-w_1}{1-\overline{w_1}w}.$$

Also, let $h:E\to E$ be a holomorphic function and a point $w_1\in E$ in order to satisfy

$$\left|\frac{h(w)-h(w_1)}{1-\overline{h(w_1)}h(w)}\right|\leq\left|\frac{w-w_1}{1-\overline{w_1}w}\right|=\left|q(w)\right|$$

and

$$|h(w)|\leq\frac{|h(w_1)|+\left|q(w)\right|}{1+|h(w_1)|\left|q(w)\right|} \tag{2.8}$$

by Schwarz-pick lemma [5]. If $v:E\to E$ is a holomorphic function and $0<|w_1|<1$, letting

$$h(w)=\frac{v(w)-v(0)}{w\left(1-\overline{v(0)}v(w\right)}$$

in (2.8), we obtain

$$\left|\frac{v(w)-v(0)}{w\left(1-\overline{v(0)}v(w)\right)}\right| \leq \frac{\left|\frac{v(w_1)-v(0)}{w_1\left(1-\overline{v(0)}v(w_1)\right)}\right| + \left|q(w)\right|}{1+\left|\frac{v(w_1)-v(0)}{w_1\left(1-\overline{v(0)}v(w_1)\right)}\right|\left|q(w)\right|}$$

and

$$|v(w)| \leq \frac{|v(0)| + |w|\,\frac{|C|+\left|q(w)\right|}{1+|C|\left|q(w)\right|}}{1+|v(0)|\,|w|\,\frac{|C|+\left|q(w)\right|}{1+|C|\left|q(w)\right|}}, \tag{2.9}$$

where

$$C = \frac{v(w_1)-v(0)}{w_1\left(1-\overline{v(0)}v(w_1)\right)}.$$

If we take

$$v(w) = \frac{f(w)}{w\frac{w-w_1}{1-\overline{w_1}w}},$$

where then

$$v(w_1) = \frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1}, \quad v(0) = \frac{f'(0)}{-w_1}$$

and

$$C = \frac{\frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1} + \frac{f'(0)}{w_1}}{z_1\left(1+\frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1}\frac{f'(0)}{w_1}\right)},$$

where $|C| \leq 1$. Let $|v(0)| = \alpha$ and

$$\mathrm{D} = \frac{\left|\frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1}\right| + \left|\frac{f'(0)}{w_1}\right|}{|w_1|\left(1+\left|\frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1}\right|\left|\frac{f'(0)}{w_1}\right|\right)}.$$

From (2.9), we get

$$\left|f(w)\right| \leq |w|\left|q(w)\right|\frac{\alpha+|w|\,\frac{\mathrm{D}+\left|q(w)\right|}{1+\mathrm{D}\left|q(w)\right|}}{1+\alpha\,|w|\,\frac{\mathrm{D}+\left|q(w)\right|}{1+\mathrm{D}\left|q(w)\right|}}$$

and

$$\frac{1-\left|f(w)\right|}{1-|w|} \geq \frac{1+\alpha\,|w|\,\frac{\mathrm{D}+\left|q(w)\right|}{1+\mathrm{D}\left|q(w)\right|} - \alpha\,|w|\left|q(w)\right| - \left|q(w)\right||w|^2\,\frac{\mathrm{D}+\left|q(w)\right|}{1+\mathrm{D}\left|q(w)\right|}}{(1-|w|)\left(1+\alpha\,|w|\,\frac{\mathrm{D}+\left|q(w)\right|}{1+\mathrm{D}\left|q(w)\right|}\right)} = \tau(w). \tag{2.10}$$

Let $\kappa(w) = 1 + \alpha |w| \frac{\mathrm{D}+\left|q(w)\right|}{1+\mathrm{D}\left|q(w)\right|}$ and $\tau(w) = 1 + \mathrm{D}\left|q(w)\right|$. Then

$$\tau(w) = \frac{1 - |w|^2 \left|q(w)\right|^2}{(1 - |w|)\,\kappa(w)\tau(w)} + \mathrm{D}\left|q(w)\right| \frac{1 - |w|^2}{(1 - |w|)\,\kappa(w)\tau(w)} + \mathrm{D}\alpha |w| \frac{1 - \left|q(w)\right|^2}{(1 - |w|)\,\kappa(w)\tau(w)}.$$

