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Linear Connections with and without Torsion, Making Parallel an Integrable Endomorphism on a Manifold

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Abstract. Our study is developed in a general framework, namely a manifold M endowed with a (1,1)-tensor field φ, which is integrable. The present paper solves the following two problems: how many linear connections with torsion and without torsion exist, having the property of being parallel with respect to φ. To count all these connections with the given properties, certain algebraic techniques and results are used throughout the paper.

To commemorate Mileva Prvanović (1929 - 2016), 90 years after her birth

0. Introduction

In 2014, Dušek and Kowalski had the idea to count the number of all real analytic affine connections with torsion which exist locally on a smooth manifold M of dimension n. In their paper [4], the families of general affine connections with torsion and with skew-symmetric Ricci tensor, or symmetric Ricci tensor, respectively, are described in terms of the number of arbitrary functions of n variables. This study was continued with a related topic in their paper [5], where they counted the number of all real analytic equiaffine connections with arbitrary torsion which exist locally on a smooth manifold M of dimension n. The families of general equiaffine connections and with skew-symmetric Ricci tensor, or with symmetric Ricci tensor, respectively, are described in terms of the number of arbitrary functions of n variables. Later on, the same authors dealt with how many torsionless affine connections exist in general dimension, (see [6]). Another interesting question was raised in [7], concerning how many Ricci flat affine connections are there with arbitrary torsion.

Also, Pripoae determined in [8] how many left invariant and bi-invariant connections there exist on Lie groups which satisfy additional geometric properties (such as torsionless, flatness, Ricci-flatness and so on). The framework of this study is the invariant geometry on Lie groups where the author investigates the existence and the non-existence of this geometries, aiming to obtain classification results.

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In mathematical literature there are many studies concerning the number of geometric objects having certain properties on manifolds. Our paper deals with the same topic, but the techniques that we use are completely different. We based our study on some algebraic methods and especially on Frobenius theorem. The main objects we deal with are linear connections which are parallel with a given integrable (1,1)-tensor field. Such (1,1)-tensor fields can be almost complex structures, almost product structures, f-structures of Kentaro Yano type, almost contact structures and so on. The linear connections which are parallel with such a (1,1)-tensor field are also of great interest in Differential Geometry, since there are many well known examples, such as the Levi-Civita connection on a Kähler manifold, on a para-Kähler manifold and so on. We discuss here both connections with torsion and without torsion.

We start our paper with an algebraic approach, followed in section 2 by applications to Frobenius theorem. In section 3 we discuss about structures on manifolds given by (1,1)-tensor fields and about connections with respect to which these structures are parallel. We expose here the main problem, which will be solved in section 4. The last section contains the main result of the paper. All geometric objects are taken to be smooth and the Einstein summation convention over repeated indices is assumed.

1. Algebraic approach

To solve the main problem stated in the next section, we need some algebraic preparations. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a real matrix of order n. Then the centralizer of A, denoted by

$$C(A) = \{X \in \mathcal{M}_n(\mathbb{R}) / XA = AX\}$$

ia a linear space.

We are going to reduce the main problem of section 3, to the following:

Algebraic problem: Compute the dimension of $C(A)$.

The answer is given by the following steps:

Step 1: We find the characteristic polynomial $P_A(\lambda)$ of A.

We recall the following:

Definition 1.1. *A polynomial whose dominant coefficient is one, is called monic.*

Then $P_A(\lambda)$ has a unique decomposition (up to the order) into some irreducible factors:

$$P_A(\lambda) = p_1^{s_1}(\lambda) \cdots p_r^{s_r}(\lambda),$$

which are powers of some monic polynomials of degree 1 or 2 with real coefficients.

Step 2: Associate a companion block matrix.

Given a polynomial factor $p^s(\lambda)$ from above, there is associated to it a companion block matrix B whose characteristic polynomial is $p^s(\lambda)$.

(i) If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is a root of $P_A(\lambda)$, then λ is represented by a block matrix B together with its conjugate $\overline{\lambda}$.

