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Existence and Uniqueness of Extremal Mild Solutions for Non-Autonomous Nonlocal Integro-Differential Equations via Monotone Iterative Technique

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Abstract. In this work, we will discuss the existence and uniqueness of extremal mild solutions for non-autonomous integro-differential equations having nonlocal condition via monotone iterative method with upper and lower solutions in an ordered Banach space X, using evolution system and measure of noncompactness.

1. Introduction

Monotone iterative technique is an effective method to find the existence and uniqueness of mild solutions. Using this method, we get monotone sequences of approximate solutions that converges to maximal and minimal mild solutions. Du [8], first used this technique to find extremal mild solutions for a differential equation. Chen and Li [4] used monotone iterative technique to establish the existence and uniqueness of mild solutions for a semilinear differential equation with nonlocal condition. In [5], Chen and Mu discussed the existence and uniqueness of mild solutions for a semilinear impulsive integro-differential equation by using monotone iterative method. Mu [15] studied the existence and uniqueness of mild solutions for a fractional evolution equation with the help of monotone iterative method. In [16], Mu and Li investigated the existence and uniqueness of mild solutions for an impulsive fractional differential equation by using monotone iterative technique. Later on, the result has been extended for nonlocal condition by Mu [17]. Kamaljeet [13] and Renu [3] used monotone iterative method to discuss the existence and uniqueness of mild solutions for nonlocal fractional differential equations having finite delay and for fractional neutral differential equations having infinite delay respectively. The technique has been used for autonomous system till now.

Yan [20] studied the existence of mild solutions for non-autonomous integro-differential equations with nonlocal conditions by using the theory of evolution system, Banach contraction principle and Schauder's fixed point theorem. Haloi et al. [11] studied existence, uniqueness and asymptotic stability of non-autonomous differential equations with deviated arguments via Banach fixed point theorem and theory of analytic semigroup. In [2], Alka et al. established the existence and uniqenss of mild solutions for non-autonomous instantaneous impulsive differential equations with iterated deviating arguments by using

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analytic semigroup theory and Banach fixed point theorem.

Nonlocal condition is a generalization of classical initial condition which is more effective to produce better results in the application of physical problems rather than classical initial condition (see e.g. [10, 19] and references therein). The existence results for nonlocal Cauchy problem was first studied by Byszewski [1]. In [7], to describe the diffusion phenomenon of a small amount of gas in a transparent tube, Deng used the nonlocal condition.

To the best of our knowledge, there is no work yet reported on the existence and uniqueness of mild solutions for non-autonomous differential equations by using monotone iterative method. Motivated by this fact, we consider the following non-autonomous integro-differential system with nonlocal condition in an ordered Banach space X :

$$\begin{aligned} x'(t) + \mathbb{A}(t)x(t) &= \mathcal{F}\Big(t, x(t), \int_0^t k(t,s)x(s)ds\Big), \quad t \in (0, b], \\ x(0) &= x_0 + \mathcal{G}(x), \end{aligned} \tag{1}$$

where $\mathbb{A}(t) : D(\mathbb{A}(t)) \subset X \to X$ is linear operator, $x_0 \in X$, $\mathcal{F}$ is X-valued function defined over $J \times X \times X$, $\mathcal{G}$ is X-valued function defined over $C(J, X)$ with $J = [0, b]$, and $k \in C(\mathbb{D}, \mathbb{R}^+)$ where $\mathbb{D} := \{(t, s) : 0 \leqslant s \leqslant t \leqslant b\}$.

We organize the article as following. In section 2, we will recall some basic theory. In section 3, we will establish the existence of extremal mild solutions for the system (1), and also we will show the uniqueness of extremal mild solutions. In last section, we will discuss an example to illustrate our results.