Since

$$\lim_{w\to -1} \kappa(w) = \lim_{w\to -1} 1 + \beta |z| \frac{\mathrm{D}+\left|q(w)\right|}{1 + \mathrm{D}\left|q(w)\right|} = 1 + \alpha,$$

$$\lim_{w\to -1} \tau(w) = \lim_{w\to -1} 1 + \mathrm{D}\left|q(w)\right| = 1 + \mathrm{D}$$

and

$$1 - \left|q(w)\right|^2 = 1 - \left|\frac{w - w_1}{1 - \overline{w_1} w}\right|^2 = \frac{\left(1 - |w_1|^2\right)\left(1 - |w|^2\right)}{\left|1 - \overline{w_1} w\right|^2}, \tag{2.11}$$

passing to the limit in (2.10) and using (2.11) gives

$$\begin{aligned} \left|f'(-1)\right| &\geq \frac{2}{(1+\alpha)\,(1+\mathrm{D})}\left(1 + \frac{1 - |w_1|^2}{|1 + w_1|^2} + \mathrm{D}+\alpha\mathrm{D}\frac{1 - |w_1|^2}{|1 + w_1|^2}\right) \\ &= 1 + \frac{1 - |w_1|^2}{|1 + w_1|^2} + \frac{1 - \alpha}{1 + \alpha}\left(1 + \frac{1 - \mathrm{D}}{1 + \mathrm{D}}\frac{1 - |w_1|^2}{|1 + w_1|^2}\right). \end{aligned}$$

Moreover, since

$$\begin{aligned} \frac{1-\alpha}{1+\alpha} &= \frac{1 - |v(0)|}{1 + |v(0)|} = \frac{1 - \left|\frac{f'(0)}{w_1}\right|}{1 + \left|\frac{f'(0)}{w_1}\right|} = \frac{|w_1| - \left|f'(0)\right|}{|w_1| + \left|f'(0)\right|} \\ &= \frac{\left|\frac{z_1 - i}{z_1 + i}\right| - \frac{\left|f'(i)\right|}{\Im f(i)}}{\left|\frac{z_1 - i}{z_1 + i}\right| + \frac{\left|f'(i)\right|}{\Im f(i)}}, \end{aligned}$$

$$\begin{aligned} \frac{1-\mathrm{D}}{1+\mathrm{D}} &= \frac{1 - \frac{\left|\frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1}\right| + \left|\frac{f'(0)}{w_1}\right|}{|w_1|\left(1 + \left|\frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1}\right|\left|\frac{f'(0)}{w_1}\right|\right)}}{1 + \frac{\left|\frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1}\right| + \left|\frac{f'(0)}{w_1}\right|}{|w_1|\left(1 + \left|\frac{f'(w_1)\left(1-|w_1|^2\right)}{w_1}\right|\left|\frac{f'(0)}{w_1}\right|\right)}} \\ &= \frac{1 - \frac{\left|\frac{|z+i|^2}{(2i)^2}\frac{f'(z_1)}{\Im f(i)}\left(1 - \left|\frac{z_1 - i}{z_1 + i}\right|^2\right)\right| + \left|\frac{f'(i)}{\Im f(i)}\right|}{\left|\frac{z_1 - i}{z_1 + i}\right|\left(1 + \left|\frac{|z+i|^2}{(2i)^2}\frac{f'(z_1)}{\Im f(i)}\left(1 - \left|\frac{z_1 - i}{z_1 + i}\right|^2\right)\right|\left|\frac{f'(i)}{\Im f(i)}\right|\right)}}{1 + \frac{\left|\frac{|z+i|^2}{(2i)^2}\frac{f'(z_1)}{\Im f(i)}\left(1 - \left|\frac{z_1 - i}{z_1 + i}\right|^2\right)\right| + \left|\frac{f'(i)}{\Im f(i)}\right|}{\left|\frac{z_1 - i}{z_1 + i}\right|\left(1 + \left|\frac{|z+i|^2}{(2i)^2}\frac{f'(z_1)}{\Im f(i)}\left(1 - \left|\frac{z_1 - i}{z_1 + i}\right|^2\right)\right|\left|\frac{f'(i)}{\Im f(i)}\right|\right)}}, \end{aligned}$$

