# Relative Multifractal Box-Dimensions

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**Abstract.** Given two probability measures $\mu$ and $\nu$ on $\mathbb{R}^n$. We define the upper and lower relative multifractal box-dimensions of the measure $\mu$ with respect to the measure $\nu$ and investigate the relationship between the multifractal box-dimensions and the relative multifractal Hausdorff dimension, the relative multifractal pre-packing dimension. We also, calculate the relative multifractal spectrum and establish the validity of multifractal formalism. As an application, we study the behavior of projections of measures obeying to the relative multifractal formalism.

## 1. Introduction and statement of the results

Multifractal analysis was first introduced by Mandelbrot in the context of turbulence [40, 41] and then studied as a mathematical tool in increasingly general settings. It's concerned with describing the local singular behavior of measures or functions. More precisely, given a finite measure $\mu$ on $\mathbb{R}^n$, we define the local dimension (or the pointwise Hölder exponent) of $\mu$ at a point $x$, by :

$$\alpha_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ denotes the closed ball of center $x$ and radius $r$. The aim of multifractal analysis of measures is to relate the Hausdorff and packing dimensions of a level set of the local dimension of $\mu$, to the Legendre transform of some concave (convex) function [5–9, 15, 44–48, 50]. This is done by calculating the functions $f_\mu(\alpha) = \dim_H \left\{ x : \ \alpha_\mu(x) = \alpha \right\}$ and $F_\mu(\alpha) = \dim_P \left\{ x : \ \alpha_\mu(x) = \alpha \right\}$ for $\alpha \geq 0$, where $\dim_H$ and $\dim_P$ denote respectively the Hausdorff and packing dimensions (see [40, 41, 53]).

One of the main problems about multifractal analysis is to understand the multifractal spectrum and the Rényi dimensions, and their relationship with each other. During the past 20 years there has been an enormous interest in computing the multifractal spectra of measures in the mathematical literature. And within the last 15 years the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^n$, exhibiting some degree of self-similarity have been computed rigorously, see the paper [44], the textbooks [28, 52] and the references therein. In an attempt to develop a general theoretical framework for studying the multifractal structure of arbitrary measures, Olsen [44] and Pesin [51] suggested various ways of defining an auxiliary measure in very general settings.

Let $\mu, \nu$ be two probability measures on a metric space $X$. In [14], Billingsley applied methods from ergodic theory to calculate the size of sets

$$E(\alpha) = \left\{ x \in \text{supp}\,\mu \cap \text{supp}\,\nu : \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log \nu(B(x,r))} = \alpha \right\}.$$

Cajar [16] also studied these sets in the code space. Anyone who is familiar with multifractal analysis will recognize this as a form of multifractal analysis. In several recent papers on multifractal analysis, this type of multifractal analysis has re-emerged as mathematicians and physicists have begun to discuss the idea of performing multifractal analysis with respect to an arbitrary reference measure. Cole introduced Billingsley's concept of Hausdorff, packing measures and dimensions about two measures in probability space to multifractal analysis, i.e., he studied the set of points which has a given local dimension with respect to an arbitrary probability measure. More specifically, for $\mu$ and $\nu$ be two compactly supported Borel probability measures on $\mathbb{R}^n$, the upper and lower $\nu$-local dimensions of $\mu$ are defined as

$$\underline{\alpha}_{\mu,\nu}(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log \nu(B(x,r))} \quad \text{and} \quad \overline{\alpha}_{\mu,\nu}(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log \nu(B(x,r))}.$$

When $\underline{\alpha}_{\mu,\nu}(x) = \overline{\alpha}_{\mu,\nu}(x)$ we refer to the common value as the $\nu$-local dimension of $\mu$ at $x$, and we denote it by $\alpha_{\mu,\nu}(x)$. Cole calculate, for $\alpha \geq 0$, the size of the set

$$E(\alpha) = E_{\mu,\nu}(\alpha) = \left\{ x \in \text{supp}\,\mu \cap \text{supp}\,\nu : \ \alpha_{\mu,\nu}(x) = \alpha \right\}$$

where $\text{supp}\,\mu$ is the topological support of $\mu$. In several recent papers, many authors have begun to discuss the idea of performing multifractal analysis with respect to an arbitrary reference measure [1, 2, 9, 18–22, 24, 27, 38, 55, 61, 63].

