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Exponential Inequalities Under Sub-Linear Expectations with Applications to Strong Law of Large Numbers

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Abstract. In this paper, we give some exponential inequalities for extended independent random variables under sub-linear expectations. As an application, we obtain the strong convergence rate $O(n^{-1/2}\ln^{1/2} n)$ for the strong law of large numbers under sub-linear expectations, which generalizes some corresponding ones under the classical linear expectations.

1. Introduction

The classical exponential inequalities and strong law of large numbers are based on the linearity of expectations and probability measures. However, such an additivity assumption is not feasible in many areas of applications because many uncertain phenomena cannot be well modelled by using additive probabilities or additive expectations. More specifically, motivated by some problems in mathematical economics, statistics, quantum mechanics and finance, a number of papers have used non-additive probabilities (called capacities) and nonlinear expectations (for example Choquet integral/expectation, g-expectation) to describe and interpret the phenomena which are generally nonadditive (see Chen [1], Peng [2]). Peng [3]-[5] introduced the general framework of the sub-linear expectation in a general function space by relaxing the linear property of the classical expectation to the sub-additivity and positive homogeneity. Under this framework, many limit theorems have been established recently, including the central limit theorem and weak law of large numbers (cf. Peng [4] and [6]), the small derivation and Chung's law of the iterated logarithm (cf. Zhang [7]), the exponential inequalities and the laws of the iterated algorithm (cf. Zhang [8]), the moment inequalities for the maximum partial sums (cf. Zhang [9]). In addition, Chen [10] investigated kinds of strong laws of large numbers for capacities, Zhang [11] obtained the moment inequalities and the Kolmogorov type exponential inequalities, Wu and Chen [12] researched the invariance principles for the law of the iterated logarithm, Wu and Jiang [13] established the strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations, Wu et al. [14] investigated the approximations of inverse moments for double-indexed weighted sums of random variables and obtained the convergence

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rate of approximations under sub-linear expectations, and so on. In this work, we will further study the probability limit properties for partial sums of random variables under the sub-linear expectations, especially the exponential inequalities for unbounded random variables and applications to the convergence rate of the strong law of large numbers.

Now we use the notations of Peng [4]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \forall x, y \in \mathbb{R}^n,$$

for some $C > 0$ and $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables. In this case we denote $X \in \mathcal{H}$.

Definition 1.1. *A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}} := [-\infty, \infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) Monotonicity: If $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;

(b) Constant preserving: $\hat{\mathbb{E}}[c] = c$, where c is a constant;

(c) Sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$, whenever $\hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \lambda > 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\mathcal{E}}[X] = -\hat{\mathbb{E}}[-X], \ \ \forall X \in \mathcal{H}.$$

Obviously, for all $X \in \mathcal{H}$, $\hat{\mathcal{E}}[X] \leq \hat{\mathbb{E}}[X]$. We also call $\hat{\mathbb{E}}[X]$ and $\hat{\mathcal{E}}[X]$ the upper expectation and lower expectation of X, respectively.

The concepts of independence and identical distribution were introduced by Peng [4] and [5] as follows.

Definition 1.2. *(i) (**Identical distribution**) Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if*

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)], \ \ \forall \varphi \in C_{l,Lip}(\mathbb{R}_n),$$

whenever the sub-expectations are finite.

*(ii) (**Independence**) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}$ if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$, we have*

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]|_{x=X}],$$

whenever $\bar{\varphi}(x) := \hat{\mathbb{E}}[|\varphi(x, Y)|] < \infty$ for all x and $\hat{\mathbb{E}}[|\varphi(X)|] < \infty$.

*(iii) (**Independent random variables**) A sequence $\{X_n, n \geq 1\}$ of random variables is said to be independent, if X_{i+1} is independent to $(X_1, \ldots, X_i)$ for each $i \geq 1$.*

Zhang [11] introduced the following concept of extended independence under sub-linear expectations.

Definition 1.3. *A sequence $\{X_n, \ n \geq 1\}$ of random variables is said to be extended independent, if*

$$\hat{\mathbb{E}}\left[\prod_{i=1}^{n} \varphi_i(X_i)\right] = \prod_{i=1}^{n} \hat{\mathbb{E}}[\varphi_i(X_i)], \ \ \forall \ n \geq 2, \ \forall \ 0 \leq \varphi_i(x) \in C_{l,Lip}(\mathbb{R}).$$

An array $\{X_{ni}, \ 1 \leq i \leq n, \ n \geq 1\}$ is said to be rowwise extended independent, if for any fixed $n \geq 1$, $\{X_{ni}, \ 1 \leq i \leq n\}$ are extended independent random variables.

It can be showed that the independence implies the extended independence. It is obvious that, if $\{X_n, n \geq 1\}$ is a sequence of independent (extended independent) random variables and $f_1(x), f_2(x), \ldots \in C_{l,Lip}(\mathbb{R})$, then $\{f_n(X_n), \ n \geq 1\}$ is also a sequence of independent (extended independent) random variables.

Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\phi) = 0, \ \mathbb{V}(\Omega) = 1 \textit{ and } V(A) \leq \mathbb{V}(B), \quad \forall \ A \subset B \textit{ and } A, \ B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A\bigcup B)\le V(A)+V(B),\ \forall\ A,\ B\in\mathcal{G}$ and $A\bigcup B\in\mathcal{G}$.