2. Preliminaries

Now, we recall some basic theory which is useful to prove our main results.
Let $(X, \|\cdot\|, \leqslant)$ is a partially ordered complete norm space, $\mathcal{P} = \{x \in X : x \geqslant 0\}$ (0 is the zero element of X) is a positive cone of X. The cone $\mathcal{P}$ is known as normal if there is a real number $\mathcal{N} > 0$ such that $0 \leqslant x_1 \leqslant x_2 \Rightarrow \|x_1\| \leqslant \mathcal{N}\|x_2\|$, for all $x_1, x_2 \in X$, the smallest value of such $\mathcal{N}$ is called normal constant. Let $C(J, X)$ be space of all continuous maps from J to X, with sup norm. For $x_1, x_2 \in C(J, X)$, $x_1 \leqslant x_2 \Leftrightarrow x_1(t) \leqslant x_2(t)$, $\forall\, t \in J$. For $\nu, \omega \in C(J, X)$ with $\nu \leqslant \omega$, we will use the notation $[\nu, \omega] := \{x \in C(J, X) : \nu \leqslant x \leqslant \omega\}$ for an interval in $C(J, X)$, and $[\nu(t), \omega(t)] := \{x \in X : \nu(t) \leqslant x \leqslant \omega(t)\}(t \in J)$ for an interval in X. Let us denote $C^1(J, X) = \{x \in C(J, X) : x' \text{ exists on } J, x' \in C(J, X),\ x(t) \in D(\mathbb{A})\ (t \geqslant 0)\}$, and $\mathcal{L}^p(J, X)(1 \leqslant p < \infty)$ be the Banach space with norm $\|x\|_{\mathcal{L}^p(J,X)} = (\int_0^b \|x(t)\|^p dt)^{\frac{1}{p}}$. For our convenience we denote $\mathcal{K}x(t) := \int_0^t k(t, s)x(s)ds$, and $K^* := \sup\limits_{(t,s)\in\mathbb{D}} k(t, s)$.

First, we recall the definition and some basic properties of evolution system. For more details, we refer [9] and [18].

Definition 2.1. *([18]) Let X be a Banach space. A two parameter family of bounded linear operators* $\mathcal{S}(t_1, t_2), 0 \leqslant t_2 \leqslant t_1 \leqslant b$ *on X is known as evolution system, if :*

1. $\mathcal{S}(s, s) = I$, *where I is the identity operator.*
2. $\mathcal{S}(t_1, t_2)\mathcal{S}(t_2, t_3) = \mathcal{S}(t_1, t_3)$ *for* $0 \leqslant t_3 \leqslant t_2 \leqslant t_1 \leqslant b$.
3. $(t_1, t_2) \to \mathcal{S}(t_1, t_2)$ *is strongly continuous for* $0 \leqslant t_2 \leqslant t_1 \leqslant b$.

For the family of linear operators $\{\mathbb{A}(t) : t \in J\}$ on X, we impose the following assumptions :

(A1) $\mathbb{A}(t)$ is closed operator, the domain of $\mathbb{A}(t)$ is independent of t, and dense in X.

(A2) The resolvent of $\mathbb{A}(t)$ exists for $Re(\vartheta) \leqslant 0$, $t \in J$, and there exists a positive constant ς such that $\|\mathcal{R}(\vartheta; t)\| \leqslant \frac{\varsigma}{|\vartheta|+1}$.

(A3) There exist positive constants K, and $\rho \in (0, 1]$ such that $\|[\mathbb{A}(\tau_1) - \mathbb{A}(\tau_2)]\mathbb{A}^{-1}(\tau_3)\| \leqslant K|\tau_1 - \tau_2|^\rho$ for any $\tau_1, \tau_2, \tau_3 \in J$.

Theorem 2.2. *([18]) Suppose that the assumptions* $(A1)$-$(A3)$ *hold, then* $-\mathbb{A}(t)$ *generates a unique evolution system* $\{\mathcal{S}(t_1, t_2) : 0 \leqslant t_2 \leqslant t_1 \leqslant b\}$, *which satisfies the following properties :*

(i) There exists a positive constant $\mathcal{M}$ *such that* $\|\mathcal{S}(t_1, t_2)\| \leqslant \mathcal{M}$, $0 \leqslant t_2 \leqslant t_1 \leqslant b$.

(ii) For $0 \leqslant t_2 < t_1 \leqslant b$, *the derivative* $\frac{\partial \mathcal{S}(t_1,t_2)}{\partial t_1}$ *exists in strong operator topology, is strongly continuous, and belongs to* $B(X)$ *(set of all bounded linear operators on* X*). Moreover,*

$$\frac{\partial \mathcal{S}(t_1, t_2)}{\partial t_1} + \mathbb{A}(t_1)\mathcal{S}(t_1, t_2) = 0, \ 0 \leqslant t_2 < t_1 \leqslant b.$$

Proposition 2.3. *([20]) The family of operators* $\{\mathcal{S}(t_1, t_2), \ t_2 < t_1\}$ *is continuous in* t_1 *uniformly for* t_2 *with respect to operator norm.*