For instance, to a monic polynomial $\lambda^2 + a\lambda + b$ with real coefficients, but non-real roots, the companion block matrix is

$$B = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}.$$

(ii) If $\lambda \in \mathbb{R}$ is an eigenvalue of A, whose order of multiplicity s is equal to the dimension of the eigenspace $V(\lambda)$ of λ, then the Jordan form of A contains s blocks of order 1, namely (λ). Hence the companion block matrix is

$$B = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda \end{pmatrix}.$$

(iii) If λ is an eigenvalue of A whose order of multiplicity is different from $dimV(\lambda)$, then one has a flag space, since eigenvectors generate principal vectors.

From the above (i), (ii) and (iii), we draw the following:

Conclusion 1.2. *If we denote by B the companion block matrix of λ, then from the Frobenius theorem, the dimension contribution of B is:*

$$\sum_{i=1}^{k}(2i-1)deg f_i, \tag{1.1}$$

where $f_1, \ldots, f_k$ are invariant monic factors associated to B such that f_i divides f_{i-1}, $i = \overline{2,k}$.

Step 3: $dimC(A)$ is the sum of the contribution given by each root λ of $P_A(\lambda)$.

Note that when $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then its contribution is taken together with that of its conjugate $\overline{\lambda}$.

2. Applications of Frobenius theorem

In this section, we use the formula (1.1) in order to compute the dimension of the centralizer $C(A)$ for any matrix A of order 2 or 3, to show in detail how this procedure works.

Example 2.1. *Let $A \in \mathcal{M}_2(\mathbb{R})$ and let the roots of $P_A(\lambda)$ be $\lambda_{1,2} \in \mathbb{C}$.*

Case I. $\lambda_{1,2} \in \mathbb{R}$

1) If $\lambda_1 \neq \lambda_2$, then $P_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$. Since for each $i = 1, 2$, one has $deg(\lambda - \lambda_i) = 1$, then from (1.1), the contribution of λ_i is $(2 \cdot 1 - 1) \cdot 1$ and from section 1, Step 3, we obtain

$dimC(A) = 2.$

2) If $\lambda_1 = \lambda_2$, then the order of multiplicity of λ_1 is 2.

a) If $dimV(\lambda_1) \neq 2$, then the companion block martix of λ_1 is $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ and its characteristic polynomial $(\lambda - \lambda_1)^2$ is of degree 2. By (1.1), we obtain $dimC(A) = (2 \cdot 1 - 1) \cdot 2 = 2$.

b) If $dimV(\lambda_1) = 2$, then the companion block martix of λ_1 is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$, which decomposes into 2 blocks, both equal to (λ_1) and its characteristic polynomial is $f_1 f_2$, where $f_1 = f_2 = \lambda - \lambda_1$. Note that f_2 divides f_1 and $deg f_1 = deg f_2 = 1$. From (1.1), we obtain $dimC(A) = (2 \cdot 1 - 1) \cdot 1 + (2 \cdot 2 - 1) \cdot 1 = 4$.

Case II. If $\lambda_{1,2} \in \mathbb{C} \setminus \mathbb{R}$,

then $P_A(\lambda) = \lambda^2 + a\lambda + b$ is a polynomial of degree 2, irreducible over $\mathbb{R}$ (where $a = -(\lambda_1 + \lambda_2)$ and $b = \lambda_1\lambda_2 \in \mathbb{R}$). From (1.1), we have $dimC(A) = (2 \cdot 1 - 1) \cdot 2 = 2$.

Example 2.2. *Let $A \in \mathcal{M}_3(\mathbb{R})$ and let the roots of $P_A(\lambda)$ be $\lambda_{1,2,3} \in \mathbb{C}$.*

Case I. $\lambda_{1,2,3} \in \mathbb{R}$

1) If all $\lambda_{1,2,3}$ are distinct, then $P_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$. Since for each $i = \overline{1,3}$, one has $deg(\lambda - \lambda_i) = 1$, then from (1.1), the contribution of λ_i is $(2 \cdot 1 - 1) \cdot 1$ and from section 1, Step 3, we obtain $dimC(A) = 3$.

