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Oscillatory and Asymptotic Behavior of Solutions for Second-Order Mixed Nonlinear Integro-Dynamic Equations with Maxima on Time Scales

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Abstract. This paper is concerned with the oscillatory and asymptotic behavior for solutions of the following second-order mixed nonlinear integro-dynamic equations with maxima on time scales

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + \int_0^t a(t,s)f(s,x(s))\Delta s + \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s) = 0,$$

where

$$z(t) = x(t) + p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t)), t \in [0, +\infty)_{\mathbb{T}}.$$

The oscillatory behavior of this equation hasn't been discussed before, also our results improve and extend some results established by Grace et al. [2] and [8].

1. Introduction.

In recent years, there have been many activities concerning the oscillation and nonoscillation of dynamic equations on time scales, since Hilger introduced the theory of time scales to unify continuous and discrete calculus. We refer the reader to the books [6, 7], also the papers [2-5, 8] and the references cited therein.

The qualitative theory of differential equations with "maxima" received very little attention, respect, for instance, the problems connected to minimizers of variational functionals (see e.g.[11]), even though such equations often arise in the problem of automatic regulation of various real systems, see for example [9, 10], also the research on oscillation theory for integro-dynamic equations is limited due to lack of available techniques.

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In this paper we deal with the oscillatory and asymptotic behavior of solutions for the second-order mixed nonlinear integro-dynamic equations with maxima of the form:

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+\int_{0}^{t}a(t,s)f(s,x(s))\Delta s+\sum_{i=1}^{n}q_i(t)\max_{s\in[\tau_i(t),\xi_i(t)]}x^{\alpha}(s)=0, \tag{1.1}$$

where

$$z(t)=x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)). \tag{1.2}$$

We take $\mathbb{T}\subseteq\mathbb{R}$ to be an arbitrary time scale with $0\in\mathbb{T}$ and $sup\mathbb{T}=+\infty$.
Subject to the following hypotheses:
(H_1) $\mathbb{T}$ is an unbounded above time scale. We define the time scale interval $[t_0,+\infty)_{\mathbb{T}}$ by $[t_0,+\infty)_{\mathbb{T}}=[t_0,+\infty)\bigcap\mathbb{T}$.
(H_2) $\eta_1,\eta_1,\tau_i,\xi_i:\mathbb{T}\to\mathbb{T}$ are rd-continuous functions such that $\eta_1(t)\le t\le\eta_2(t)$, $\tau_i(t)\le t\le\xi_i(t)$,$i=1,2,...,n$ and $lim_{t\to+\infty}\eta_1(t)=+\infty=lim_{t\to+\infty}\tau_i(t)$.
(H_3) p_1,p_2,q_i and r are non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$ such that $r(t)>0$, $i=1,2,...,n$ considering, when either

$$lim_{t\to+\infty}L(t,t_0):=lim_{t\to+\infty}\int_{t_0}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}=+\infty, \tag{1.3}$$

or

$$lim_{t\to+\infty}L(t,t_0)<+\infty. \tag{1.4}$$

(H_4) $a(t,s):\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a rd-continuous function such that

$$a(t,s)>0,\ a^{\Delta_t}(t,s)<0 \text{ and } \sup_{t\ge t_0}\int_{0}^{t_0}a(t,s)\Delta s:=k_1<+\infty.$$

(H_5) $f\in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ such that, $f^{\Delta_t}(t,x(t))>0$ and $x(t)f(t,x(t))\ge m(t)|x(t)|^{\beta+1}>0$, $x\ne 0$ for non trivial solutions x, where $m(t):\mathbb{T}\to(0,+\infty)$ is a positive rd-continuous function and β is a quotient of odd positive integers.
(H_6) α and γ are quotients of odd positive integers.
Throughout this paper, we assume that:

$$g_{-}(t)=\frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}}\int_{t_4}^{t}a(t,s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s-c_3^{\alpha}\sum_{i=1}^{n}q_i(t)\max_{s\in[\tau_i(t),\xi_i(t)]}Q^{\alpha}(s), \tag{1.5}$$

$$g_{+}(t)=\frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}}\int_{t_4}^{t}a(t,s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s-c_3^{\alpha}\sum_{i=1}^{n}q_i(t)\min_{s\in[\tau_i(t),\xi_i(t)]}Q^{\alpha}(s), \tag{1.6}$$

$$h_{-}(t)=\frac{c_4}{r(t)}-\frac{c_3^{\alpha}}{r(t)}\int_{t_4}^{t}\sum_{i=1}^{n}q_i(s)\max_{s\in[\tau_i(t),\xi_i(t)]}Q^{\alpha}(s)\Delta s,$$

$$h_+(t) = \frac{c_4}{r(t)} - \frac{c_3^\alpha}{r(t)} \int_{t_4}^{t} \sum_{i=1}^{n} q_i(s) \min_{s\in[\tau_i(t),\xi_i(t)]} Q^\alpha(s)\Delta s,$$

$$h_*(t) = \max\{0, h_+(t), h_-(t)\}, g^*(t) = \max\{g_-(t), g_+(t), 0\},$$

where

$$Q(t) := 1 - p_1(t) - p_2(t)\frac{R(\eta_2(t))}{R(t)} > 0, \tag{1.7}$$

and $R(t)$ is a positive rd- continuous function.

By a solution of (1.1), we mean a nontrivial real valued Δ- differentiable function $x(t)$ satisfying (1.1) for $t \in \mathbb{T}$.

Definition 1.1. *A solution $x(t)$ of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. i.e.*
for every $t_0 > 0$, we have

$$\inf_{t\geq t_0} x(t) < 0 < \sup_{t\geq t_0} x(t),$$

otherwise, it is called nonoscillatory.

Eq. (1.1) is said to be oscillatory if all of its solutions are oscillatory. We concentrate our study on those solutions of E.q. (1.1) which are not identically vanishing eventually.
In what follows, we provide some previous studies which are special cases of our equation. In 2013 S. R. Grace et al.[8] studied the asymptotic behavior of non-oscillatory solutions of the following second order integro-dynamic equation

$$(r(t)x^\Delta(t))^\Delta + \int_0^t a(t,s)f(s,x(s))\Delta s = 0, \tag{1.8}$$

then, In 2014 S. R. Grace et al.[2] studied the oscillatory and asymptotic behavior of the following second order integro-dynamic equation

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + \int_0^t a(t,s)f(s,x(s))\Delta s = 0. \tag{1.9}$$

Noting that Eqs.(1.8) and (1.9) are special cases of our Eq. (1.1) when taking $q_i(t) = 0 = p_1(t) = p_2(t)$, and so the results of [2] and [8] can't be applied to Eq. (1.1). Also to the best of our knowledge, there are no papers in the literature dealing with neutral integro dynamic equations with "maxima"on time scales. To fill this gap, we initiate in this paper the study of neutral integro dynamic equations with "maxima"on time scales. New results are established and an example is presented.

2. Basic Lemmas.

In this section, we give some lemmas that play an important role in the proofs of our results.

Lemma 2.1. *[1] If X and Y are nonnegative real numbers, then*

$$X^{\lambda} + (\lambda - 1)Y^{\lambda} - \lambda XY^{\lambda-1} \geq 0 \, for \, \lambda > 1,$$

and

$$X^{\lambda} - (1 - \lambda)Y^{\lambda} - \lambda XY^{\lambda-1} \leq 0 \, for \, \lambda < 1,$$

with equality holding if and only if $X = Y$ or $\lambda = 1$.

Lemma 2.2. *If $f(s)$ and $a(u,s)$ are rd-continuous functions, then*

$$\int_{t_0}^{t}\int_{t_0}^{u} a(u,s)f(s)\Delta s\Delta u = \int_{t_0}^{t}(ta(t,s) - \sigma(s)a(\sigma(s),s))f(s)\Delta s - \int_{t_0}^{t}\sigma(u)\int_{t_0}^{u} a^{\Delta_u}(u,s)f(s)\Delta s\Delta u. \quad (2.1)$$

Proof.
Let $F(u) := \int_{t_0}^{u} a(u,s)f(s)\Delta s$, and $g(u) := u$, then Theorem 5.37 in [7], leads to

$$F^{\Delta}(u) = a(\sigma(u),u)f(u) + \int_{t_0}^{u} a^{\Delta_u}(u,s)f(s)\Delta s.$$

Now by using, $Fg^{\Delta} = [Fg]^{\Delta} - F^{\Delta}g^{\sigma}$, then (2.1) holds.