Theorem 2.4. *([18]) Suppose that the assumptions* $(A1)$-$(A3)$ *hold and* $\mathcal{F}$ *satisfies uniform Hölder continuity on* J *with exponent* $\alpha \in (0, 1]$, *then the unique solution of the following linear Cauchy problem*

$$\begin{aligned} x'(t) + \mathbb{A}(t)x(t) &= \mathcal{F}(t), \quad t \in (0, b], \\ x(0) &= x_0 \in X, \end{aligned} \tag{2}$$

is given as

$$x(t) \;=\; \mathcal{S}(t, 0)x_0 + \int_0^t \mathcal{S}(t, \eta)\mathcal{F}(\eta)d\eta. \tag{3}$$

Definition 2.5. *A mild solution of (1) is a function* $x \in C(J, X)$ *satisfying the following integral equation*

$$x(\varrho) = \mathcal{S}(\varrho, 0)(x_0 + \mathcal{G}(x)) + \int_0^\varrho \mathcal{S}(\varrho, \eta)\mathcal{F}(\eta, x(\eta), \mathcal{K}x(\eta))d\eta, \ \varrho \in J.$$

Definition 2.6. *An evolution system* $\mathcal{S}(t, s)$ *is called positive if* $\mathcal{S}(t, s)y \geqslant 0$, *for all* $y \in \mathcal{P}$ *and* $0 \leqslant s \leqslant t \leqslant b$.

Definition 2.7. $\omega_0 \in C^1(J, X)$ *is called lower solution for the system (1), if*

$$\begin{aligned} \omega_0'(t) + \mathbb{A}(t)\omega_0(t) &\leqslant \mathcal{F}\Big(t, \omega_0(t), \int_0^t k(t, s)\omega_0(s)ds\Big), \quad t \in (0, b], \\ \omega_0(0) &\leqslant x_0 + \mathcal{G}(\omega_0). \end{aligned} \tag{4}$$

If the inequalities of (4) are opposite, solution is known as upper solution.

Now, we state the definition and some properties of Kuratowski measure of noncompactness. For more details, we refer [6] and [12] .

Definition 2.8. *If* Υ *is a complete norm space and* $M(\Upsilon)$ *is a collection of subsets of* Υ, *which are bounded, then the function* $\mu : M(\Upsilon) \to [0, \infty)$ *defined as following*

$$\mu(S) = \inf\{\varepsilon > 0 : S \subset \cup_{j=1}^n S_j, \ diam(S_j) < \varepsilon \ (j = 1, 2, \ldots, n \in \mathbb{N})\},$$

is known as Kuratowski measure of noncompactness.

Lemma 2.9. *If* $\mathbb{X}_1$ *and* $\mathbb{X}_2$ *be complete norm spaces and* $C, D \subset \mathbb{X}_1$ *be bounded, then the following properties are satisfied :*

(i) D *is precompact if and only if* $\mu(D) = 0$.

(ii) $\mu(C \cup D) = \max\{\mu(C), \mu(D)\}$.

(iii) $\mu(C + D) \leqslant \mu(C) + \mu(D)$.

(iv) $\varphi : dom(\varphi) \subset \mathbb{X}_1 \to \mathbb{X}_2$ *satisfies Lipschitz continuity with Lipschitz constant L, then* $\mu(\varphi(S)) \leqslant L\mu(S)$, $S \subset dom(\varphi)$ *is bounded.*

Lemma 2.10. *If* Υ *is a complete norm space, and* $S \subset C(J, \Upsilon)$, $S(t) = \{f(t) : f \in S\}(t \in J)$. *Then boundedness and equicontinuity of S in* $C(J, \Upsilon)$ *implies that* $\mu(S(t))$ *is continuous on J, moreover* $\mu(S) = \max_{t \in J} \mu(S(t))$.

Lemma 2.11. *Suppose* $\mathbb{X}$ *is complete norm space and* $\{f_n\} \subset C(J, \mathbb{X})$ *is a bounded sequence, then* $\mu(\{f_n(t)\} \in \mathcal{L}^1(J, \mathbb{X})$, *moreover*

$$\mu\Big(\Big\{\int_0^t f_n(\eta)d\eta\Big\}_{n=1}^{\infty}\Big) \leqslant 2\int_0^t \mu(\{f_n(\eta)\}_{n=1}^{\infty})d\eta.$$

3. Main Results

First, we will show the existence of extremal mild solutions for (1), then the uniqueness will be discussed. Let us define $Q : C(J, X) \to C(J, X)$ in the following way :

$$Qx(\varrho) = \mathcal{S}(\varrho, 0)(x_0 + \mathcal{G}(x)) + \int_0^{\varrho} \mathcal{S}(\varrho, \eta)\mathcal{F}(\eta, x(\eta), \mathcal{K}x(\eta))d\eta, \ \varrho \in J. \tag{5}$$

To prove that the system (1) has a mild solution, we need to show the operator Q has a fixed point.