Let $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ be a sub-linear space. It is natural to define the capacity of a set A to be the sub-linear expectation of the indicator function I_A of A. However, I_A may be not in $\mathcal{H}$. So, we denote a pair $(\mathbb{V},\ \mathcal{V})$ of capacities by

$$\mathbb{V}(A):=\inf\{\hat{\mathbb{E}}[\xi]: I_A\le\xi,\xi\in\mathcal{H}\},\mathcal{V}(A):=\mathbb{V}(A^c),\forall A\in\mathcal{F},$$

where A^c is the complement set of A. Then

$$\hat{\mathbb{E}}[f]\le\mathbb{V}(A)\le\hat{\mathbb{E}}[g],\ \hat{\mathcal{E}}[f]\le\hat{\mathcal{E}}[A]\le\hat{\mathcal{E}}[g],\ if\ f\le I_A\le g,\ f,\ g\in\mathcal{H}.$$

Also, we define the Choquet integrals/expecations $(C_{\mathbb{V}},C_{\mathcal{V}})$ by

$$C_V[X]=\int_0^\infty V(X\ge t)dt+\int_{-\infty}^0[V(X\ge t)-1]dt$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively.

In this work, we will establish some exponential inequalities for identically distributed extended independent random variables under the sub-linear expectations. In addition, by using these exponential inequalities, we will further investigate the strong law of large numbers under sub-linear expectations with the strong convergence rate $O(n^{-1/2}\ln^{1/2}n)$. The results obtained in the paper will generalize some corresponding ones under the classical linear expectations.

Throughout the paper, let C represent a positive constant which may vary in different places. Denote $\log x=\ln\max(x,e)$, $S_n=\sum_{k=1}^n X_k$, $B_n=\sum_{k=1}^n\hat{\mathbb{E}}[X_k^2]$ and $M_{n,p}=\sum_{k=1}^n\hat{\mathbb{E}}[|X_k|^p]$, $M_{n,p,+}=\sum_{k=1}^n\hat{\mathbb{E}}[(X_k^+)^p]$. $a_n=O(b_n)$ stands for $a_n\le Cb_n$, where $\{a_n,n\ge1\}$ and $\{b_n,n\ge1\}$ are two sequences of positive constants.

2. Main results and their proofs

Let $\{X_n,n\ge1\}$ be a sequence of random variables and $\{c_n,n\ge1\}$ be a sequence of positive numbers such that $\lim\limits_{n\to\infty}c_n=\infty$. For fixed $n\ge1$, denote for $1\le i\le n$ that

$$\begin{aligned}&X_{i,n}=-c_nI(X_i<-c_n)+X_iI(-c_n\le X_i\le c_n)+c_nI(X_i>c_n),\\&Y_{i,n}=(X_i-c_n)I(X_i>c_n),\qquad Z_{i,n}=(X_i+c_n)I(X_i<-c_n).\end{aligned}\tag{2.1}$$

It is easy to check that $X_{i,n}+Y_{i,n}+Z_{i,n}=X_i$ for $1\le i\le n,\ n\ge1$ and $X_{i,n}$ are bounded by c_n for each fixed $n\ge1$. If $\{X_n,n\ge1\}$ are extended independent random variables, then $\{X_{i,n},1\le i\le n\}$, $\{Y_{i,n},1\le i\le n\}$ and $\{Z_{i,n},1\le i\le n\}$ are all extended independent random variables for each fixed $n\ge1$ under sub-linear expectations.

Before we state the main results of the paper, we first list some important lemmas, which will be used to prove the main results of the paper

Lemma 2.1.(Borel-Cantelli lemma, cf. Zhang [9]) *Let $\{A_n,n\ge1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If $\sum_{n=1}^\infty V(A_n)<\infty$, then*

$$V(A_n\ i.o.)=0,\quad where\quad \{A_n\ i.o.\}=\bigcap_{n=1}^\infty\bigcup_{i=n}^\infty A_i.$$

Lemma 2.2. *Let $X_1,\ X_2,\ \cdots,X_n$ be extended independent random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_i]\le0$ for all $i=1,2,\cdots,n$.*

(a) For all $x,y>0$,

$$\mathbb{V}(S_n\ge x)\le\mathbb{V}\left(\max_{k\le n}X_k\ge y\right)+\exp\left\{-\frac{x^2}{2(xy+B_n)}\left(1+\frac{2}{3}\ln(1+\frac{xy}{B_n})\right)\right\}.\tag{2.2}$$

(b) For any $p \geq 2$, there exists a constant $C_p \geq 1$ such that for all $x > 0$ and $0 < \delta \leq 1$,

$$\mathbb{V}(S_n \geq x) \leq C_p \delta^{-2p} \frac{M_{n,p,+}}{x^p} + \exp\left\{-\frac{x^2}{2B_n(1+\delta)}\right\}. \tag{2.3}$$