Lemma 2.3. *[6] (Gronwall's Inequality) Let $p \in \mathfrak{R}^+$. Also, assume that y and $f \in C_{rd}$. If*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t) for \, all \, t \in \mathbb{T},$$

then

$$y(t) \leq y(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau for \, all \, t, t_0 \in \mathbb{T}.$$

3. Main Results.

Theorem 3.1. *Let conditions (1.3) and $H_1 - H_6$ hold with $\beta > 1, \gamma \geq 1$. Also, suppose that there exist positive rd-continuous functions $N(t)$ and $R(t)$ such that for all t_4 sufficiently large such that $t_4 \geq t_3 > t_0$, we have*

$$\frac{R(t)}{r^{\frac{1}{\gamma}}(t)\int_{t_3}^{t}\frac{1}{r^{\frac{1}{\gamma}}(s)}\Delta s} - R^{\Delta}(t) \leq 0, \quad (3.1)$$

$$\limsup_{t\to+\infty}\int_{t_4}^{t}(\frac{1}{r(u)}\int_{t_4}^{u} g_+(s)\Delta s)^{\frac{1}{\gamma}}\Delta u] < +\infty, \quad (3.2)$$

then every nonoscillatory solution $x(t)$ of Eq. (1.1) satisfies

$$\mid x(t) \mid= O[A_1 e_{p(t)}(t,t_4) + \int_{t_4}^{t} e_{p(t)}(t,\sigma(v))f(v)\Delta v],$$

where

$$p(t) = \frac{1}{\gamma r(t)}\int_{t_4}^{t} a(\sigma(s),s)\sigma(s)N(s)\Delta s,$$

and

$$f(t) = \frac{c_4}{r(t)^{\frac{1}{\gamma}}} + (1 - \frac{1}{\gamma}).$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$ and there exists $t_2 \geq t_1 + \max\{\eta_1, \tau_i, i = 1, 2, ..., n\}$, such that $x(\eta_1(t)) > 0$ and $x(\tau_i(t)) > 0$ for all $t \geq t_2$. Now from (1.1), we have

$$\begin{aligned}(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} &= -\int_0^t a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s),\\ &= -\int_0^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s), \qquad (3.3)\end{aligned}$$

choosing $t_3 > t_2$ sufficiently large, then from H_5, we can find $k \geq 0$ such that

$$\int_0^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t_3} a(t,s)f(s,x(s))\Delta s := k,$$

so (3.3) can be written as

$$\begin{aligned}(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} &= -k - \int_{t_3}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s),\\ &< -\int_{t_3}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s) < 0. \qquad (3.4)\end{aligned}$$

Then, $r(t)(z^{\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_3, +\infty)_{\mathbb{T}}$. Now we claim that $r(t)(z^{\Delta}(t))^{\gamma} > 0$ on $[t_3, +\infty)_{\mathbb{T}}$. Therefore assume that this is not true. Then there is $t_3^* \in [t_3, +\infty)_{\mathbb{T}}$, such that $G_1 := r(t_3^*)(z^{\Delta}(t_3^*))^{\gamma} < 0$, by using the fact that $r(t)(z^{\Delta}(t))^{\gamma}$ is decreasing, we have

$$z^{\Delta}(t) \leq \frac{G_1^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)},$$

integrating from t_3^* to t and using condition (1.3), we get

$$z(t) \leq z(t_3^*) + G_1^{\frac{1}{\gamma}} \int_{t_3^*}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} \to -\infty \text{ as } t \to \infty.$$

Hence, $z(t)$ is eventually negative. This is a contradiction. Then,

$$z(t) > 0,\ z^{\Delta}(t) > 0 \text{ and } (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} < 0 \ \text{ for } t \in [t_3, +\infty)_{\mathbb{T}}. \qquad (3.5)$$

Using the fact that $z(t)$ is increasing, then

$$z(t) > z(t_3) := c_3. \qquad (3.6)$$

Now integrating $z^{\Delta}(t)$ from t_3 to t and using (3.5), we obtain

$$\begin{aligned}z(t) &= z(t_3) + \int_{t_3}^{t} \frac{[r(s)(z^{\Delta}(s))^{\gamma}]^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)}\Delta s\\ &\geq r^{\frac{1}{\gamma}}(t)z^{\Delta}(t)L(t,t_3),\end{aligned}$$

where $L(t,t_3) := \int_{t_3}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}$, hence by condition (3.1), we have

$$\begin{aligned}\Big(\frac{z(t)}{R(t)}\Big)^{\Delta} &= \frac{z^{\Delta}(t)R(t) - z(t)R^{\Delta}(t)}{R(t)R^{\sigma}(t)}\\ &\leq \frac{z(t)}{R(t)R^{\sigma}(t)}\Big[\frac{R(t)}{r^{\frac{1}{\gamma}}(t)L(t,t_3)} - R^{\Delta}(t)\Big] \leq 0,\end{aligned} \tag{3.7}$$

then z/R is a non-increasing function. From the definition of $z(t)$ (see(1.2)), (3.5), (3.6) and (3.7), we see that

$$\begin{aligned}x(t) &= z(t) - p_1(t)x(\eta_1(t)) - p_2(t)x(\eta_2(t))\\ &\geq z(t) - p_1(t)z(\eta_1(t)) - p_2(t)z(\eta_2(t))\\ &= (1 - p_1(t)\frac{z(\eta_1(t))}{z(t)} - p_2(t)\frac{z(\eta_2(t))}{z(t)})z(t)\\ &\geq (1 - p_1(t) - p_2(t)\frac{R(\eta_2(t))}{R(t)})z(t) = Q(t)z(t) \geq c_3 Q(t) \text{ for all } t \geq t_3,\end{aligned}$$

where $Q(t) := (1 - p_1(t) - p_2(t)\frac{R(\eta_2(t))}{R(t)})$. Then

$$\max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s) \geq \max_{s\in[\tau_i(t),\xi_i(t)]} c_3^{\alpha} Q^{\alpha}(s) = c_3^{\alpha} \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s). \tag{3.8}$$

Choosing t_4 sufficiently large such that $t_4 \geq t_3$ and using H_5 in (3.4), we can write that

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq -\int_{t_4}^{t} a(t,s)m(s)x^{\beta}(s)\Delta s - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s), \tag{3.9}$$

substituting from (3.8) in the previous inequality, we get

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq -\int_{t_4}^{t} a(t,s)m(s)x^{\beta}(s)\Delta s - c_3^{\alpha}\sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s). \tag{3.10}$$

Letting $N(t)$ be a positive rd-continuous function, hence (3.10) can be written as

$$\begin{aligned}(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq \int_{t_4}^{t} a(t,s)[N(s)x(s) - m(s)x^{\beta}(s)]\Delta s - \int_{t_4}^{t} a(t,s)N(s)x(s)\Delta s\\ -c_3^{\alpha}\sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s).\end{aligned} \tag{3.11}$$

Applying Lemma 2.1, with $\lambda = \beta, X = m^{\frac{1}{\beta}}(s)x(s)$ and $Y = [\frac{N(s)}{\beta m^{\frac{1}{\beta}}(s)}]^{\frac{1}{\beta-1}}$, we have

$$N(s)x(s) - m(s)x^{\beta}(s) \leq \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}} N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s). \tag{3.12}$$

Substituting from (3.12) into (3.11), gives

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq g_{-}(t) - \int_{t_4}^{t} a(t,s)N(s)x(s)\Delta s, \tag{3.13}$$

where

$$g_{-}(t)=\frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}}\int_{t_4}^{t}a(t,s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s-c_3^{\alpha}\sum_{i=1}^{n}q_i(t)\max_{s\in[\tau_i(t),\xi_i(t)]}Q^{\alpha}(s).$$