2) If $\lambda_1 = \lambda_2 \neq \lambda_3$, then the order of multiplicity of λ_1 is 2.

a) If $dimV(\lambda_1) \neq 2$, then the contribution of λ_1 is 4 (as in Example 2.1, Case I, 2a). Since from (1.1), the contribution of λ_3 is $(2 \cdot 1 - 1) \cdot 1 = 1$, then from section 1, Step 3, we have $dimC(A) = 4 + 1 = 5$.

b) If $dimV(\lambda_1) = 2$, then the contribution of λ_1 is 2 (as in Example 2.1, Case I, 2b). Since from (1.1), the contribution of λ_3 is $(2 \cdot 1 - 1) \cdot 1 = 1$, then from section 1, Step 3, we have $dimC(A) = 2 + 1 = 3$.

3) If $\lambda_1 = \lambda_2 = \lambda_3$, then the order of multiplicity of λ_1 is 3.

a) If $dimV(\lambda_1) = 3$, then the companion block martix of λ_1 is $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$, which contains 3 blocks equal to (λ_1) and $P_A(\lambda) = f_1 f_2 f_3$, where $f_1 = f_2 = f_3 = \lambda - \lambda_1$. We note that f_3 divides f_2, which divides f_1 and $degf_1 = degf_2 = degf_3$. From (1.1), we obtain $dimC(A) = (2 \cdot 1 - 1) \cdot 1 + (2 \cdot 2 - 1) \cdot 1 + (2 \cdot 3 - 1) \cdot 1 = 9$.

b) If $dimV(\lambda_1) = 2$, then the companion block martix of λ_1 is $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$ and $P_A(\lambda) = f_1 f_2$, where $f_1 = (\lambda - \lambda_1)^2$ and $f_2 = \lambda - \lambda_1$. We note that f_2 divides f_1, $degf_1 = 2$ and $degf_2 = 1$. From (1.1), we have $dimC(A) = (2 \cdot 1 - 1) \cdot 2 + (2 \cdot 2 - 1) \cdot 1 = 2 + 3 = 5$.

c) If $dimV(\lambda_1) = 1$, then the companion block martix of λ_1 is $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$ and $P_A(\lambda) = (\lambda - \lambda_1)^3$ is of degree 3. From (1.1), we have $dimC(A) = (2 \cdot 1 - 1) \cdot 3 = 3$.

Case II. If $\lambda_{1,2} \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda_3 \in \mathbb{R}$,
then $P_A(\lambda) = (\lambda^2 + a\lambda + b)(\lambda - \lambda_3)$ and (as in Example 2.1, Case II) the contribution of λ_1 together with $\lambda_2 = \overline{\lambda_1}$, is 2. From (1.1), the contribution of λ_3 is $(2 \cdot 1 - 1) \cdot 1 = 1$. From section 1, Step 3, we obtain $dimC(A) = 2 + 1 = 3$.

3. Structures on manifolds

On a manifold M, let $\mathcal{F}(M)$ be the ring of all smooth real functions on M and let $\chi(M)$ be the $\mathcal{F}(M)$ - module of all vector fields on M. Then any (1,1)-tensor field φ on M can be viewed as an $\mathcal{F}(M)$-endomorphism $\varphi : \chi(M) \to \chi(M)$. In the setting of G-structures, we recall here the following notion (see the Example 1.6, page 17 from [3] and also page 77 from [1]):

Definition 3.1. *Let M be an n-dimensional manifold endowed with a (1,1)-tensor field φ. Then φ is called integrable if around any point of M, there exists a local chart $(x^1, \ldots, x^n)$ with respect to which φ has constant coefficients φ^h_j, $j, h = \overline{1, n}$, given by:*

$$\varphi(\frac{\partial}{\partial x^j}) = \varphi^h_j(\frac{\partial}{\partial x^h}). \tag{3.1}$$

Remark 3.2.

(i) If φ is integrable, then there exists an atlas of local charts $(x^1, \ldots, x^n)$ and a real matrix $F = \left[\varphi^h_j\right]_{j,h=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$ whose entries are given by the relation (3.1). In other words, from Definition 3.1 it follows that the matrix F of φ is the same in any local chart of the atlas.