(c) We have for $x > 0$, $r > 0$ and $p \geq 2$ that

$$\mathbb{V}(S_n^+ \geq x) \leq \mathbb{V}\left(\max_{k \leq n} X_k^+ \geq \frac{x}{r}\right) + e^r \left(\frac{rB_n}{rB_n + x^2}\right), \tag{2.4}$$

$$C_{\mathbb{V}}\left[(S_n^+)^p\right] \leq p^p C_{\mathbb{V}}\left[(\max_{k \leq n} X_k^+)^p\right] + C_P B_n^{p/2}. \tag{2.5}$$

In particular,

$$\mathbb{V}(S_n \geq x) \leq (1+e)\frac{B_n}{x^2}. \tag{2.6}$$

Proof. It is obvious that, if $\{X_n,\ n \geq 1\}$ is a sequence of extended independent random variables, then they are extended negatively dependent with $K = 1$. Similar to the proof of Theorem 3.1 in Zhang [11], we can obtain Lemma 2.2 immediately. □

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of extended independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_i] \leq 0$ for each $i \geq 1$. If there exists a sequence of positive numbers $\{c_n, n \geq 1\}$ such that $|X_i| \leq c_i$ for each $i \geq 1$, then for any $t > 0$ and $n \geq 1$,*

$$\hat{\mathbb{E}}\left[\exp\left(t\sum_{i=1}^n X_i\right)\right] \leq \exp\left\{\frac{t^2}{2}\sum_{i=1}^n e^{tc_i}\hat{\mathbb{E}}[X_i^2]\right\}. \tag{2.7}$$

Proof. It is easy to check that $e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}$ for all $x \in \mathbb{R}$. Thus, by $\hat{\mathbb{E}}[X_i] \leq 0$ and $|X_i| \leq c_i$ for each $i \geq 1$, we have that for any $t > 0$,

$$\begin{aligned}\hat{\mathbb{E}}\left[e^{tX_i}\right] &\leq 1 + t\hat{\mathbb{E}}[X_i] + \frac{1}{2}t^2\hat{\mathbb{E}}\left[X_i^2 e^{t|X_i|}\right] \leq 1 + \frac{1}{2}t^2\hat{\mathbb{E}}\left[X_i^2 e^{t|X_i|}\right] \\ &\leq 1 + \frac{1}{2}t^2 e^{tc_i}\hat{\mathbb{E}}\left[X_i^2\right] \leq \exp\left\{\frac{1}{2}t^2 e^{tc_i}\hat{\mathbb{E}}[X_i^2]\right\}. \end{aligned} \tag{2.8}$$

By the definition of extended independent random variables and (2.8), we can see that

$$\hat{\mathbb{E}}\left[\exp\left(t\sum_{i=1}^n X_i\right)\right] = \prod_{i=1}^n \hat{\mathbb{E}}\left[e^{tX_i}\right] \leq \exp\left\{\frac{t^2}{2}\sum_{i=1}^n e^{tc_i}\hat{\mathbb{E}}[X_i^2]\right\}. \tag{2.9}$$

This completes the proof of the lemma. □

Noting that $\hat{\mathbb{E}}[X_{i,n}^2] \leq \hat{\mathbb{E}}[X_i^2]$ for $1 \leq i \leq n$ and $n \geq 1$, we can therefore get the following corollary for $\{X_{i,n}, 1 \leq i \leq n, n \geq 1\}$.

Corollary 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of extended independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\{X_{i,n}, 1 \leq i \leq n, n \geq 1\}$ be defined by (2.1). Then for any $t > 0$ and $n \geq 1$,*

$$\hat{\mathbb{E}}\left[\exp\left(t\sum_{i=1}^n \left(X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right)\right)\right] \leq \exp\left\{2t^2 e^{2tc_n}\sum_{i=1}^n \hat{\mathbb{E}}[X_i^2]\right\}. \tag{2.10}$$

Proof It is easily seen that $\{X_{i,n} - \hat{\mathbb{E}}[X_{i,n}], 1 \le i \le n, n \ge 1\}$ are still extended independent random variables with $\hat{\mathbb{E}}\left[X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right] = 0$ and $|X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]| \le 2c_n$ for each $1 \le i \le n$ and $n \ge 1$. By Lemma 2.3, C_r-inequality and Jensen's inequality, we have

$$\begin{aligned}
\hat{\mathbb{E}}\left[\exp\left(t\sum_{i=1}^{n}\left(X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right)\right)\right] &\le \exp\left\{\frac{t^2}{2}e^{2tc_n}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left(X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right)^2\right]\right\} \\
&\le \exp\left\{\frac{t^2}{2}e^{2tc_n}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[2\left(X_{i,n}^2 + \hat{\mathbb{E}}^2[X_{i,n}]\right)\right]\right\} \\
&\le \exp\left\{2t^2e^{2tc_n}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[X_{i,n}^2\right]\right\} \\
&\le \exp\left\{2t^2e^{2tc_n}\sum_{i=1}^{n}\hat{\mathbb{E}}[X_i^2]\right\}, \qquad (2.11)
\end{aligned}$$

which yields the desired result (2.10). □

Now, we present the main results of the paper. The first one is the exponential inequality for $\{X_{i,n}, 1 \le i \le n, n \ge 1\}$.