Integrating the previous inequality from t_4 to t, leads to

$$(z^{\Delta}(t))^{\gamma}\leq\frac{r(t_4)(z^{\Delta}(t_4))^{\gamma}}{r(t)}-\frac{1}{r(t)}\int_{t_4}^{t}\int_{t_4}^{u}a(u,s)N(s)x(s)\Delta s\Delta u+\frac{1}{r(t)}\int_{t_4}^{t}g_{-}(s)\Delta s, \tag{3.14}$$

by using Lemma 2.2, H_4, and taking $g^{*}(t)=\max\{g_{-}(t),g_{+}(t),0\}$, we have

$$z^{\Delta}(t)\leq\Big[\frac{c_4}{r(t)}+\frac{1}{r(t)}\int_{t_4}^{t}g^{*}(s)\Delta s+\frac{1}{r(t)}\int_{t_4}^{t}\sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s\Big]^{\frac{1}{\gamma}}, \tag{3.15}$$

where $c_4=r(t_4)[z^{\Delta}(t_4)]^{\gamma}$. By employing $(a+b)^{\lambda}\leq a^{\lambda}+b^{\lambda}$ for $a\geq 0$, $b\geq 0$ and $\lambda\leq 1$, thus (3.15) becomes

$$z^{\Delta}(t)\leq(\frac{c_4}{r(t)})^{\frac{1}{\gamma}}+(\frac{1}{r(t)}\int_{t_4}^{t}g^{*}(s)\Delta s)^{\frac{1}{\gamma}}+(\frac{1}{r(t)}\int_{t_4}^{t}\sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s)^{\frac{1}{\gamma}}. \tag{3.16}$$

Integrating the above inequality from t_4 to t and taking A_1 as upper bound for

$$z(t_4)+\int_{t_4}^{t}(\frac{1}{r(u)}\int_{t_4}^{u}g^{*}(s)\Delta s)^{\frac{1}{\gamma}}\Delta u,$$

we have

$$x(t)\leq z(t)\leq A_1+\int_{t_4}^{t}(\frac{c_4}{r(s)})^{\frac{1}{\gamma}}\Delta s+\int_{t_4}^{t}(\frac{1}{r(u)}\int_{t_4}^{u}\sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s)^{\frac{1}{\gamma}}\Delta u.$$

Again using Lemma 2.1, with $X=\frac{1}{r(u)}\int_{t_4}^{u}\sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s$, $\lambda=\frac{1}{\gamma}$, and $Y=1$, then the previous inequality can be written as

$$\begin{aligned}x(t)&\leq A_1+\int_{t_4}^{t}(\frac{c_4}{r(s)})^{\frac{1}{\gamma}}\Delta s+(1-\frac{1}{\gamma})\int_{t_4}^{t}\Delta u+\int_{t_4}^{t}\frac{1}{\gamma r(u)}\int_{t_4}^{u}\sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s\Delta u,\\&\leq A_1+\int_{t_4}^{t}(\frac{c_4}{r(s)})^{\frac{1}{\gamma}}\Delta s+(1-\frac{1}{\gamma})t+\int_{t_4}^{t}\frac{1}{\gamma r(u)}\int_{t_4}^{u}\sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s\Delta u.\end{aligned}$$

Let $u(t)$ equals the right hand side of the previous inequality, then we have

$$u^{\Delta}(t)=(\frac{c_4}{r(t)})^{\frac{1}{\gamma}}+(1-\frac{1}{\gamma})+\frac{1}{\gamma r(t)}\int_{t_4}^{t}\sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s,u(t_4)=A_1,$$

hence $u(t)$ is increasing and since $x(t) \leq u(t)$, then we have

$$u^{\Delta}(t) \leq f(t) + p(t)u(t),$$

where $f(t) := (\frac{c_4}{r(t)})^{\frac{1}{\gamma}} + (1 - \frac{1}{\gamma})$ and $p(t) = \frac{1}{\gamma r(t)} \int_{t_4}^{t} \sigma(s)a(\sigma(s), s)N(s)\Delta s$. Using Lemma 2.3, leads to

$$x(t) \leq u(t) \leq A_1 e_{p(t)}(t, t_4) + \int_{t_4}^{t} e_{p(t)}(t, \sigma(v))f(v)\Delta v,$$

then, $x(t) = O[A_1 e_{p(t)}(t, t_4) + \int_{t_4}^{t} e_{p(t)}(t, \sigma(v))f(v)\Delta v]$.

If $x(t)$ is an eventually negative solution of Eq. (1.1), then we can see that the transformation $y = -x, y > 0$ transforms Eq. (1.1) into

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} - \int_{0}^{t} a(t, s)f(s, -y(s))\Delta s + \sum_{i=1}^{n} q_i(t) \min_{s \in [\tau_i(t), \xi_i(t)]} y^{\alpha}(s) = 0,$$

where

$$v(t) = y(t) + p_1(t)y(\eta_1(t)) + p_2(t)y(\eta_2(t)).$$

Thus,

$$\begin{aligned}(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} &= \int_{0}^{t} a(t, s)f(s, -y(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s \in [\tau_i(t), \xi_i(t)]} y^{\alpha}(s), \\ &= \int_{0}^{t_2} a(t, s)f(s, x(s))\Delta s + \int_{t_2}^{t} a(t, s)f(s, x(s))\Delta s + \sum_{i=1}^{n} q_i(t) \min_{s \in [\tau_i(t), \xi_i(t)]} y^{\alpha}(s), \end{aligned} \tag{3.17}$$

choosing $t_4 > t_2$ sufficiently large, then from H_5, we can find $k_1 \leq 0$ such that

$$\int_{0}^{t_2} a(t, s)f(s, x(s))\Delta s + \int_{t_2}^{t_4} a(t, s)f(s, x(s))\Delta s := k_1,$$

so (3.17) can be written as

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} \leq -\int_{t_4}^{t} a(t, s)m(s)y^{\beta}(s)\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s \in [\tau_i(t), \xi_i(t)]} y^{\alpha}(s).$$

It follows in a similar manner that $-x(t) = O[A_1 e_{p(t)}(t, t_4) + \int_{t_4}^{t} e_{p(t)}(t, \sigma(v))f(v)\Delta v]$. This completes the proof.

Corollary 3.1. *Let all assumptions of Theorem 3.1 hold and*

$$\limsup_{t \to +\infty} \frac{1}{e_{p(t)}(t, t_4)} \int_{t_4}^{t} e_{p(t)}(t, \sigma(v))[(\frac{c_4}{r(v)})^{\frac{1}{\gamma}} + (1 - \frac{1}{\gamma})]\Delta v < +\infty, \tag{3.18}$$

then every non-oscillatory solution, satisfies

$$|x(t)| = O(e_{p(t)}(t, t_4)).$$

Theorem 3.2. *Let conditions (1.4), (3.18) hold and* $0 \le p_1(t) + p_2(t) \le p_* < 1$. *Also, let all assumptions of Theorem 3.1 hold except condition (1.3). If for sufficiently large* t_4*, we have*

$$\limsup_{t\to+\infty} \frac{1}{e_{p(t)}(t,t_4)} \int_{t_4}^{t} \Big(\frac{1}{r(v)} \int_{t_4}^{v} a(\sigma(s),s)\sigma(s)m(s)A^{\beta}(s)\Delta s\Big)^{\frac{1}{\gamma}} \Delta v < +\infty, \tag{3.19}$$

then every non-oscillatory solution, satisfies

$$\limsup_{t\to+\infty} \frac{|x(t)|}{e_{p(t)}(t,t_4)} < +\infty \text{ or } \lim_{t\to+\infty} x(t) = 0.$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then proceeding similar to the proof of Theorem 3.1, we have $r(t)(z^{\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_3,+\infty)_{\mathbb{T}}$, and there exists $t_5 \in [t_3,+\infty)_{\mathbb{T}}$ such that:

$$z^{\Delta}(t) > 0 \text{ or } z^{\Delta}(t) < 0 \text{ for } t \in [t_5,+\infty)_{\mathbb{T}}.$$

We consider each of the following two cases separately.
Case 1. The proof is similar to that of Theorem 3.1, so it is omitted.
Case 2.

$$z(t) > 0,\ z^{\Delta}(t) < 0 \text{ and } (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} < 0 \text{ for } t \in [t_5,+\infty)_{\mathbb{T}}. \tag{3.20}$$

In this case, we have

$$\lim_{t\to+\infty} z(t) = l,\ l \ge 0.$$

Case 2.I
If $l = 0$, and since we have $0 < x(t) \le z(t)$, then $\lim_{t\to+\infty} x(t) = 0$.
Case 2.II
If $l > 0$, then for any $\epsilon > 0$, we have $l < z(t) < l + \epsilon$, eventually.
Take $0 < \epsilon < l(1 - p_*)/p_*$. Then

$$\begin{aligned} x(t) &= z(t) - p_1(t)x(\eta_1(t)) - p_2(t)x(\eta_2(t)), \\ &\ge l - p_1(t)(l+\epsilon) - p_2(t)(l+\epsilon), \\ &\ge l - p_*(l+\epsilon), \end{aligned}$$

where $p_1 + p_2 \le p_* < 1$. Hence

$$x(t) \ge m_1(l+\epsilon) > m_1 z(t), \tag{3.21}$$

where

$$m_1 := \frac{l}{l+\epsilon} - p_* = \frac{l(1-p_*) - \epsilon p_*}{l+\epsilon} > 0.$$