(ii) In particular, let J (resp. P) be an almost complex (resp. almost product) structure on M, that is $J^2 = -Id$ (resp. $P^2 = Id$ and $P \neq \pm Id$). Then J (resp. P) is a complex (resp. product) structure on M if and only if one of the following equivalent conditions is satisfied:

(a) J (resp. P) is integrable;

(b) The Nijenhuis tensor field associated to J (resp. P) vanishes identically;

(c) There exists an atlas of local charts on M, with respect to which the matrix $\left[J^h_j\right]_{j,h=\overline{1,n}}$ of J (resp. $\left[P^h_j\right]_{j,h=\overline{1,n}}$ of P), associated from the relation (3.1), is given by $\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} I_p & 0 \\ 0 & I_{n-p} \end{pmatrix}$), where $2k = n$ (resp. p is the dimension of the eigenspace corresponding to the eigenvalue 1 of P).

We recall here the following:

Definition 3.3. *Let (M, φ) be a manifold endowed with a (1,1)-tensor field. A linear connection ∇ on M is a φ-connection if φ is parallel with respect to ∇, that is*

$$\nabla \varphi = 0. \tag{3.2}$$

Remark 3.4.

(i) The relation (3.2) can be written as:

$$(\nabla_X\varphi)Y \overset{\text{Not}}{=} \nabla_X(\varphi Y) - \varphi(\nabla_X Y) = 0, \forall X, Y \in \chi(M). \tag{3.3}$$

(ii) In local coordinates, the relation (3.3) can be written as follows:

$$\begin{aligned}
&(\nabla_{\frac{\partial}{\partial x^i}}\varphi)\frac{\partial}{\partial x^j} = 0 \Leftrightarrow \nabla_{\frac{\partial}{\partial x^i}}(\varphi\frac{\partial}{\partial x^j}) = \varphi\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\\
&\Leftrightarrow \nabla_{\frac{\partial}{\partial x^i}}(\varphi^h_j\frac{\partial}{\partial x^h}) = \varphi(\Gamma^h_{ij}\frac{\partial}{\partial x^h}) \Leftrightarrow\\
&\Leftrightarrow \frac{\partial\varphi^k_j}{\partial x^i}\frac{\partial}{\partial x^k} + \varphi^h_j\Gamma^k_{ih}\frac{\partial}{\partial x^k} = \Gamma^h_{ij}\varphi^k_h\frac{\partial}{\partial x^k}\\
&\Leftrightarrow \frac{\partial\varphi^k_j}{\partial x^i} + \varphi^h_j\Gamma^k_{ih} = \Gamma^h_{ij}\varphi^k_h, i, j = \overline{1,n},
\end{aligned} \tag{3.4}$$

where $(\Gamma^k_{ij})_{i,j,k=\overline{1,n}}$ denote the Christoffel coefficients of ∇.

(iii) The existence of a φ-connection is studied in mathematical literature in several contexts. For instance, the Levi-Civita connection of the metric g on a Kähler manifold (M, g, J) (resp. para-Kähler manifold (M, g, P)) is a J-connection (resp. a P-connection).

Different from the existence problem, we state now the following:

General problem On a manifold (M, φ) endowed with a (1,1)-tensor field, how many φ-connections exist?

In the present paper we solve the following:

Main problem If (M, φ) is a manifold endowed with an integrable (1,1)-tensor field, how many φ-connections exist?

4. F-connections

In this section, (M, φ) denotes an n-dimensional manifold endowed with an integrable (1,1)-tensor field. From Definition 3.1, there exists a matrix $F \in \mathcal{M}_n(\mathbb{R})$ and an atlas on M with local coordinates $(x^1, \ldots, x^n)$ such that:

$$F = \left[\varphi^h_j\right]_{j,h=\overline{1,n}} \tag{4.1}$$

where φ^h_j, $j, h = \overline{1,n}$ are given by (3.1). The relation (4.1) shows that the coefficients φ^h_j, $j, h = \overline{1,n}$, are constant.

Under the above notations, it follows from (3.4) that a linear connection ∇ is a φ-connection if and only if its Christoffel coefficients $(\Gamma^h_{ij})_{i,j,h=\overline{1,n}}$ satisfy:

$$\varphi^h_j\Gamma^k_{ih} = \Gamma^h_{ij}\varphi^k_h, i, j, k = \overline{1,n}, \tag{4.2}$$

in the above local coordinates.