Theorem 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of extended independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and $\{X_{i,n}, 1 \le i \le n, n \ge 1\}$ be defined by (2.1). Denote $B_n^2 = \sum_{i=1}^{n}\hat{\mathbb{E}}[X_i^2]$ for each $n \ge 1$. Then for any $\varepsilon > 0$ such that $\varepsilon \le \frac{2eB_n^2}{c_n}$ and $n \ge 1$,*

$$\begin{aligned}
&\mathbb{V}\left(\left\{\sum_{i=1}^{n}\left(X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right) \ge \varepsilon\right\}\bigcup\left\{\sum_{i=1}^{n}\left(X_{i,n} - \hat{\mathcal{E}}[X_{i,n}]\right) \le -\varepsilon\right\}\right) \\
\le\ & 2\exp\left\{-\frac{\varepsilon^2}{8eB_n^2}\right\}. \qquad (2.12)
\end{aligned}$$

Proof By Markov's inequality and Corollary 2.1, we have that for any $t > 0$,

$$\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right) \ge \varepsilon\right) &\le e^{-t\varepsilon}\hat{\mathbb{E}}\left[\exp\left(t\sum_{i=1}^{n}\left(X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right)\right)\right] \\
&\le \exp\left\{-t\varepsilon + 2t^2e^{2tc_n}B_n^2\right\}.
\end{aligned}$$

Taking $t = \frac{\varepsilon}{4eB_n^2}$, and noting that $2tc_n \le 1$, we can obtain

$$\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right) \ge \varepsilon\right) \le \exp\left\{-\frac{\varepsilon^2}{8eB_n^2}\right\}. \qquad (2.13)$$

Since $\{-X_{i,n}, 1 \le i \le n, n \ge 1\}$ is still a sequence of extended independent random variables, we have by (2.13) that

$$\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{i,n} - \hat{\mathcal{E}}[X_{i,n}]\right) \le -\varepsilon\right) &= \mathbb{V}\left(\sum_{i=1}^{n}\left(-X_{i,n} - \hat{\mathbb{E}}[-X_{i,n}]\right) \ge \varepsilon\right) \\
&\le \exp\left\{-\frac{\varepsilon^2}{8eB_n^2}\right\}. \qquad (2.14)
\end{aligned}$$

Combining (2.13) and (2.14), we can get that

$$\begin{aligned}
&\mathbb{V}\left(\left\{\sum_{i=1}^{n}\left(X_{i,n}-\hat{\mathbb{E}}[X_{i,n}]\right)\geq\varepsilon\right\}\bigcup\left\{\sum_{i=1}^{n}\left(X_{i,n}-\hat{\mathcal{E}}[X_{i,n}]\right)\leq-\varepsilon\right\}\right)\\
\leq\;&\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{i,n}-\hat{\mathbb{E}}[X_{i,n}]\right)\geq\varepsilon\right)+\mathbb{V}\left(\sum_{i=1}^{n}\left(X_{i,n}-\hat{\mathcal{E}}[X_{i,n}]\right)\leq-\varepsilon\right)\\
\leq\;&2\exp\left\{-\frac{\varepsilon^2}{8eB_n^2}\right\}. \qquad (2.15)
\end{aligned}$$

This completes the proof of the theorem. □

For identically distributed extended independent random variables $\{X_n, n\geq 1\}$, we can get the following corollary by Theorem 2.1.

Corollary 2.2. *Let $\{X_n, n\geq 1\}$ be a sequence of identically distributed extended independent random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ and $\{X_{i,n}, 1\leq i\leq n, n\geq 1\}$ be defined by (2.1). Then for any $\varepsilon>0$ such that $\varepsilon\leq\frac{2e\hat{\mathbb{E}}[X_1^2]}{c_n}$ and $n\geq 1$,*

$$\mathbb{V}\left(\left\{\sum_{i=1}^{n}\left(X_{i,n}-\hat{\mathbb{E}}[X_{i,n}]\right)\geq n\varepsilon\right\}\bigcup\left\{\sum_{i=1}^{n}\left(X_{i,n}-\hat{\mathcal{E}}[X_{i,n}]\right)\leq-n\varepsilon\right\}\right)\leq 2\exp\left\{-\frac{n\varepsilon^2}{8e\hat{\mathbb{E}}[X_1^2]}\right\}.$$

Proof By Theorem 2.1 ,we have

$$\begin{aligned}
&\mathbb{V}\left(\left\{\sum_{i=1}^{n}\left(X_{i,n}-\hat{\mathbb{E}}[X_{i,n}]\right)\geq n\varepsilon\right\}\bigcup\left\{\sum_{i=1}^{n}\left(X_{i,n}-\hat{\mathcal{E}}[X_{i,n}]\right)\leq-n\varepsilon\right\}\right)\\
\leq\;&2\exp\left\{-\frac{n^2\varepsilon^2}{8e\sum_{i=1}^{n}\hat{\mathbb{E}}[X_i^2]}\right\}=2\exp\left\{-\frac{n\varepsilon^2}{8e\hat{\mathbb{E}}[X_1^2]}\right\}.
\end{aligned}$$

The proof is completed. □

For arrays $\{Y_{i,n}, 1\leq i\leq n, n\geq 1\}$ and $\{Z_{i,n}, 1\leq i\leq n, n\geq 1\}$, we have the following exponential inequalities.