Since $r(t)(z^{\Delta}(t))^{\gamma}$ is strictly decreasing, then

$$z^{\Delta}(s) \le \Big(\frac{r(t)(z^{\Delta}(t))^{\gamma}}{r(s)}\Big)^{\frac{1}{\gamma}}, s \in [t,+\infty)_{\mathbb{T}}.$$

Integrating for s from t to ζ and letting $\zeta \to +\infty$, we have

$$z(t) \ge -r^{\frac{1}{\gamma}}(t)z^{\Delta}(t) \int_{t}^{+\infty} \frac{\Delta u}{r^{\frac{1}{\gamma}}(u)} \ge -r^{\frac{1}{\gamma}}(t_5)z^{\Delta}(t_5) \int_{t}^{+\infty} \frac{\Delta u}{r^{\frac{1}{\gamma}}(u)} \text{ for all } t \ge t_5,$$

hence,

$$z(t) \geq c_5 A(t), \tag{3.22}$$

where, $c_5 = -r^{\frac{1}{\gamma}}(t_5)z^{\Delta}(t_5) > 0$ and $A(t) = \int_t^{+\infty} \frac{\Delta u}{r^{\frac{1}{\gamma}}(u)}$. Combining (3.21) and (3.22), we get

$$x(t) \geq c_6 A(t), \tag{3.23}$$

where $c_6 = m_1 c_5$. Substituting from (3.23) into (3.9), we obtain

$$\begin{aligned}
(r(t)(-z^{\Delta}(t))^{\gamma})^{\Delta} &\geq \int_{t_5}^{t} a(t,s)m(s)x^{\beta}(s)\Delta s + \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s),\\
&\geq c_6^{\beta} \int_{t_5}^{t} a(t,s)m(s)A^{\beta}(s)\Delta s + c_6^{\alpha} \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} A^{\alpha}(s),\\
&\geq B \int_{t_5}^{t} a(t,s)m(s)A^{\beta}(s)\Delta s + B \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} A^{\alpha}(s),
\end{aligned}$$

where $B = \min\{c_6^{\alpha}, c_6^{\beta}\}$, now integrate the above inequality from t_5 to t, we obtain

$$r(t)(-z^{\Delta}(t))^{\gamma} \geq B \int_{t_5}^{t} \int_{t_5}^{u} a(u,s)m(s)A^{\beta}(s)\Delta s\Delta u + B \int_{t_5}^{t} \sum_{i=1}^{n} q_i(s) \max_{s\in[\tau_i(t),\xi_i(t)]} A^{\alpha}(u)\Delta s, \tag{3.24}$$

by using lemma 2.2 and H_4, we obtain

$$\begin{aligned}
&\int_{t_5}^{t} \int_{t_5}^{u} a(u,s)m(s)A^{\beta}(s)\Delta s\Delta u = \int_{t_5}^{t} (ta(t,s) - \sigma(s)a(\sigma(s),s))m(s)A^{\beta}(s)\Delta s\\
&- \int_{t_5}^{t} \sigma(u) \int_{t_5}^{u} a^{\Delta_u}(u,s)m(s)A^{\beta}(s)\Delta s\Delta u \geq - \int_{t_5}^{t} \sigma(s)a(\sigma(s),s))m(s)A^{\beta}(s)\Delta s,
\end{aligned}$$

thus (3.24), can be written in the form

$$z^{\Delta}(t) \leq \Big[\frac{B}{r(t)} \int_{t_5}^{t} \Big[\sigma(s)a(\sigma(s),s)m(s)A^{\beta}(s)\Big]\Delta s.$$

Integrating the latter inequality from t_5 to t and using $x(t) < z(t)$, we obtain

$$x(t) \leq z(t_5) + \int_{t_5}^{t} \Big[\frac{B}{r(v)} \int_{t_5}^{v} \Big[\sigma(s)a(\sigma(s),s)m(s)A^{\beta}(s)\Big]\Delta s\Big]^{\frac{1}{\gamma}}\Delta v,$$

using condition (3.19), we see that

$$\limsup_{t\to+\infty} \frac{x(t)}{e_{p(t)}(t,t_5)} < +\infty.$$

If $x(t)$ is an eventually negative solution of Eq. (1.1), then we can see that the transformation $y = -x, y > 0$ transforms Eq. (1.1) into

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} - \int_0^t a(t,s)f(s,-y(s))\Delta s + \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) = 0,$$

where

$$v(t) = y(t) + p_1(t)y(\eta_1(t)) + p_2(t)y(\eta_2(t)).$$

Thus,

$$\begin{aligned}(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} &= \int_0^t a(t,s)f(s,-y(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s),\\ &= \int_0^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t} a(t,s)f(s,x(s))\Delta s + \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s), \end{aligned} \tag{3.25}$$

choosing $t_3 > t_2$ sufficiently large, then from H_5, we can find $k_1 \le 0$ such that

$$\int_0^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t_3} a(t,s)f(s,x(s))\Delta s := k_1,$$

so, (3.25) can be written as

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} \le -\int_{t_3}^{t} a(t,s)m(s)y^{\beta}(s)\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s).$$

It follows in a similar manner that $\limsup_{t\to+\infty} \frac{-x(t)}{e_{p(t)}(t,t_5)} < +\infty$. This completes the proof.

Corollary 3.2. *Let condition (1.4) and* $0 \le p_1(t) + p_2(t) \le p_* < 1$ *hold. Let all assumptions of Theorem 3.1 hold except condition (1.3). If for sufficiently large* t_4

$$\limsup_{t\to+\infty} \int_{t_4}^{t} \frac{1}{\gamma r(v)} \int_{t_4}^{t} a(\sigma(s),s)\sigma(s)N(s)\Delta s\Delta v < +\infty, \tag{3.26}$$

$$\limsup_{t\to+\infty} \int_{t_3}^{t} e_{p(t)}(t,\sigma(v))[(\frac{c_4}{r(v)})^{\frac{1}{\gamma}} + (1-\frac{1}{\gamma})]\Delta v < +\infty, \tag{3.27}$$

and

$$\limsup_{t\to+\infty} \int_{t_4}^{t} (\frac{1}{r(v)} \int_{t_4}^{v} a(\sigma(s),s)\sigma(s)m(s)A^{\beta}(s)\Delta s)^{\frac{1}{\gamma}} \Delta v < +\infty, \tag{3.28}$$

then every non-oscillatory solution, satisfies

$$|x(t)| = O(1) \text{ or } \lim_{t\to+\infty} x(t) = 0.$$

Theorem 3.3. *Assume that conditions (1.4), (3.1), (3.2), and (3.26) hold for $\beta > 1$ and $\gamma = 1$, also let $0 \leq p_1(t) + p_2(t) \leq p_* < 1$. And if*

$$\int_{t_4}^{+\infty} \frac{1}{r(u)}\Big[\int_{t_4}^{u} \sigma(s)a(\sigma(s),s)m(s)A^{\beta}(s) - \sum_{i=1}^{n} q_i(s) \max_{s\in[\tau_i(t),\xi_i(t)]} A^{\alpha}(u)\Delta s\Big]\Delta u < +\infty, \tag{3.29}$$

then every nonoscillatory solution $x(t)$ of Eq. (1.1) satisfies

$$\limsup_{t\to+\infty} |x(t)| < +\infty \text{ or } \lim_{t\to+\infty} x(t) = 0.$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then proceeding similar to the proof of Theorem 3.1, we have $r(t)z^{\Delta}(t)$ is strictly decreasing on $[t_3, +\infty)_{\mathbb{T}}$, and there exists $t_5 \in [t_3, +\infty)_{\mathbb{T}}$ such that:

$$z^{\Delta}(t) > 0 \text{ or } z^{\Delta}(t) < 0 \text{ for } t \in [t_5, +\infty)_{\mathbb{T}}.$$

We consider each of the following two cases separately.
Case 1.

$$z(t) > 0,\ z^{\Delta}(t) > 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5, +\infty)_{\mathbb{T}}.$$