$$\frac{1-\mathrm{D}}{1+\mathrm{D}}=\frac{\left|\frac{z_1-i}{z_1+i}\right|\left(1+\left|\frac{\frac{|z+i|^2}{(2i)^2}\frac{f'(z_1)}{\Im f(i)}\left(1-\left|\frac{z_1-i}{z_1+i}\right|^2\right)}{\frac{z_1-i}{z_1+i}}\right|\left|\frac{\frac{f'(i)}{\Im f(i)}}{\frac{z_1-i}{z_1+i}}\right|\right)-\left|\frac{\frac{|z+i|^2}{(2i)^2}\frac{f'(z_1)}{\Im f(i)}\left(1-\left|\frac{z_1-i}{z_1+i}\right|^2\right)}{\frac{z_1-i}{z_1+i}}\right|-\left|\frac{\frac{f'(i)}{\Im f(i)}}{\frac{z_1-i}{z_1+i}}\right|}{\left|\frac{z_1-i}{z_1+i}\right|\left(1+\left|\frac{\frac{|z+i|^2}{(2i)^2}\frac{f'(z_1)}{\Im f(i)}\left(1-\left|\frac{z_1-i}{z_1+i}\right|^2\right)}{\frac{z_1-i}{z_1+i}}\right|\left|\frac{\frac{f'(i)}{\Im f(i)}}{\frac{z_1-i}{z_1+i}}\right|\right)+\left|\frac{\frac{|z+i|^2}{(2i)^2}\frac{f'(z_1)}{\Im f(i)}\left(1-\left|\frac{z_1-i}{z_1+i}\right|^2\right)}{\frac{z_1-i}{z_1+i}}\right|+\left|\frac{\frac{f'(i)}{\Im f(i)}}{\frac{z_1-i}{z_1+i}}\right|}$$

$$=\frac{\left|\frac{z_1-i}{z_1+i}\right|^2+\frac{|z+i|^2}{4}\left|\frac{f'(z_1)}{\Im f(i)}\right|\left(1-\left|\frac{z_1-i}{z_1+i}\right|^2\right)\left|\frac{f'(i)}{\Im f(i)}\right|-\left|\frac{|z+i|^2}{4}\left|\frac{f'(z_1)}{\Im f(i)}\right|\left(1-\left|\frac{z_1-i}{z_1+i}\right|^2\right)\right|-\left|\frac{f'(i)}{\Im f(i)}\right|}{\left|\frac{z_1-i}{z_1+i}\right|^2+\frac{|z+i|^2}{4}\left|\frac{f'(z_1)}{\Im f(i)}\right|\left(1-\left|\frac{z_1-i}{z_1+i}\right|^2\right)\left|\frac{f'(i)}{\Im f(i)}\right|+\left|\frac{|z+i|^2}{4}\left|\frac{f'(z_1)}{\Im f(i)}\right|\left(1-\left|\frac{z_1-i}{z_1+i}\right|^2\right)\right|-\left|\frac{f'(i)}{\Im f(i)}\right|},$$

and

$$1-\left|\frac{z_1-i}{z_1+i}\right|^2=\frac{4\Im z_1}{|z_1+i|^2}$$

$$\left|1+\frac{z_1-i}{z_1+i}\right|^2=\frac{4|z_1|^2}{|z_1+i|^2},$$

we take

$$\left|f'(-1)\right|\geq\left(1+\frac{\Im z_1}{|z_1|^2}+\frac{\Im f(i)|z_1-i|-\left|f'(i)\right||z_1+i|}{\Im f(i)|z_1-i|+\left|f'(i)\right||z_1+i|}\right.$$

$$\left.\times\left[1+\frac{\left(\Im f(i)\right)^2\left(1-\frac{4\Im z_1}{|z_1+i|^2}\right)+\left|f'(z_1)\right|\Im z_1\left|f'(i)\right|-\Im f(i)\left|f'(z_1)\right|\Im z_1-\Im f(i)\left|f'(z_1)\right|}{\left(\Im f(i)\right)^2\left(1-\frac{4\Im z_1}{|z_1+i|^2}\right)+\left|f'(z_1)\right|\Im z_1\left|f'(i)\right|+\Im f(i)\left|f'(z_1)\right|\Im z_1+\Im f(i)\left|f'(z_1)\right|}\frac{\Im z_1}{|z_1|^2}\right]\right).$$

From definition of $f(w)$, we have

$$f'(w)=\frac{\frac{2i}{(1-w)^2}f'\left(i\frac{1+w}{1-w}\right)\left(f(i)-\overline{f(i)}\right)}{\left(f\left(i\frac{1+w}{1-w}\right)-\overline{f(i)}\right)^2}$$

and

$$\left|f'(-1)\right|=\frac{\left|f'(0)\right|}{\Im f(i)}.$$

Thus, we obtain the inequality (2.7).

Now, we shall show that the inequality (2.7) is sharp.