Theorem 2.2. *Let $\{X_n, n\geq 1\}$ be a sequence of identically distributed extended independent random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ with $\lim_{c\to\infty}\hat{\mathbb{E}}[(|X_1|^2-c)^+]=0$. Let $\{Y_{i,n}, 1\leq i\leq n, n\geq 1\}$ and $\{Z_{i,n}, 1\leq i\leq n, n\geq 1\}$ be defined by (2.1). Assume that there exists a $\delta>0$ satsifying $\sup_{|\tau|\leq\delta}\hat{\mathbb{E}}[e^{\tau X_1^2}]\leq M_\delta<\infty$, where M_δ is a positive constant depending only on δ. Then for any $\varepsilon>0$, $\tau\in(0,\delta]$, and all n large enough,*

$$\begin{aligned}
&\mathbb{V}\left(\left\{\sum_{i=1}^{n}\left(Y_{i,n}+\hat{\mathbb{E}}[-Y_{i,n}]\right)\geq n\varepsilon\right\}\bigcup\left\{\sum_{i=1}^{n}\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\leq-n\varepsilon\right\}\right)\\
\leq\;&\frac{32(1+e)(\tau c_n^2+1)M_\delta}{n\tau\varepsilon^2e^{\tau c_n^2}}, \qquad (2.16)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{V}\left(\left\{\sum_{i=1}^{n}\left(Z_{i,n}+\hat{\mathbb{E}}[-Z_{i,n}]\right)\geq n\varepsilon\right\}\bigcup\left\{\sum_{i=1}^{n}\left(Z_{i,n}-\hat{\mathbb{E}}[Z_{i,n}]\right)\leq-n\varepsilon\right\}\right)\\
\leq\;&\frac{32(1+e)(\tau c_n^2+1)M_\delta}{n\tau\varepsilon^2e^{\tau c_n^2}}. \qquad (2.17)
\end{aligned}$$

Proof We first prove (2.16). Note that

$$\begin{aligned}
& \mathbb{V}\left(\sum_{i=1}^{n}\left(Y_{i,n}+\hat{\mathbb{E}}[-Y_{i,n}]\right)\geq n\varepsilon\right) \\
\leq\ & \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]+\hat{\mathbb{E}}[Y_{i,n}]+\hat{\mathbb{E}}[-Y_{i,n}]\right)\geq \varepsilon\right) \\
=\ & \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\geq \varepsilon-\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\mathbb{E}}[Y_{i,n}]+\hat{\mathbb{E}}[-Y_{i,n}]\right)\right).
\end{aligned} \tag{2.18}$$

By (2.1) and $\lim_{c\to\infty}\hat{\mathbb{E}}[(|X|-c)^+]=0$, we have

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\mathbb{E}}[Y_{i,n}]+\hat{\mathbb{E}}[-Y_{i,n}]\right) \\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\mathbb{E}}[(X_i-c_n)I(X_i>c_n)]+\hat{\mathbb{E}}[-(X_i-c_n)I(X_i>c_n)]\right) \\
\leq\ & 2\hat{\mathbb{E}}[|X_1-c_n|I(X_1>c_n)]=2\hat{\mathbb{E}}[(X_1-c_n)^+] \\
\leq\ & 2\hat{\mathbb{E}}[(|X_1|-c_n)^+]\to 0,\quad as\ n\to\infty.
\end{aligned} \tag{2.19}$$

It follows from (2.18) and (2.19) that for all n large enough,

$$\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n}\left(Y_{i,n}+\hat{\mathbb{E}}[-Y_{i,n}]\right)\geq n\varepsilon\right) &\leq \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n}\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\geq \frac{\varepsilon}{2}\right) \\
&= \mathbb{V}\left(\sum_{i=1}^{n}\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\geq \frac{n\varepsilon}{2}\right).
\end{aligned}$$

By Markov's inequality, (2.6), C_r-inequality and Jensen's inequality, we can get that for all n large enough,

$$\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n}\left(Y_{i,n}+\hat{\mathbb{E}}[-Y_{i,n}]\right)\geq n\varepsilon\right) &\leq \mathbb{V}\left(\sum_{i=1}^{n}\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\geq \frac{n\varepsilon}{2}\right) \\
&\leq \frac{4(1+e)}{n^2\varepsilon^2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)^2\right] \\
&\leq \frac{4(1+e)}{n^2\varepsilon^2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[2\left(Y_{i,n}^2+\hat{\mathbb{E}}^2[Y_{i,n}]\right)\right] \\
&\leq \frac{16(1+e)\sum_{i=1}^{n}\hat{\mathbb{E}}[Y_{i,n}^2]}{n^2\varepsilon^2}=\frac{16(1+e)\hat{\mathbb{E}}[Y_{1,n}^2]}{n\varepsilon^2}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n}\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\leq -n\varepsilon\right) &= \mathbb{V}\left(-\sum_{i=1}^{n}\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\geq n\varepsilon\right) \\
&\leq \mathbb{V}\left(\sum_{i=1}^{n}\left(-Y_{i,n}-\hat{\mathbb{E}}[-Y_{i,n}]\right)\geq \frac{n\varepsilon}{2}\right) \\
&\leq \frac{4(1+e)}{n^2\varepsilon^2}\sum_{i=1}^{n}\hat{\mathbb{E}}\left[2\left(Y_{i,n}^2+\hat{\mathbb{E}}^2[Y_{i,n}]\right)\right] \\
&\leq \frac{16(1+e)\hat{\mathbb{E}}[Y_{1,n}^2]}{n\varepsilon^2}.
\end{aligned}$$