Proceeding similar to that of Theorem 3.1 with $\gamma = 1$, then (3.16), can be written as:

$$z^{\Delta}(t) \leq \frac{c_4}{r(t)} + \frac{1}{r(t)}\int_{t_5}^{t} g_-(s)\Delta s + \frac{1}{r(t)}\int_{t_5}^{t} \sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s. \tag{3.30}$$

Integrating the above inequality from t_5 to t, we get

$$x(t) \leq z(t) \leq z(t_5) + \int_{t_5}^{t} \frac{1}{r(u)}\int_{t_5}^{u} g_-(s)\Delta s\Delta u + \int_{t_5}^{t} \frac{c_4}{r(s)}\Delta s + \int_{t_5}^{t} \frac{1}{r(u)}\int_{t_5}^{u} \sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s\Delta u. \tag{3.31}$$

By using conditions (3.2) and (1.4), we can take A_2 as an upper bound for

$$z(t_5) + \int_{t_5}^{t} \frac{1}{r(u)}\int_{t_5}^{u} g_-(s)\Delta s\Delta u + \int_{t_5}^{t} \frac{c_4}{r(s)}\Delta s,$$

hence, we obtain

$$x(t) \leq z(t) \leq A_2 + \int_{t_5}^{t} \frac{1}{r(u)}\int_{t_5}^{u} \sigma(s)a(\sigma(s),s)N(s)x(s)\Delta s\Delta u.$$

Taking limsup as $t \to +\infty$ to the above inequality and using condition (3.26), we have

$$\limsup_{t\to+\infty} x(t) < +\infty.$$

Case 2.

$$z(t) > 0,\ z^{\Delta}(t) < 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5, +\infty)_{\mathbb{T}}. \tag{3.32}$$

In this case, we have

$$\lim_{t\to+\infty} z(t) = l,\ \ l \geq 0.$$

Case 2.I

If $l = 0$, and since we have $0 < x(t) \leq z(t)$, then $\lim_{t\to+\infty} x(t) = 0$.

Case 2.II

If $l > 0$, then proceeding similar to the proof of Theorem 3.2 taking $\gamma = 1$, then (3.24) can be written as

$$r(t)(-z^{\Delta}(t)) \geq B\int_{t_5}^{t}\int_{t_5}^{u} a(u,s)m(s)A^{\beta}(s)\Delta s\Delta u + B\int_{t_5}^{t}\sum_{i=1}^{n} q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]} A^{\alpha}(u)\Delta s,$$

which implies

$$z^{\Delta}(t) \leq \frac{-B}{r(t)}\int_{t_5}^{t}\int_{t_5}^{u} a(u,s)m(s)A^{\beta}(s)\Delta s - \frac{B}{r(t)}\int_{t_5}^{t}\sum_{i=1}^{n} q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]} A^{\alpha}(u)\Delta s,$$

by using lemma 2.2 and H_4, we obtain

$$z^{\Delta}(t) \leq \frac{B}{r(t)}\int_{t_5}^{t}\Big[\sigma(s)a(\sigma(s),s)m(s)A^{\beta}(s) - \sum_{i=1}^{n} q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]} A^{\alpha}(u)\Big]\Delta s.$$

Integrating the latter inequality from t_5 to t and using $x(t) < z(t)$, we obtain

$$\begin{aligned} x(t) \leq z(t_5) + \int_{t_5}^{t}\frac{B}{r(v)}\int_{t_5}^{v}\Big[&\sigma(s)a(\sigma(s),s)m(s)A^{\beta}(s) \\ &- \sum_{i=1}^{n} q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]} A^{\alpha}(u)\Big]\Delta s\Delta v, \end{aligned} \tag{3.33}$$

hence, taking limsup as $t \to +\infty$ and using condition (3.29), leads to

$$\limsup_{t\to+\infty} x(t) < +\infty.$$

If $x(t)$ is an eventually negative solution of Eq. (1.1), then we can see that the transformation $y = -x, y > 0$ transforms Eq. (1.1) into

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} - \int_{0}^{t} a(t,s)f(s,-y(s))\Delta s + \sum_{i=1}^{n} q_i(t)\min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) = 0,$$

where

$$v(t) = y(t) + p_1(t)y(\eta_1(t)) + p_2(t)y(\eta_2(t)).$$

Thus,

$$\begin{aligned} (r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} &= \int_{0}^{t} a(t,s)f(s,-y(s))\Delta s - \sum_{i=1}^{n} q_i(t)\min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s), \\ &= \int_{0}^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t} a(t,s)f(s,x(s))\Delta s + \sum_{i=1}^{n} q_i(t)\min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s), \end{aligned} \tag{3.34}$$

choosing $t_3 > t_2$ sufficiently large, then from H_5, we can find $k_1 \leq 0$ such that

$$\int_0^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t_3} a(t,s)f(s,x(s))\Delta s := k_1,$$

so (3.34) can be written as

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} \leq -\int_{t_3}^{t} a(t,s)m(s)y^{\beta}(s)\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s).$$

It follows in a similar manner that $\limsup_{t\to+\infty} -x(t) < +\infty$. This completes the proof.

Theorem 3.4. *Let all assumptions of Theorem 3.3 hold, such that*

$$\liminf_{t\to+\infty} \int_{t_4}^{t} \frac{1}{r(u)} \int_{t_4}^{u} g_{-}(s)\Delta s\Delta u = -\infty, \tag{3.35}$$

and

$$\liminf_{t\to+\infty} \int_{t_4}^{t} \frac{1}{r(\mu)} \Big[\int_{t_4}^{\mu} \sigma(s)a(\mu,s)m(s)A^{\beta}(s) - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} A^{\alpha}(u)\Delta s \Big]\Delta\mu = -\infty. \tag{3.36}$$

Then every solution $x(t)$ of Eq. (1.1) is oscillatory or $\lim_{t\to+\infty} x(t) = 0$.

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then proceeding similar to the proof of Theorem 3.1, we have $r(t)z^{\Delta}(t)$ is strictly decreasing on $[t_3,+\infty)_{\mathbb{T}}$, and there exists $t_5 \in [t_3,+\infty)_{\mathbb{T}}$ such that:

$$z^{\Delta}(t) > 0 \ \text{ or } \ z^{\Delta}(t) < 0 \ \text{ for } \ t \in [t_5,+\infty)_{\mathbb{T}}.$$

We consider each of the following two cases separately.
Case 1.

$$z(t) > 0,\ z^{\Delta}(t) > 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5,+\infty)_{\mathbb{T}}.$$

Proceeding similar to that of Theorem 3.3, till we reach (3.31), hence

$$x(t) \leq z(t) \leq z(t_5) + \int_{t_5}^{t} \frac{1}{r(u)} \int_{t_5}^{u} g_{-}(s)\Delta s\Delta u + \int_{t_5}^{t} \frac{c_4}{r(s)}\Delta s + \int_{t_5}^{t} \frac{1}{r(u)} \int_{t_5}^{u} \sigma(s)a(u,s)N(s)x(s)\Delta s\Delta u.$$

Since all the assumptions of Theorem 3.3 hold, then we have the last two integrals of the above inequality are bounded. Finally take *liminf* as $t \to +\infty$ and using (3.35), we get a contradiction with $x(t)$ is positive.
Case 2.

$$z(t) > 0,\ z^{\Delta}(t) < 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5,+\infty)_{\mathbb{T}}.$$

In this case, we have

$$\lim_{t\to+\infty} z(t) = l,\ \ l \geq 0.$$

Case 2.I
If $l = 0$, and since we have $0 < x(t) \le z(t)$, then $\lim_{t\to+\infty} x(t) = 0$.
Case 2.II
If $l > 0$, then proceeding similar to that case in Theorem 3.3, till we reach (3.33), we get

$$x(t) \le z(t_5) + \int_{t_5}^{t} \frac{B}{r(v)} \int_{t_5}^{v} \Big[\sigma(s)a(v,s)m(s)A^{\beta}(s) - q(s) \max_{u\in[s,\tau(s)]} A^{\alpha}(u)\Big]\Delta s\Delta v,$$

then taking liminf as $t \to +\infty$ and using (3.36), we get a contradiction with $x(t)$ is positive. If $x(t)$ is an eventually negative solution of Eq. (1.1), the proof is similar. So it is omitted. This completes the proof.