Theorem 3.1. *Suppose X is a partially ordered complete norm space with normal positive cone* $\mathcal{P}$, *the assumptions* (A1)-(A3) *hold and* $-\mathbb{A}(t)$ *generates a positive evolution system* $\mathcal{S}(t, s)$ *on X,* $\mathcal{F}$ *is continuous from* $J \times X \times X \to X$, $x_0 \in X$, *and* ω_0, $\nu_0 \in C^1(J, X)$ *with* $\omega_0 \leqslant \nu_0$ *are lower and upper solutions respectively for (1). Moreover assume the following* :

(H1) *For* $t \in J$, *we have*

$$\mathcal{F}(t, y_1, x_1) \leqslant \mathcal{F}(t, y_2, x_2),$$

where $y_1, y_2 \in X$ *with* $\omega_0(t) \leqslant y_1 \leqslant y_2 \leqslant \nu_0(t)$, *and* $\mathcal{K}\omega_0(t) \leqslant x_1 \leqslant x_2 \leqslant \mathcal{K}\nu_0(t)$.

(H2) *There exists a constant* $\mathcal{L} > 0$ *such that for all* $t \in J$,

$$\mu(\{\mathcal{F}(t, y_n, x_n)\}) \leqslant \mathcal{L}(\mu(\{y_n\}) + \mu(\{x_n\})),$$

where $\{y_n\} \subset [\omega_0(t), \nu_0(t)]$ *and* $\{x_n\} \subset [\mathcal{K}\omega_0(t), \mathcal{K}\nu_0(t)]$ *are monotone increasing or decreasing sequences.*

(H3) $\mathcal{G} : C(J, X) \to X$ *is a continuous increasing compact function.*

Then, the system (1) has extremal mild solutions in the interval $[\omega_0, \nu_0]$, *provided that*

$$\Lambda_1 := 2\mathcal{M}\mathcal{L}b(1 + bK^*) < 1.$$

Proof. Let us denote $I = [\omega_0, \nu_0]$. For any $x \in I$, $(H1)$ implies

$$\mathcal{F}(\varrho, \omega_0(\varrho), \mathcal{K}\omega_0(\varrho)) \leqslant \mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho)) \leqslant \mathcal{F}(\varrho, \nu_0(\varrho), \mathcal{K}\nu_0(\varrho)).$$

Therefore, from the normality of $\mathcal{P}$ we get a constant $c > 0$, such that

$$\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| \leqslant c, \ x \in I. \tag{6}$$

First, we will prove that the map $Q : I \to C(J, X)$ is continuous. Let $\{x_n\}$ be a sequence in I such that $x_n \to x \in I$. Since $\mathcal{G}$, $\mathcal{F}$ are continuous, so $\mathcal{G}x_n \to \mathcal{G}x$, and $\mathcal{F}(\varrho, x_n(\varrho), \mathcal{K}x_n(\varrho)) \to \mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))$ for

$\varrho \in J$, and from (6) we get that $\|\mathcal{F}(\varrho, x_n(\varrho), \mathcal{K}x_n(\varrho)) - \mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| \leqslant 2c$. So, by Lebesgue dominated convergence theorem, we estimate

$$\begin{aligned}\|Qx_n(t) - Qx(t)\| &\leqslant \mathcal{M}\|\mathcal{G}x_n - \mathcal{G}x\| \\ &\quad + \mathcal{M}\int_0^t \|\mathcal{F}(\varrho, x_n(\varrho), \mathcal{K}x_n(\varrho)) - \mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Thus Q is continuous map on I.

Next, we will prove $Q : I \to I$ is monotone increasing. Let $x_1, x_2 \in I$, $x_1 \leqslant x_2$. Using the positivity of $\mathcal{S}(t,s)$, the hypotheses $(H1)$ and $(H3)$, it is easy to see that $Qx_1 \leqslant Qx_2$. Suppose $\omega_0'(\eta) + \mathbb{A}(\eta)\omega_0(\eta) = h(\eta)$, Definition 2.7 implies $h(\eta) \leqslant \mathcal{F}(\eta, \omega_0(\eta), \mathcal{K}\omega_0(\eta))$ for $\eta \in J$, and $\omega_0(0) \leqslant x_0 + \mathcal{G}(\omega_0)$. Therefore, for any $t \in J$, Theorem (2.4) yields

$$\begin{aligned}\omega_0(t) &= \mathcal{S}(t,0)\omega_0(0) + \int_0^t \mathcal{S}(t,\eta)h(\eta)d\eta \\ &\leqslant \mathcal{S}(t,0)(x_0 + \mathcal{G}(\omega_0)) + \int_0^t \mathcal{S}(t,\eta)\mathcal{F}(\eta, \omega_0(\eta), \mathcal{K}\omega_0(\eta))d\eta \\ &= Q\omega_0(t).\end{aligned}$$