Therefore, it remains only to estimate $\hat{\mathbb{E}}[Y_{1,n}^2]$. For arbitrary random variable X, it can be verified that, if $\lim_{c\to\infty}\hat{\mathbb{E}}[(|X|^2-c)^+]=0$, then $\lim_{c\to\infty}\hat{\mathbb{E}}[(|X|-c)^+]=0$, and thus, $\hat{\mathbb{E}}[|X|]\le C_{\mathbb{V}}[|X|]$. One can refer to Lemma 3.9 of Zhang [9], or Zhang [11] for instance. Noting that

$$0\le \lim_{c\to\infty}\hat{\mathbb{E}}\left[\left\{(X_1-c_n)^2 I(X_1>c_n)-c\right\}^+\right]\le \lim_{c\to\infty}\hat{\mathbb{E}}[(|X_1|^2-c)^+]=0,$$

we have $\hat{\mathbb{E}}[(X_1-c_n)^2I(X_1>c_n)]\le C_{\mathbb{V}}[(X_1-c_n)^2I(X_1>c_n)]$. Hence

$$\begin{aligned}
\hat{\mathbb{E}}[Y_{1,n}^2] &\le C_{\mathbb{V}}\left[(X_1-c_n)^2I(X_1>c_n)\right]\\
&= \int_0^\infty \mathbb{V}((X_1-c_n)^2I(X_1>c_n)\ge t)dt\\
&\le \int_0^\infty \mathbb{V}(X_1^2I(X_1>c_n)\ge t)dt\\
&= \int_0^{c_n^2}\mathbb{V}(X_1^2>c_n^2)dt+\int_{c_n^2}^\infty \mathbb{V}(X_1^2\ge t)dt\\
&\le \frac{c_n^2\mathbb{E}[e^{\tau X_1^2}]}{e^{\tau c_n^2}}+\frac{\mathbb{E}[e^{\tau X_1^2}]}{\tau e^{\tau c_n^2}}\\
&\le \frac{(\tau c_n^2+1)\mathbb{E}[e^{\tau X_1^2}]}{\tau e^{\tau c_n^2}}.
\end{aligned}$$

Noting that $\sup_{|\tau|\le\delta}\hat{\mathbb{E}}[e^{\tau X_1^2}]\le M_\delta<\infty$, we have that for all n large enough,

$$\begin{aligned}
&\mathbb{V}\left(\left\{\sum_{i=1}^n\left(Y_{i,n}+\hat{\mathbb{E}}[-Y_{i,n}]\right)\ge n\varepsilon\right\}\bigcup\left\{\sum_{i=1}^n\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\le -n\varepsilon\right\}\right)\\
\le\ &\mathbb{V}\left(\sum_{i=1}^n\left(Y_{i,n}+\hat{\mathbb{E}}[-Y_{i,n}]\right)\ge n\varepsilon\right)+\mathbb{V}\left(\sum_{i=1}^n\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\le -n\varepsilon\right)\\
\le\ &\frac{32(1+e)(\tau c_n^2+1)M_\delta}{n\tau\varepsilon^2e^{\tau c_n^2}}.
\end{aligned}$$

For (2.17), the proof is similar to the case for (2.16) and is omitted.

The proof is completed. □

By Theorem 2.2, we can get the following corollary immediately.

Corollary 2.3. *Let $\{X_n, n\ge 1\}$ be a sequence of identically distributed extended independent random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$ with $\lim_{c\to\infty}\hat{\mathbb{E}}[(|X_1|^2-c)^+]=0$. Let $\{Y_{i,n}, 1\le i\le n, n\ge 1\}$ and $\{Z_{i,n}, 1\le i\le n, n\ge 1\}$ be defined by (2.1). Assume that $\hat{\mathbb{E}}[e^{\delta X_1^2}]<\infty$ for some $\delta>0$. Then for any $\varepsilon>0$ and for all n large enough,*

$$\begin{aligned}
&\mathbb{V}\left(\left\{\sum_{i=1}^n\left(Y_{i,n}+\hat{\mathbb{E}}[-Y_{i,n}]\right)\ge n\varepsilon\right\}\bigcup\left\{\sum_{i=1}^n\left(Y_{i,n}-\hat{\mathbb{E}}[Y_{i,n}]\right)\le -n\varepsilon\right\}\right)\\
\le\ &\frac{32(1+e)(\delta c_n^2+1)\hat{\mathbb{E}}[e^{\delta X_1^2}]}{n\delta\varepsilon^2e^{\delta c_n^2}},
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{V}\left(\left\{\sum_{i=1}^n\left(Z_{i,n}+\hat{\mathbb{E}}[-Z_{i,n}]\right)\ge n\varepsilon\right\}\bigcup\left\{\sum_{i=1}^n\left(Z_{i,n}-\hat{\mathbb{E}}[Z_{i,n}]\right)\le -n\varepsilon\right\}\right)\\
\le\ &\frac{32(1+e)(\delta c_n^2+1)\hat{\mathbb{E}}[e^{\delta X_1^2}]}{n\delta\varepsilon^2e^{\delta c_n^2}}.
\end{aligned}$$