Theorem 3.5. *Let condition (1.3), (3.1) and $H_1 - H_6$ hold with $\beta < 1, \gamma \ge 1$. Also, suppose that there exists a positive rd- continuous function $R(t)$ such that for sufficiently large $t_4 > t_0$, then every nonoscillatory solution $x(t)$ of Eq. (1.1) satisfies*

$$\mid x(t) \mid = O[A_3 e_{d(t)}(t,t_4) + \int_{t_4}^{t} e_{d(t)}(t,\sigma(v))E(v)\Delta v],$$

where

$$d(t) = \frac{1}{r(t)} \int_{t_4}^{t} [a(\sigma(s),s)\sigma(s)m(s)]^{\frac{1}{\beta}}\Delta s,$$

and

$$E(t) = h_*^{\frac{1}{\gamma}}(t) + (1-\frac{1}{\gamma}) + \frac{1-\beta}{\gamma}\frac{t}{r(t)}.$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Proceeding similar to the proof of Theorem 3.1, till we reach (3.10), then we have

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \le -\int_{t_3}^{t} a(t,s)m(s)x^{\beta}(s)\Delta s - c_3^{\alpha}\sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s).$$

Integrating the previous inequality from t_3 to t, leads to

$$\begin{aligned}(z^{\Delta}(t))^{\gamma} &\le \frac{r(t_3)(z^{\Delta}(t_3))^{\gamma}}{r(t)} - \frac{1}{r(t)}\int_{t_3}^{t}\int_{t_3}^{u} a(u,s)m(s)x^{\beta}(s)\Delta s\Delta u \\ &\quad - \frac{c_3^{\alpha}}{r(t)}\int_{t_3}^{t}\sum_{i=1}^{n} q_i(s) \max_{s\in[\tau_i(s),\xi_i(s)]} Q^{\alpha}(s)\Delta s, \\ &\le h_{-}(t) - \frac{1}{r(t)}\int_{t_3}^{t}\int_{t_3}^{u} a(u,s)m(s)x^{\beta}(s)\Delta s\Delta u,\end{aligned}$$

where

$$h_{-}(t) = \frac{c_4}{r(t)} - \frac{c_3^{\alpha}}{r(t)}\int_{t_3}^{t}\sum_{i=1}^{n} q_i(s) \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s)\Delta s,$$

and c_4 is as defined in Theorem 3.1. Using Lemma 2.2 and H_4, implies

$$z^{\Delta}(t) \leq \Big[h_{-}(t) + \frac{1}{r(t)}\int_{t_3}^{t} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s\Big]^{\frac{1}{\gamma}}. \tag{3.37}$$

Taking $h_*(t) = \max\{0, h_+(t), h_-(t)\}$, then

$$z^{\Delta}(t) \leq \Big[h_{*}(t) + \frac{1}{r(t)}\int_{t_3}^{t} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s\Big]^{\frac{1}{\gamma}}.$$

Applying $(a+b)^{\lambda} \leq a^{\lambda} + b^{\lambda}$ for $a \geq 0$, $b \geq 0$ and $\lambda \leq 1$, thus the previous inequality can be written as

$$z^{\Delta}(t) \leq h_{*}^{\frac{1}{\gamma}}(t) + \Big[\frac{1}{r(t)}\int_{t_3}^{t} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s\Big]^{\frac{1}{\gamma}}.$$

Integrating the above inequality from t_3 to t, leads to

$$x(t) \leq z(t) \leq z(t_3) + \int_{t_3}^{t} h_{*}^{\frac{1}{\gamma}}(s)\Delta s + \int_{t_3}^{t}\Big[\frac{1}{r(u)}\int_{t_3}^{u} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s\Big]^{\frac{1}{\gamma}}\Delta u. \tag{3.38}$$

Using Lemma 2.1, with $X = \frac{1}{r(u)}\int_{t_3}^{u} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s$, $\lambda = \frac{1}{\gamma}$, and $Y = 1$, then we have

$$\Big[\frac{1}{r(u)}\int_{t_3}^{u} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s\Big]^{\frac{1}{\gamma}} \leq (1-\frac{1}{\gamma}) + \frac{1}{\gamma r(u)}\int_{t_3}^{u} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s,$$

substituting from the previous inequality into (3.38), we obtain

$$\begin{aligned} x(t) &\leq z(t_3) + \int_{t_3}^{t} h_{*}^{\frac{1}{\gamma}}(s)\Delta s + (1-\frac{1}{\gamma})\int_{t_3}^{t}\Delta v + \int_{t_3}^{t}\frac{1}{\gamma r(u)}\int_{t_3}^{u} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s\Delta u, \\ &\leq z(t_3) + \int_{t_3}^{t} h_{*}^{\frac{1}{\gamma}}(s)\Delta s + (1-\frac{1}{\gamma})t + \int_{t_3}^{t}\frac{1}{\gamma r(u)}\int_{t_3}^{u} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s\Delta u. \end{aligned} \tag{3.39}$$

Again using Lemma 2.2, with $X = [\sigma(s)a(\sigma(s),s)m(s)]^{\frac{1}{\beta}}x(s)$, $\lambda = \beta$, and $Y = 1$, we obtain

$$\sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s) \leq (1-\beta) + \beta[\sigma(s)a(\sigma(s),s)m(s)]^{\frac{1}{\beta}}x(s),$$

substituting from the above inequality into (3.39), we get

$$
\begin{aligned}
x(t) \leq z(t) \leq z(t_3) &+ \int_{t_3}^{t} h_*^{\frac{1}{\gamma}}(s)\Delta s + (1-\frac{1}{\gamma})t + \frac{(1-\beta)}{\gamma}\int_{t_3}^{t}\frac{1}{r(u)}\int_{t_3}^{u}\Delta s\Delta u \\
&+ \int_{t_3}^{t}\frac{\beta}{\gamma r(u)}\int_{t_3}^{u}[\sigma(s)a(\sigma(s),s)m(s)]^{\frac{1}{\beta}}x(s)\Delta s\Delta u, \\
&\leq z(t_3) + \int_{t_3}^{t} h_*^{\frac{1}{\gamma}}(s)\Delta s + (1-\frac{1}{\gamma})t + \frac{(1-\beta)}{\gamma}\int_{t_3}^{t}\frac{u}{r(u)}\Delta u \\
&+ \int_{t_3}^{t}\frac{\beta}{\gamma r(u)}\int_{t_3}^{u}[\sigma(s)a(\sigma(s),s)m(s)]^{\frac{1}{\beta}}x(s)\Delta s\Delta u. \qquad (3.40)
\end{aligned}
$$

Let $u(t)$ equals the right hand side of inequality (3.40), thus

$$u^{\Delta}(t) = h_*^{\frac{1}{\gamma}}(t) + (1-\frac{1}{\gamma}) + \frac{(1-\beta)}{\gamma}\frac{t}{r(t)} + \frac{\beta}{\gamma r(t)}\int_{t_3}^{t}[\sigma(s)a(\sigma(s),s)m(s)]^{\frac{1}{\beta}}x(s)\Delta s,$$

hence $u(t)$ is increasing and since $x(t) \leq u(t)$, then we have

$$u^{\Delta}(t) < E(t) + d(t)u(t),$$

where $E(t) := h_*^{\frac{1}{\gamma}}(t) + (1-\frac{1}{\gamma}) + \frac{(1-\beta)}{\gamma}\frac{t}{r(t)}$ and $d(t) = \frac{\beta}{\gamma r(t)}\int_{t_3}^{t}[\sigma(s)a(\sigma(s),s)m(s)]^{\frac{1}{\beta}}$. Using Lemma 2.3, we get

$$x(t) \leq u(t) \leq A_3 e_{d(t)}(t,t_3) + \int_{t_3}^{t} e_{d(t)}(t,\sigma(v))E(v)\Delta v,$$

then, $x(t) = O[A_3 e_{d(t)}(t,t_3) + \int_{t_3}^{t} e_{d(t)}(t,\sigma(v))E(v)\Delta v]$.