For any fixed $i \in \{1, \ldots, n\}$, we denote by G_i the matrix $(\Gamma^h_{ij})_{j,h=\overline{1,n}}$.

Since $F \in \mathcal{M}_n(\mathbb{R})$, let $C(F)$ be its centralizer. Hence, (4.2) becomes:

$$FG_i = G_iF, i = \overline{1,n}, \tag{4.3}$$

or equivalently, $G_i \in C(F)$, $i = \overline{1,n}$.

Let $q(F) = dimC(F)$, as computed in section 1.

We conclude with the following solution to the main problem:

Theorem 4.1. *Let (M, φ) be an n-dimensional manifold endowed with an integrable (1,1)-tensor field, whose associated matrix is F. Then all φ-connections*

(i) with torsion in dimension n depend locally on $nq(F)$ arbitrary functions of n variables;

(ii) without torsion in dimension $n > q(F)$ depend locally on at most $nq(F)$ arbitrary functions of n variables;

(iii) without torsion in dimension $n \leq q(F)$ depend locally on at most $n(q(F) + n)/2$ arbitrary functions of n variables.

Proof. Let $(\Gamma^h_{ij})_{i,j,h=\overline{1,n}}$ be the Christoffel symbols of a φ-connection. Any $(\Gamma^h_{ij})_{i,j,h=\overline{1,n}}$ can be seen as a cubic matrix or else, as n ordinary matrices $G_i = (\Gamma^h_{ij})_{j,h=\overline{1,n}}$, indexed by $i = \overline{1,n}$. We saw above that G_i run in a $q(F)$-dimensional space for any $i = \overline{1,n}$. Hence (i) is shown. For torsion-free connections, one has the symmetry of Γ^h_{ij} with respect to i and j. Then (ii) is proved. When $q(F) \geq n$, we are looking for the maximum dimension of symmetric φ-connections. For this purpose, to the n^2 entries from the diagonal $i = j \in \{1, \dots, n\}$ of the matrix $(\Gamma^h_{ii})_{i,h=\overline{1,n}}$, we add the $(q(F) - n)n/2$ entries outside this diagonal (where we divided by 2, based on the symmetry of Γ^h_{ij} with respect to i and j). Hence we obtain $n^2 + (q(F) - n)n/2 = n(q(F) + n)/2$, which shows (iii) and complete the proof. □

In the following example we show that the maximum is reached.

Example 4.2. *If $n = 2$ and φ is the identity, than $q(I) = n^2 = 4 > n$. One can see that all φ-connections without torsion depend locally on $(1 + 2)2 = 6$ arbitrary functions, while $n(q(I) + n)/2 = 6$. For the φ-connections with torsion, we obtain $n^3 = nq(I) = 8$.*

This example can be generalized for any dimension. In the next example the maximum is not reached.

Example 4.3. *In dimension $n = 2$, any almost complex structure is integrable and its canonical form is: $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A straightforward computation yields:*

$$C(J) = \{A \in \mathcal{M}_2(\mathbb{R}); A = A(a, b) \in \mathbb{R}, a, b \in \mathbb{R}\},$$

where $A(a, b)$ denotes $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Hence $q(J) = 2$, which is exactly the 2-dimensional Case II from Example 2.1. One has $G_i = A(a_i, b_i), i = 1, 2$, which yields that:

(i) any general complex linear connection depends on $2 \cdot 2 = 4$ coefficients (denoted above by $a_i, b_i, i = 1, 2$);

(ii) any torsion-free complex linear connection depends on 2 coefficients (denoted above by a_1, b_1, since from the symmetry condition with respect to i and j, we obtain $a_2 = -b_1$ and $b_2 = a_1$).

Remark 4.4. *The above study on J and its centralizer $C(J)$ is similar to the discussion for almost tangent structures from [2].*

This research of the present paper will be continued in a forthcoming work, where we will count all linear connections with respect to which a given semi-Riemannian metric is parallel. The role played here by the (1,1)-tensor field will be replaced there by a semi-Riemannian metric.

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