Proof It is easily seen that $\sup_{|\tau|\le\delta}\hat{\mathbb{E}}[e^{\tau X_1^2}] \le \hat{\mathbb{E}}[e^{\delta X_1^2}] \triangleq M_\delta < \infty$, which implies the desired results immediately from Theorem 2.2. □

Theorem 2.3. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed extended independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\lim_{c\to\infty}\hat{\mathbb{E}}[(|X_1|^2 - c)^+] = 0$ and $\hat{\mathbb{E}}[e^{\delta X_1^2}] < \infty$ for some $\delta > 0$. Let $\{c_n, n \ge 1\}$ be a sequence of positive numbers such that*

$$0 < c_n \le \left(\frac{ne\hat{\mathbb{E}}[X_1^2]}{2\delta}\right)^{1/4} \quad for\ some\ n \ge n_0, \tag{2.20}$$

where n_0 is a positive integer. Denote $\varepsilon_n = \sqrt{8\delta e\hat{\mathbb{E}}[X_1^2]c_n^2/n}$. Then for any $n \ge n_0$,

$$\begin{aligned}
&\mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^n\left(X_i - \hat{\mathbb{E}}[X_i]\right) \ge 3\varepsilon_n\right\}\bigcup\left\{\frac{1}{n}\sum_{i=1}^n\left(X_i - \hat{\mathcal{E}}[X_i]\right) \le -3\varepsilon_n\right\}\right)\\
\le\ & 2\left(1 + \frac{4(1+e)(\delta c_n^2+1)\hat{\mathbb{E}}[e^{\delta X_1^2}]}{2\delta^2 e\hat{\mathbb{E}}[X_1^2]c_n^2}\right)e^{-\delta c_n^2}.
\end{aligned}\tag{2.21}$$

Proof It is easy to check that $\varepsilon_n c_n \le 2e\hat{\mathbb{E}}[X_1^2]$ for $n \ge n_0$ and $\frac{n\varepsilon_n^2}{8e\hat{\mathbb{E}}[X_1^2]} = \delta c_n^2$. Noting that

$$\hat{\mathbb{E}}[X_i] \ge \hat{\mathbb{E}}[X_{i,n}] - \hat{\mathbb{E}}[-Y_{i,n}] - \hat{\mathbb{E}}[-Z_{i,n}]$$

and

$$\hat{\mathcal{E}}[X_i] \le \hat{\mathcal{E}}[X_{i,n}] + \hat{\mathbb{E}}[Y_{i,n}] + \hat{\mathbb{E}}[Z_{i,n}],$$

we have by Corollaries 2.2 and 2.3 that

$$\begin{aligned}
&\mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^n\left(X_i - \hat{\mathbb{E}}[X_i]\right) \ge 3\varepsilon_n\right\}\bigcup\left\{\frac{1}{n}\sum_{i=1}^n\left(X_i - \hat{\mathcal{E}}[X_i]\right) \le -3\varepsilon_n\right\}\right)\\
\le\ & \mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^n\left(X_{i,n} - \hat{\mathbb{E}}[X_{i,n}]\right) \ge \varepsilon_n\right\}\bigcup\left\{\frac{1}{n}\sum_{i=1}^n\left(X_{i,n} - \hat{\mathcal{E}}[X_{i,n}]\right) \le -\varepsilon_n\right\}\right)\\
& +\mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^n\left(Y_{i,n} + \hat{\mathbb{E}}[-Y_{i,n}]\right) \ge \varepsilon_n\right\}\bigcup\left\{\frac{1}{n}\sum_{i=1}^n\left(Y_{i,n} - \hat{\mathbb{E}}[Y_{i,n}]\right) \le -\varepsilon_n\right\}\right)\\
& +\mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^n\left(Z_{i,n} + \hat{\mathbb{E}}[-Z_{i,n}]\right) \ge \varepsilon_n\right\}\bigcup\left\{\frac{1}{n}\sum_{i=1}^n\left(Z_{i,n} - \hat{\mathbb{E}}[Z_{i,n}]\right) \le -\varepsilon_n\right\}\right)\\
\le\ & 2\exp\left\{-\frac{n\varepsilon_n^2}{8e\hat{\mathbb{E}}[X_1^2])}\right\} + \frac{64(1+e)(\delta c_n^2+1)\hat{\mathbb{E}}[e^{\delta X_1^2}]}{n\delta\varepsilon_n^2}e^{-\delta c_n^2}\\
=\ & 2\left(1 + \frac{4(1+e)(\delta c_n^2+1)\hat{\mathbb{E}}[e^{\delta X_1^2}]}{\delta^2 e\hat{\mathbb{E}}[X_1^2]c_n^2}\right)e^{-\delta c_n^2},
\end{aligned}$$

which yields (2.21). This completes the proof of the theorem. □

Taking $c_n = \sqrt{\ln n}$ and $\delta > 1$ in Theorem 2.3, we can get the following result.