Corollary 3.3. *Let all assumptions of Theorem 3.5 hold and*

$$\limsup_{t\to+\infty}\frac{1}{e_{d(t)}(t,t_4)}\int_{t_4}^{t} e_{d(t)}(t,\sigma(v))[h_*^{\frac{1}{\gamma}}(v) + (1-\frac{1}{\gamma}) + \frac{(1-\beta)}{\gamma}\frac{v}{r(v)}]\Delta v < +\infty, \qquad (3.41)$$

then every non-oscillatory solution, satisfies

$$|x(t)| = O(e_{d(t)}(t,t_4)).$$

Theorem 3.6. *Let conditions (1.4), (3.19) and (3.41) hold, also let all assumptions of Theorem (3.5) hold except condition (1.3), then every non-oscillatory solution, satisfies*

$$\limsup_{t\to+\infty}\frac{|x(t)|}{e_{d(t)}(t,t_4)} < +\infty \text{ or } \lim_{t\to+\infty}x(t) = 0.$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then proceeding similar to the proof of Theorem 3.1, we have $r(t)(z^{\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_3,+\infty)_{\mathbb{T}}$, and there exists $t_5 \in [t_3,+\infty)_{\mathbb{T}}$ such that:

$$z^{\Delta}(t) > 0 \;\text{ or }\; z^{\Delta}(t) < 0 \;\text{ for }\; t \in [t_5,+\infty)_{\mathbb{T}}.$$

We consider each of the following two cases separately.
Case 1. The proof is similar to that of Theorem 3.5, so it is omitted.
Case 2.

$$z(t) > 0,\; z^{\Delta}(t) < 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5,+\infty)_{\mathbb{T}}. \tag{3.42}$$

In this case, the proof is similar to that of Theorem 3.2, so it is omitted. This completes the proof.

Corollary 3.4. *Let conditions (1.4) and (3.28) hold. Let all assumptions of Theorem 3.5 hold except condition (1.3). If for all sufficiently large* t_4

$$\limsup_{t\to+\infty} \int_{t_4}^{t} \frac{\beta}{\gamma r(v)} \int_{t_4}^{v} [\sigma(s)a(\sigma(s),s)m(s)]^{\frac{1}{\beta}} \Delta v < +\infty, \tag{3.43}$$

and

$$\limsup_{t\to+\infty} \int_{t_3}^{t} e_{d(t)}(t,\sigma(v))[h_*^{\frac{1}{\gamma}}(v) + (1-\frac{1}{\gamma}) + \frac{(1-\beta)}{\gamma}\frac{v}{r(v)}]\Delta v < +\infty, \tag{3.44}$$

then every non-oscillatory solution, satisfies

$$|x(t)| = O(1) \text{ or } \lim_{t\to+\infty} x(t) = 0.$$

Theorem 3.7. *Assume that conditions (1.4), (3.1), (3.43) and (3.29) hold for* $\beta > 1$ *and* $\gamma = 1$*, also let* $0 \le p_1(t) + p_2(t) \le p_* < 1$*. And if*

$$\limsup_{t\to+\infty} \int_{t_4}^{t} h_+(s)\Delta s < +\infty, \tag{3.45}$$

then every nonoscillatory solution $x(t)$ *of Eq. (1.1) satisfies*

$$\limsup_{t\to+\infty} |x(t)| < +\infty \text{ or } \lim_{t\to+\infty} x(t) = 0.$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then proceeding similar to the proof of Theorem 3.1, we have $r(t)z^{\Delta}(t)$ is strictly decreasing on $[t_3,+\infty)_{\mathbb{T}}$, and there exists $t_5 \in [t_3,+\infty)_{\mathbb{T}}$ such that:

$$z^{\Delta}(t) > 0 \;\text{ or }\; z^{\Delta}(t) < 0 \;\text{ for }\; t \in [t_5,+\infty)_{\mathbb{T}}.$$

We consider each of the following two cases separately.
Case 1.

$$z(t) > 0,\; z^{\Delta}(t) > 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5,+\infty)_{\mathbb{T}}.$$

Proceeding similar to that of Theorem 3.5 with $\gamma = 1$, then (3.37), can be written as:

$$z^{\Delta}(t) \le h_-(t) + \frac{1}{r(t)} \int_{t_5}^{t} \sigma(s)a(\sigma(s),s)m(s)x^{\beta}(s)\Delta s. \tag{3.46}$$

Using Lemma 2.2, with $X = [\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} x(s)$, $\lambda = \beta$, and $Y = 1$, then we obtain

$$\sigma(s)a(\sigma(s), s)m(s)x^{\beta}(s) \leq (1-\beta) + \beta[\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} x(s),$$

substituting from the above inequality into (3.46), we get

$$\begin{aligned} z^{\Delta}(t) &\leq h_{-}(t) + \frac{(1-\beta)}{r(t)} \int_{t_5}^{t} \Delta s + \frac{\beta}{r(t)} \int_{t_5}^{t} [\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} x(s) \Delta s, \\ &\leq h_{-}(t) + \frac{(1-\beta)t}{r(t)} + \frac{\beta}{r(t)} \int_{t_5}^{t} [\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} x(s) \Delta s, \end{aligned} \tag{3.47}$$

integrating the above inequality from t_5 to t, we obtain

$$x(t) \leq z(t) \leq z(t_5) + \int_{t_5}^{t} h_{-}(s)\Delta s + (1-\beta) \int_{t_5}^{t} \frac{s}{r(s)} \Delta s + \int_{t_5}^{t} \frac{\beta}{r(v)} \int_{t_5}^{v} [\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} x(s) \Delta s \Delta v, \tag{3.48}$$

from (3.29) and (3.45), we can take A_4 an upper bound for

$$z(t_5) + \int_{t_5}^{t} h_{-}(s)\Delta s + (1-\beta) \int_{t_5}^{t} \frac{s}{r(s)} \Delta s,$$

thus (3.48), can be written as:

$$z(t) \leq A_4 + \int_{t_5}^{t} \frac{\beta}{r(v)} \int_{t_5}^{v} [\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} x(s) \Delta s \Delta v.$$

Let $u(t)$ equals the right hand side of the above inequality , thus

$$u^{\Delta}(t) = \frac{\beta}{r(t)} \int_{t_5}^{t} [\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} x(s) \Delta s, u(t_5) = A_4,$$

hence $u(t)$ is increasing and since $x(t) \leq u(t)$, then we have

$$u^{\Delta}(t) < \frac{\beta}{r(t)} \int_{t_5}^{t} [\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} \Delta s u(t).$$

Using Lemma 2.3, we get

$$x(t) \leq A_4 e_{d(t)}(t, t_5),$$

taking limsup as $t \to +\infty$ and using condition (3.43), we have

$$\limsup_{t \to +\infty} x(t) < +\infty.$$

Case 2.

$$z(t) > 0, \; z^{\Delta}(t) < 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5, +\infty)_{\mathbb{T}}. \tag{3.49}$$

In this case, we have

$$\lim_{t\to+\infty} z(t) = l, \;\; l \geq 0.$$

Case 2.I
If $l = 0$, and since we have $0 < x(t) \leq z(t)$, then $\lim_{t\to+\infty} x(t) = 0$.
Case 2.II
The proof is similar to that of Theorem 3.3, so it is omitted. This completes the proof.

Theorem 3.8. *Let all assumptions of Theorem 3.7 hold, such that (3.36) and*

$$\liminf_{t\to+\infty} \int_{t_4}^{t} h_-(s)\Delta s = -\infty, \tag{3.50}$$

hold, then every solution $x(t)$ of Eq. (1.1) is oscillatory or $\lim_{t\to+\infty} x(t) = 0$.