Since $v(w)=\frac{f(w)}{w\frac{w-w_1}{1-\overline{w_1}w}}$ is a holomorphic function in the unit disc E and $|v(w)|\leq 1$ for $|w|<1$. we obtain

$$\left|f'(0)\right|\leq|w_1|$$

and

$$\left|f'(w_1)\right|\leq\frac{|w_1|}{1-|w_1|^2}.$$

We take $w_1\in(-1,0)$ and arbitrary two numbers x and y. Let

$$\mathtt{M}=\frac{\frac{x\left(1-|w_1|^2\right)}{w_1}+\frac{x}{w_1}}{w_1\left(1+xy\frac{1-|w_1|^2}{w_1^2}\right)}=\frac{1}{w_1^2}\frac{y\left(1-|w_1|^2\right)+x}{1+xy\frac{1-|w_1|^2}{w_1^2}}.$$

The auxiliary function

$$h(w)=w\frac{w-w_1}{1-\overline{w_1}w}\frac{\frac{-x}{w_1}+w\frac{\mathtt{M}+\frac{w-w_1}{1-\overline{w_1}w}}{1+\mathtt{M}\frac{w-w_1}{1-\overline{w_1}w}}}{1-\frac{x}{w_1}w\frac{\mathtt{M}+\frac{w-w_1}{1-\overline{w_1}w}}{1+\mathtt{M}\frac{w-w_1}{1-\overline{w_1}w}}}$$

is holomorphic in E and $|h(z)| < 1$ for $z \in E$. Let

$$f(w) = w\frac{w - w_1}{1 - \overline{w_1}w}\frac{\frac{-x}{w_1} + w\frac{\mathtt{M}+\frac{w-w_1}{1-\overline{w_1}w}}{1+\mathtt{M}\frac{w-w_1}{1-\overline{w_1}w}}}{1 - \frac{x}{w_1}w\frac{\mathtt{M}+\frac{w-w_1}{1-\overline{w_1}w}}{1+\mathtt{M}\frac{w-w_1}{1-\overline{w_1}w}}}. \tag{2.12}$$

From (2.12), with the simple calculations, we obtain

$$\begin{aligned} f'(-1) &= 1 + \frac{1-w_1^2}{(1-w_1)^2} + \frac{\left(1+\frac{1-w_1^2}{(1-w_1)^2}\frac{1-\mathtt{M}^2}{(1+\mathtt{M})^2}\right)\left(1-\frac{x}{w_1}\right)+\frac{x}{w_1}\left(1+\frac{1-w_1^2}{(1-w_1)^2}\frac{1-\mathtt{M}^2}{(1+\mathtt{M})^2}\right)\left(1-\frac{x}{w_1}\right)}{\left(1-\frac{a}{z_1}\right)^2} \\ &= 1 + \frac{1-w_1^2}{(1-w_1)^2} + \frac{x+w_1}{-x+w_1}\left(1 + \frac{1-w_1^2}{(1-w_1)^2}\frac{w_1^2+xy\left(1-w_1^2\right)-y\left(1-w_1^2\right)-x}{w_1^2+ab\left(1-w_1^2\right)+y\left(1-w_1^2\right)+x}\right). \end{aligned}$$

Since $\left|f'(-1)\right| = \frac{\left|f'(0)\right|}{\Im f(i)}$, we take

$$\begin{aligned} \left|f'(0)\right| &= \Im f(i)\left(1 + \frac{1-\left(\frac{z_1-i}{z_1+i}\right)^2}{\left(1-\frac{z_1-i}{z_1+i}\right)^2} + \frac{x+\frac{z_1-i}{z_1+i}}{-x+\frac{z_1-i}{z_1+i}}\right. \\ &\times \left.\left[1 + \frac{1-\left(\frac{z_1-i}{z_1+i}\right)^2}{\left(1-\frac{z_1-i}{z_1+i}\right)^2}\frac{\left(\frac{z_1-i}{z_1+i}\right)^2+xy\left(1-\left(\frac{z_1-i}{z_1+i}\right)^2\right)-y\left(1-\left(\frac{z_1-i}{z_1+i}\right)^2\right)-x}{z_1^2+ab\left(1-\left(\frac{z_1-i}{z_1+i}\right)^2\right)+y\left(1-\left(\frac{z_1-i}{z_1+i}\right)^2\right)+x}\right]\right). \end{aligned}$$

Thus, since $\Re z_1 = 0$ and $0 < \Im z_1 < 1$, choosing suitable signs of the numbers x and y, the last the last equality shows that (2.7) is sharp. □

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