Hence, $\omega_0 \leqslant Q\omega_0$. In the same way, we get $Qv_0 \leqslant v_0$. Let $u \in I$, so we have $\omega_0 \leqslant Q\omega_0 \leqslant Qu \leqslant Qv_0 \leqslant v_0$, that means $Qu \in I$. Therefore, $Q : I \to I$ is monotone increasing.

Now, we will show $Q(I)$ is equicontinuous on J. For $x \in I$ and $\eta_1, \eta_2 \in J$ with $\eta_1 < \eta_2$, we have

$$\begin{aligned}\|Qx(\eta_2) - Qx(\eta_1)\| &\leqslant \|\mathcal{S}(\eta_2, 0) - \mathcal{S}(\eta_1, 0)\|\|x_0 + \mathcal{G}x\| \\ &\quad + \int_0^{\eta_1} \|\mathcal{S}(\eta_2, \varrho) - \mathcal{S}(\eta_1, \varrho)\|\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\quad + \int_{\eta_1}^{\eta_2} \|\mathcal{S}(\eta_2, \varrho)\|\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\leqslant I_1 + I_2 + I_3.\end{aligned}$$

For $\eta_1 = 0$, it is easy to see that $I_2 = 0$. For $\eta_1 > 0$ and $\epsilon > 0$ small enough, we obtain

$$\begin{aligned}I_2 &\leqslant \int_0^{\eta_1 - \epsilon} \|\mathcal{S}(\eta_2, \varrho) - \mathcal{S}(\eta_1, \varrho)\|\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\quad + \int_{\eta_1 - \epsilon}^{\eta_1} \|\mathcal{S}(\eta_2, \varrho) - \mathcal{S}(\eta_1, \varrho)\|\|\mathcal{F}(\varrho, x(\varrho), \mathcal{K}x(\varrho))\| d\varrho \\ &\leqslant c(\eta_1 - \epsilon) \sup_{\varrho \in [0, \eta_1 - \epsilon]} \|\mathcal{S}(\eta_2, \varrho) - \mathcal{S}(\eta_1, \varrho)\| + 2\mathcal{M}c\epsilon. \\ &\to 0 \quad \text{as} \quad \eta_2 \to \eta_1,\ \epsilon \to 0,\end{aligned}$$

by using the continuity of $\{\mathcal{S}(\eta, \varrho) : \varrho < \eta\}$ in η in uniform operator topology. Also It is clear from the expression of I_1, I_3 that $I_1 \to 0$, $I_3 \to 0$ as $\eta_2 \to \eta_1$. As a result $\|Qx(\eta_2) - Qx(\eta_1)\| \to 0$ as $\eta_2 \to \eta_1$, independently of $x \in I$. Hence $Q(I)$ is equicontinuous on J.

Now we define the sequences

$$\omega_n = Q\omega_{n-1} \quad \text{and} \quad v_n = Qv_{n-1}, \quad n \in \mathbb{N}, \tag{7}$$

monotonicity of Q implies

$$\omega_0 \leqslant \omega_1 \leqslant \cdots \omega_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0. \tag{8}$$

Let $S = \{\omega_n\}$ and $S_0 = \{\omega_{n-1}\}$. Then $S_0 = S \cup \{\omega_0\}$ and $\mu(S_0(t)) = \mu(S(t))$, $t \in J$. Observe that $\mu(\mathcal{S}(t,0)(x_0)) = 0 = \mu(\mathcal{S}(t,0)\mathcal{G}(\omega_{n-1}))$ for $\{x_0\}$ is compact set, $\mathcal{G}$ is compact map and $\mathcal{S}(t,0)$ is bounded. Also, with the help of Lemma 2.10 and Lemma 2.11, we observe that

$$\begin{aligned}
\mu\Big(\{\mathcal{K}\omega_{n-1}(\eta)\}\Big) &= \mu\Big(\int_0^\eta k(\eta,s)\omega_{n-1}(s)ds\Big)\\
&\leqslant K^*\mu\Big(\int_0^\eta \omega_{n-1}(s)ds\Big)\\
&\leqslant 2K^*\int_0^\eta \mu(\omega_{n-1}(s))ds\\
&\leqslant 2K^*\eta \sup_{s\in[0,\eta]} \mu(S_0(s)).
\end{aligned}$$