Theorem 2.4. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed extended independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\lim_{c\to\infty}\hat{\mathbb{E}}[(|X_1|^2 - c)^+] = 0$ and $\hat{\mathbb{E}}[e^{\delta X_1^2}] < \infty$ for some $\delta > 1$. Denote $\varepsilon_n = \sqrt{(8\delta e\hat{\mathbb{E}}[X_1^2]\ln n)/n}$. Then*

$$\sum_{n=1}^\infty \mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^n\left(X_i - \hat{\mathbb{E}}[X_i]\right) \ge 3\varepsilon_n\right\}\bigcup\left\{\frac{1}{n}\sum_{i=1}^n(X_i - \hat{\mathcal{E}}[X_i]) \le -3\varepsilon_n\right\}\right) < \infty. \tag{2.22}$$

Proof It follows from (2.21) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{\mathbb{E}}[X_i]\right) \geq 3\varepsilon_n\right\} \bigcup \left\{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{\mathcal{E}}[X_i]\right) \leq -3\varepsilon_n\right\}\right) \\
\leq\ & C\sum_{n=1}^{\infty}\left(1 + \frac{4(1+e)(\delta c_n^2+1)\hat{\mathbb{E}}[e^{\delta X_1^2}]}{\delta^2 e\hat{\mathbb{E}}[X_1^2]c_n^2}\right)e^{-\delta c_n^2} \leq C\sum_{n=1}^{\infty} e^{-\delta c_n^2} = C\sum_{n=1}^{\infty} n^{-\delta} < \infty,
\end{aligned}$$

which implies (2.22). The proof is completed. □

Corollary 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed extended independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\lim_{c\to\infty} \hat{\mathbb{E}}[(|X_1|^2 - c)^+] = 0$, $\hat{\mathbb{E}}[X_i] = \hat{\mathcal{E}}[X_i] = 0$ and $\hat{\mathbb{E}}[e^{\delta X_1^2}] < \infty$ for some $\delta > 1$. Denote $\varepsilon_n = \sqrt{(8\delta e\hat{\mathbb{E}}[X_1^2]\ln n)/n}$. Then*

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq 3\varepsilon_n\right) < \infty. \tag{2.23}$$

Remark 2.1. The Borel-Cantelli lemma implies that $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges almost surely in capacity $\mathcal{V}$ with convergence rate $O(n^{-1/2}\ln^{1/2} n)$ under the conditions of Corollary 2.4.

Taking $c_n = \sqrt{\log n \cdot \log\log n}$ and $\delta > 0$ in Theorem 2.3, we can get the following result.

Theorem 2.5. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed extended independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\lim_{c\to\infty} \hat{\mathbb{E}}[(|X_1|^2 - c)^+] = 0$ and $\hat{\mathbb{E}}[e^{\delta X_1^2}] < \infty$ for some $\delta > 0$. Denote $\varepsilon_n = \sqrt{(8\delta e\hat{\mathbb{E}}[X_1^2]\log n \cdot \log\log n)/n}$. Then*

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{\mathbb{E}}[X_i]\right) \geq 3\varepsilon_n\right\} \bigcup \left\{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{\mathcal{E}}[X_i]\right) \leq -3\varepsilon_n\right\}\right) < \infty. \tag{2.24}$$

Proof It follows from (2.21) again that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{V}\left(\left\{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{\mathbb{E}}[X_i]\right) \geq 3\varepsilon_n\right\} \bigcup \left\{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{\mathcal{E}}[X_i]\right) \leq -3\varepsilon_n\right\}\right) \\
\leq\ & C\sum_{n=1}^{\infty}\left(1 + \frac{4(1+e)(\delta c_n^2+1)\hat{\mathbb{E}}[e^{\delta X_1^2}]}{\delta^2 e\hat{\mathbb{E}}[X_1^2]c_n^2}\right)e^{-\delta c_n} \leq C\sum_{n=1}^{\infty} e^{-\delta c_n^2} \leq C\sum_{n=1}^{\infty} e^{-2\log n} < \infty,
\end{aligned}$$

since $-\delta c_n^2 \leq -2\log n$ for all n large enough. Hence, the desired results (2.24) is proved. □

Corollary 2.5. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed extended independent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\lim_{c\to\infty} \hat{\mathbb{E}}[(|X_1|^2 - c)^+] = 0$, $\hat{\mathbb{E}}[X_i] = \hat{\mathcal{E}}[X_i] = 0$ and $\hat{\mathbb{E}}[e^{\delta X_1^2}] < \infty$ for some $\delta > 0$. Denote $\varepsilon_n = \sqrt{(8\delta e\hat{\mathbb{E}}[X_1^2]\log n \cdot \log\log n)/n}$. Then*

$$\sum_{n=1}^{\infty} \mathbb{V}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq 3\varepsilon_n\right) < \infty. \tag{2.25}$$

Remark 2.2. The Borel-Cantelli lemma implies that $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges almost surely in capacity $\mathcal{V}$ with convergence rate $O(n^{-1/2}\log^{1/2} n \cdot \log^{1/2}\log n)$ under the conditions of Corollary 2.5.

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