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then proceeding similar to the proof of Theorem 3.1, we have $r(t)z^{\Delta}(t)$ is strictly decreasing on $[t_3, +\infty)_{\mathbb{T}}$, and there exists $t_5 \in [t_3, +\infty)_{\mathbb{T}}$ such that:

$$z^{\Delta}(t) > 0 \;\text{ or }\; z^{\Delta}(t) < 0 \;\text{ for }\; t \in [t_5, +\infty)_{\mathbb{T}}.$$

We consider each of the following two cases separately.
Case 1.

$$z(t) > 0, \; z^{\Delta}(t) > 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5, +\infty)_{\mathbb{T}}.$$

Proceeding similar to that of Theorem 3.7, till we reach (3.48), hence

$$x(t) \leq z(t_5) + \int_{t_5}^{t} h_-(s)\Delta s + (1-\beta)\int_{t_5}^{t} \frac{s}{r(s)}\Delta s + \int_{t_5}^{t} \frac{\beta}{r(v)} \int_{t_5}^{v} [\sigma(s)a(\sigma(s), s)m(s)]^{\frac{1}{\beta}} x(s)\Delta s\Delta v,$$

since all the assumptions of Theorem 3.7 hold, then we have the last two integrals of the above inequality are bounded. Finally take lim inf as $t \to +\infty$ and using (3.50), we get a contradiction with $x(t)$ is positive.
Case 2.

$$z(t) > 0, \; z^{\Delta}(t) < 0 \text{ and } (r(t)z^{\Delta}(t))^{\Delta} < 0 \text{for } t \in [t_5, +\infty)_{\mathbb{T}}.$$

In this case, we have

$$\lim_{t\to+\infty} z(t) = l, \;\; l \geq 0.$$

Case 2.I
If $l = 0$, and since we have $0 < x(t) \leq z(t)$, then $\lim_{t\to+\infty} x(t) = 0$.
Case 2.II
If $l > 0$. Then proceeding similar to that case in Theorem 3.3, till we reach (3.33)

$$x(t) \leq z(t_3) + k\int_{t_3}^{t} \frac{s}{r(s)}\Delta s + \int_{t_3}^{t} \frac{B}{r(v)} \int_{t_3}^{v} \Big[\sigma(s)a(v,s)m(s)A^{\beta}(s) - q(s) \max_{u\in[s,\tau(s)]} A^{\alpha}(u)\Big]\Delta s\Delta v,$$

then taking *liminf* as $t \to +\infty$ and using (3.36), we get a contradiction with $x(t)$ is positive. If $x(t)$ is an eventually negative solution of Eq. (1.1), the proof is similar. So it is omitted. This completes the proof.

4. Example.

In this section, we give an example of second order neutral integro-dynamic equation with maxima which cannot be studied by the previous published results to illustrate our results.

Example 4.1. *For $t \in [t_0, +\infty)_{\mathbb{T}}$ with $t_3 = 2, t_4 = 4$, and taking $\mathbb{T} = \mathbb{R}$. Consider the following neutral integro dynamic equation with maxima*

$$\left[t^3[x(t) + \frac{t-4}{t}x(\eta_1(t)) + \frac{1}{2t}x(2t)]^{\Delta}\right]^{\Delta} + \int_0^t \frac{1}{t^2s^3}f(s,x(s))\Delta s + t^3 \max_{s\in[t,t+1]} x^{\alpha}(s) = 0. \tag{4.1}$$

Here we take $n = 1, \xi_1(t) = t+1, \tau_1(t) = t, \eta_1(t) \le t, p_1(t) = \frac{t-4}{t}, p_2(t) = \frac{1}{2t}, \alpha = 1, \beta = 2, \gamma = 1, a(t,s) = \frac{1}{t^2s^3}, q_1(t) = t^3$ and $m(t) = t$, hence we have

$$0 < p_1(t) + p_2(t) = \frac{2t-7}{2t} < 1 \text{ for all } t \ge 4.$$

Taking $R(t) = t$ and since $\eta_2(t) = 2t$, then

$$Q(t) = 1 - p_1(t) - p_2(t)\frac{R(\eta_2(t))}{R(t)} = \frac{3}{t} > 0.$$

Since $r(t) = t^3$, then

$$A(u) = \int_u^{+\infty} \frac{ds}{r(s)} = \int_u^{+\infty} \frac{ds}{s^3} = \frac{1}{2u^2}. \tag{4.2}$$

By taking $N(t) = 1$, we obtain

$$\begin{aligned} g_-(t) =& \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}} \int_{t_4}^t a(t,s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s - c_3^{\alpha}\sum_{i=1}^n q_i(t) \max_{u\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(u), \\ =& \frac{1}{2^2}\int_4^t \frac{1}{t^2s^3}s^{-1}ds - c_3t^3 \max_{u\in[t,t+1]} \frac{3}{u}, \\ =& \frac{1}{768t^2} - \frac{1}{12t^5} - 3c_3t^2, \end{aligned} \tag{4.3}$$

also,

$$\begin{aligned} g_+(t) =& \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}} \int_{t_4}^t a(t,s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s - c_3^{\alpha}\sum_{i=1}^n q_i(t) \min_{u\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(u), \\ =& \frac{1}{768t^2} - \frac{1}{12t^5} - \frac{3c_3t^3}{(t+1)} < \frac{1}{768t^2}. \end{aligned} \tag{4.4}$$

Now, since we have

$$\frac{R(t)}{r(t)\int_{t_3}^t \frac{1}{r(s)}\Delta s} - R^{\Delta}(t) = \frac{12-t^2}{t^2-4} < 0 \text{ for all } t \ge 4,$$

then, condition (3.1) holds. Also as

$$\limsup_{t\to+\infty}\int_{t_4}^{t}\Big(\frac{1}{r(u)}\int_{t_4}^{u}g_+(s)\Delta s)^{\frac{1}{\gamma}}\Delta u] < \limsup_{t\to+\infty}\int_{4}^{t}\frac{1}{u^3}\int_{4}^{u}\frac{1}{768s^2}dsdu,$$

$$\leq \frac{1}{768}\limsup_{t\to+\infty}\int_{4}^{t}\frac{1}{u^3}[\frac{-1}{s}]_4^u du < +\infty,$$

hence, condition (3.2) holds, also

$$\limsup_{t\to+\infty}\int_{t_4}^{t}\frac{1}{\gamma r(v)}\int_{t_4}^{v}a(\sigma(s),s)\sigma(s)N(s)\Delta s\Delta v = \limsup_{t\to+\infty}\int_{4}^{t}\frac{1}{u^3}\int_{4}^{u}\frac{1}{s^4}dsdu < +\infty,$$

then, condition (3.26) holds, besides to

$$\int_{t_4}^{+\infty}\frac{1}{r(v)}[\int_{t_4}^{v}\sigma(s)a(\sigma(s),s)m(s)A^{\beta}(s) - q(s)\max_{u\in[s,s+1]}A^{\alpha}(u)\Delta s\Delta v,$$

$$< \int_{t_4}^{+\infty}\frac{1}{r(v)}[\int_{t_4}^{v}\sigma(s)a(\sigma(s),s)m(s)A^{\beta}(s)\Delta s]\Delta v,$$

$$< \int_{4}^{+\infty}\frac{1}{v^3}\int_{4}^{v}\frac{1}{4s^7}dsdv < +\infty.$$

Also, we have

$$\liminf_{t\to+\infty}\int_{t_4}^{t}\Big(\frac{1}{r(u)}\int_{t_4}^{u}g_-(s)\Delta s)^{\frac{1}{\gamma}}\Delta u, = \liminf_{t\to+\infty}\int_{4}^{t}\frac{1}{u^3}\int_{4}^{u}\Big[\frac{1}{768s^2} - \frac{1}{12s^5} - 3c_3t^2\Big]dsdu,$$

$$= \liminf_{t\to+\infty}\int_{4}^{t}\Big[\frac{1}{u^3}[\frac{-1}{768s} + \frac{1}{48s^4}]_4^u + \frac{64c_3}{u^3} - c_3\Big]du \to -\infty \text{ as } t\to+\infty,$$

and

$$\liminf_{t\to+\infty}\int_{t_4}^{t}\frac{1}{r(\mu)}[\int_{t_4}^{\mu}\sigma(s)a(\sigma(s),s)m(s)A^{\beta}(s) - \sum_{i=1}^{n}q_i(s)\max_{s\in[\tau_i(s),\xi_i(s)]}A^{\alpha}(u)\Delta s]\Delta\mu,$$

$$= \liminf_{t\to+\infty}\int_{4}^{t}\frac{1}{v^3}\int_{4}^{v}\frac{1}{4s^7}dsdv - \liminf_{t\to+\infty}\int_{4}^{t}\frac{1}{v^3}\int_{4}^{v}\frac{s}{2}dsdv,$$

$$= \liminf_{t\to+\infty}\int_{4}^{t}\frac{1}{4v^3}[\frac{-1}{6s^6}]_4^v dv - \liminf_{t\to+\infty}\int_{4}^{t}\frac{1}{4v^3}[s^2]_4^v dv = -\infty.$$

So conditions (3.29), (3.35) and (3.36) hold. Now using Theorem 3.4, we obtain that every solution of Eq. (4.1) is oscillatory or tends to zero.

5. Conclusions.

The results of [8] and [2] can't be applied to (4.1) as $p_2(t) \neq 0 \neq p_1(t), q(t) \neq 0$. But according to Theorem 3.4, we obtain that every solution of (4.1) is oscillatory or converges to zero as $t \to +\infty$.

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