Now, from Lemma 2.11, $(H2)$, $(H3)$, (5) and (7), we get

$$\begin{aligned}
\mu(S(t)) &= \mu(\mathcal{Q}(S_0(t)))\\
&= \mu\Big(\mathcal{S}(t,0)(x_0+\mathcal{G}(\omega_{n-1})) + \int_0^t \mathcal{S}(t,\eta)\mathcal{F}(\eta,\omega_{n-1}(\eta),\mathcal{K}\omega_{n-1}(\eta))d\eta\Big)\\
&\leqslant \mu(\mathcal{S}(t,0)x_0) + \mu(\mathcal{S}(t,0)\mathcal{G}(\omega_{n-1})) + 2\mathcal{M}\int_0^t \mu\Big(\mathcal{F}(\eta,\omega_{n-1}(\eta),\mathcal{K}\omega_{n-1}(\eta))d\eta\Big)\\
&\leqslant 2\mathcal{ML}\int_0^t \Big[\mu(\{\omega_{n-1}(\eta)\}) + \mu(\{\mathcal{K}\omega_{n-1}(\eta)\})\Big]d\eta\\
&\leqslant 2\mathcal{ML}b(1+bK^*)\sup_{t\in J}\mu(S(t)).
\end{aligned} \tag{9}$$

Since $\{\mathcal{Q}\omega_{n-1}\}$ i.e. $\{\omega_n\}$ is equicontinuous, by Lemma 2.10 and (9), we obtain

$$\begin{aligned}
\mu(S) &= \sup_{t\in J}\mu(S(t))\\
&\leqslant 2\mathcal{ML}b(1+bK^*)\sup_{t\in J}\mu(S(t)) = 2\mathcal{ML}b(1+bK^*)\mu(S) = \Lambda_1\mu(S).
\end{aligned}$$

Since $\Lambda_1 < 1$, therefore $\mu(S) = 0$. Hence the set S is relatively compact in I, so there exists a convergent subsequence of $\{\omega_n\}$ in I. From (8), it is easy to see that $\{\omega_n\}$ itself is a convergent sequence, let $\omega_n \to \omega^*$ as $n \to \infty$. By (5) and (7)

$$\begin{aligned}
\omega_n(t) &= \mathcal{Q}\omega_{n-1}(t)\\
&= \mathcal{S}(t,0)(x_0+\mathcal{G}(\omega_{n-1})) + \int_0^t \mathcal{S}(t,\eta)\mathcal{F}(\eta,\omega_{n-1}(\eta),\mathcal{K}\omega_{n-1}(\eta))d\eta.
\end{aligned} \tag{10}$$

In (10), let $n \to \infty$ and use Lebesgue dominated convergence theorem, we get

$$\omega^*(t) = \mathcal{S}(t,0)(x_0+\mathcal{G}(\omega^*)) + \int_0^t \mathcal{S}(t,\eta)\mathcal{F}(\eta,\omega^*(\eta),\mathcal{K}\omega^*(\eta))d\eta.$$

So, $\omega^* = \mathcal{Q}\omega^*$ and $\omega^* \in C(J,X)$. Hence ω^* is a mild solution for (1). In the same way there exists $\nu^* \in C(J,X)$ with $\nu_n \to \nu^*$ as $n \to \infty$, and $\nu^* = \mathcal{Q}\nu^*$. Now we show ω^*, ν^* are extremal mild solutions. Let $x \in I$ and $x = \mathcal{Q}x$, then $\omega_1 = \mathcal{Q}\omega_0 \leqslant \mathcal{Q}x = x \leqslant \mathcal{Q}\nu_0 = \nu_1$. From the process of induction $\omega_n \leqslant x \leqslant \nu_n$, and $\omega_0 \leqslant \omega^* \leqslant x \leqslant \nu^* \leqslant \nu_0$ as $n \to \infty$. That means ω^* is the minimal and ν^* is the maximal mild solution for (1) in $[\omega_0, \nu_0]$. □

Theorem 3.2. *Suppose X is a partially ordered complete norm space, with normal positive cone $\mathcal{P}$ and normal constant $\mathcal{N}$, the assumptions $(H1)$, $(H3)$, $(A1)$-$(A3)$ hold and $-\mathbb{A}(t)$ generates a positive evolution system $\mathcal{S}(t,s)(0 \leqslant s \leqslant t \leqslant b)$ on X, $\mathcal{F} \in C(J\times X\times X, X)$, $x_0 \in X$, and ω_0, $\nu_0 \in C^1(J,X)$ with $\omega_0 \leqslant \nu_0$ are lower and upper solutions respectively for (1). Moreover assume the following* :

(H4) *There exists a constant* $\mathcal{L}_1 > 0$ *such that, for* $t \in J$

$$\mathcal{F}(t, y_2, x_2) - \mathcal{F}(t, y_1, x_1) \leqslant \mathcal{L}_1[(y_2 - y_1) + (x_2 - x_1)],$$

where $y_1, y_2 \in X$ *with* $\omega_0(t) \leqslant y_1 \leqslant y_2 \leqslant \nu_0(t)$, *and* $\mathcal{K}\omega_0(t) \leqslant x_1 \leqslant x_2 \leqslant \mathcal{K}\nu_0(t)$.

(H5) *There exists a constant* $\mathcal{L}_2 > 0$ *such that*

$$\mathcal{G}(y) - \mathcal{G}(x) \leqslant \mathcal{L}_2(y - x), \text{ for } x, y \in I \text{ with } x \leqslant y.$$

Then, the system (1) has a unique mild solution in $[\omega_0, \nu_0]$, *provided that*

$$\Lambda_2 := \mathcal{N}\mathcal{M}\Big[\mathcal{L}_2 + \mathcal{L}_1 b(1 + bK^*)\Big] < 1.$$

Proof. Let $\{y_n\} \subset [\omega_0(t), \nu_0(t)]$ and $\{x_n\} \subset [\mathcal{K}\omega_0(t), \mathcal{K}\nu_0(t)]$ be increasing monotone sequences. For $t \in J$ and $n, m \in \mathbb{N}$ with $n > m$, the assumptions $(H1)$ and $(H4)$ imply

$$0 \leqslant \mathcal{F}(t, y_n, x_n) - \mathcal{F}(t, y_m, x_m) \leqslant \mathcal{L}_1[(y_n - y_m) + (x_n - x_m)].$$

Since the positive cone is normal, therefore

$$\|\mathcal{F}(t, y_n, x_n) - \mathcal{F}(t, y_m, x_m)\| \leqslant \mathcal{N}\mathcal{L}_1\|(y_n - y_m) + (x_n - x_m)\|. \tag{11}$$

So by Lemma 2.9, we get

$$\mu(\{\mathcal{F}(t, y_n, x_n)\}) \leqslant \mathcal{N}\mathcal{L}_1(\mu(\{y_n\}) + \mu(\{x_n\})).$$

Hence the assumption $(H2)$ hold, and Theorem 3.1 is applicable. Therefore (1) has minimal mild solution ω^* and maximal mild solutions ν^* in $[\omega_0, \nu_0]$. From (5), $(H4)$, $(H5)$, and the positivity of the operator $\mathcal{S}(t, s)$, we get

$$\begin{aligned}
0 &\leqslant \nu^*(t) - \omega^*(t) = \mathcal{Q}\nu^*(t) - \mathcal{Q}\omega^*(t)\\
&= \mathcal{S}(t, 0)(\mathcal{G}(\nu^*) - \mathcal{G}(\omega^*)) + \int_0^t \mathcal{S}(t, \eta)[\mathcal{F}(\eta, \nu^*(\eta), \mathcal{K}\nu^*(\eta)) - \mathcal{F}(\eta, \omega^*(\eta), \mathcal{K}\omega^*(\eta))]d\eta\\
&\leqslant \mathcal{L}_2\mathcal{S}(t, 0)(\nu^* - \omega^*) + \mathcal{L}_1 \int_0^t \mathcal{S}(t, \eta)\Big[(\nu^*(\eta) - \omega^*(\eta)) + (\mathcal{K}\nu^*(\eta) - \mathcal{K}\omega^*(\eta))\Big]d\eta.
\end{aligned}$$

Since the positive cone is normal, therefore

$$\begin{aligned}
\|\nu^* - \omega^*\| &\leqslant \mathcal{N}\Big[\mathcal{L}_2\mathcal{M}\|\nu^* - \omega^*\| + \mathcal{M}\mathcal{L}_1 b\Big(\|\nu^* - \omega^*\| + \|\mathcal{K}\nu^* - \mathcal{K}\omega^*\|\Big)\Big]\\
&\leqslant \mathcal{N}\mathcal{M}\Big[\mathcal{L}_2 + \mathcal{L}_1 b(1 + bK^*)\Big]\|\nu^* - \omega^*\| = \Lambda_2\|\nu^* - \omega^*\|.
\end{aligned}$$

Since $\Lambda_2 < 1$, so $\|\nu^* - \omega^*\| = 0$, i.e. $\nu^*(t) = \omega^*(t)$, $\forall t \in J$. Thus $\nu^* = \omega^*$ is the unique mild solution for (1) in $[\omega_0, \nu_0]$. □

4. Example

Now we consider an example to show how our abstract results can be applied to a concrete problem. Consider the following partial differential equation :

$$\begin{cases}
x'(t, z) + a(t, z)\frac{\partial^2}{\partial z^2}x(t, z) = \frac{1}{25}\frac{e^{-t}}{1+e^t}x(t, z) + \int_0^t \frac{1}{50}e^{-s}x(s, z)ds, & z \in [0, \pi], \quad t \in J = [0, b],\\
x(t, 0) = 0, \ x(t, \pi) = 0 & t \in J,\\
x(0, z) = \frac{e^{x(t,z)}}{1+e^{x(t,z)}} + x_0(z), & z \in [0, \pi],
\end{cases} \tag{12}$$

where $X = \mathcal{L}^2([0,b] \times [0,\pi], \mathbb{R})$, $x_0(z) \in X$, $a(t,z)$ is continuous function and satisfies uniform Hölder continuity in t. Define

$$\mathbb{A}(t)x(t,z) = a(t,z)\frac{\partial^2}{\partial z^2}x(t,z), \tag{13}$$

with domain

$$D(\mathbb{A}) = \{x \in X : x,\ \frac{\partial x}{\partial z} \text{ are absolutely continuous, } \frac{\partial^2 x}{\partial z^2} \in X,\ x(0) = x(\pi) = 0\}.$$

It is well known that $-\mathbb{A}(t)$ generates a positive evolution system of bounded linear operators $\mathcal{S}(t,s)$ on X and satisfy the conditions $(A1)$-$(A3)$ (see [18]). Put

$$\begin{aligned} x(t)(z) &= x(t,z),\ t \in [0,b],\ z \in [0,\pi], \\ \mathcal{F}(t, x(t), \mathcal{K}x(t))(z) &= \frac{1}{25}\frac{e^{-t}}{1+e^{t}}x(t,z) + \int_0^t \frac{1}{50}e^{-s}x(s,z)ds, \\ (\mathcal{K}x(t))(z) &= \int_0^t \frac{1}{50}e^{-s}x(s,z)ds, \\ (\mathcal{G}x(t))(z) &= \frac{e^{x(t,z)}}{1+e^{x(t,z)}}. \end{aligned} \tag{14}$$

Then the system (12) can be rewritten into the abstract form of (1). Now, assume that $x_0(z) \geqslant 0$ for $z \in [0,\pi]$, and there exists a function $v(t,z) \geqslant 0$ such that

$$\begin{aligned} v'(t,z) + \mathbb{A}(t)v(t,z) &\geqslant \mathcal{F}\Big(t, v(t,z), \mathcal{K}v(t,z)\Big), \quad t \in J,\ z \in [0,\pi], \\ v(t,0) &= v(t,\pi) = 0, \quad t \in J, \\ v(0,z) &\geqslant \mathcal{G}(v(z)) + x_0(z), \quad z \in [0,\pi]. \end{aligned}$$

From the above assumptions, we have $\omega_0 = 0$ and $\nu_0 = v(t,z)$ are lower and upper solutions for the system (12). By (14), it is easy to verify that the assumptions $(H1)$ and $(H3)$ hold. Suppose $\{x_n\} \subset [\omega_0(t), \nu_0(t)]$ be a monotone increasing sequence. For $n \leqslant m$

$$\begin{aligned} \|\mathcal{F}(t, x_m, \mathcal{K}x_m) - \mathcal{F}(t, x_n, \mathcal{K}x_n)\| &\leqslant \frac{1}{25}\Big(\|x_m - x_n\| + \|\mathcal{K}x_m - \mathcal{K}x_n\|\Big), \text{ hence} \\ \mu\Big(\mathcal{F}(t, x_n, \mathcal{K}x_n)\Big) &\leqslant \frac{1}{25}\Big(\mu(\{x_n\}) + \mu(\mathcal{K}x_n)\Big). \end{aligned}$$

Therefore, assumption $(H2)$ is satisfied. So, by Theorem (3.1), we conclude that the minimal and maximal mild solutions for (12) exist between the lower solution 0 and upper solution v.

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