In [17] Cole formalised these ideas by introducing a relative formalism for the multifractal analysis of one measure with respect to another. This formalism is based on the ideas of the *classical multifractal formalism* as clarified by Halsey et al. [33], and monitors the formal treatment of this formalism donated by Peyrière and Olsen [44, 53]. Later, Ben Nasr et al. in [11, 12] developed a necessary and sufficient condition for the validity of the multifractal formalism. As an application, we can refer to the multifractal structure of one graph directed self-conformal measure with respect to another (see for example [3, 17, 21]).

In the present paper we study the level sets of the (upper or lower) $\nu$-local dimensions of a given measure $\mu$. We need to determine the functions

$$f_{\mu,\nu}(\alpha) = \dim_H E(\alpha) \quad \text{and} \quad F_{\mu,\nu}(\alpha) = \dim_P E(\alpha).$$

Generally it is very difficult to obtain the singularity spectrum $f_{\mu,\nu}(\alpha)$ directly from the definition of the Hausdorff dimension. To avoid this difficulty, we prove that the relative multifractal formalism provides a formula which link the singularity spectrum $f_{\mu,\nu}(\alpha)$ to the Legendre transform of the relative multifractal $q$-box-dimension. The upper and lower $q$-box-dimensions are then given by

$$\overline{\tau}_{\mu,\nu}(q) = \limsup_{r \to 0} \frac{\log S_{\mu,r}(q)}{-\log \zeta_\nu(r)} \quad \text{and} \quad \underline{\tau}_{\mu,\nu}(q) = \liminf_{r \to 0} \frac{\log S_{\mu,r}(q)}{-\log \zeta_\nu(r)}.$$

When the limit above exists for all $q$ we speak of the $q$-box dimension $\tau_{\mu,\nu}(q)$, where

$$S_{\mu,r}(q) = \sup \left\{ \sum_i \mu(B(x_i, r))^q \right\}$$

with the supremum is taken over all centered packings $\left( B(x_i, r) \right)_i$ of $\text{supp}\,\mu$, and

$$\zeta_\nu(r) = \sup_{x \in \text{supp}\,\mu \,\cap\, \text{supp}\,\nu} \nu\left( B(x,r) \right).$$

First, we define the lower and upper relative multifractal box-dimensions and the relative Rényi dimensions. We compare them to the relative multifractal Hausdorff dimension and the multifractal pre-packing dimension. Secondly, we calculate the $\nu$-multifractal spectrum, where $\nu$ denotes the Borel probability measure on $n$-dimensional linear space, and then, we can research the validity of the relative multifractal formalism by means kind of analogue to those used in the validity of the classical multifractal formalism. As an application, we study the behavior of projections of measures obeying to the relative multifractal formalism.

## 2. Preliminaries

We denote the family of Borel probability measures on $\mathbb{R}^n$ by $\mathcal{P}(\mathbb{R}^n)$. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$. For $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$, we define the generalized packing pre-measure relatively to $\mu$ and $\nu$

$$\overline{\mathcal{P}}_{\mu,\nu,\delta}^{q,t}(E) = \sup \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t,$$

where the supremum is taken over all centered $\delta$-packings of $E$,

$$\overline{\mathcal{P}}_{\mu,\nu}^{q,t}(E) = \inf_{\delta > 0} \overline{\mathcal{P}}_{\mu,\nu,\delta}^{q,t}(E).$$

In a similar way, we define the generalized Hausdorff pre-measure relatively to $\mu$ and $\nu$

$$\overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,t}(E) = \inf \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t,$$

where the infimum is taken over all centered $\delta$-coverings of $E$, and

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,t}(E) = \sup_{\delta > 0} \overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,t}(E),$$

with the conventions $\infty \cdot 0 = 0 \cdot \infty = 0$, $0^q = \infty$ for $q \le 0$ and $0^q = 0$ for $q > 0$.

The function $\overline{\mathcal{H}}_{\mu,\nu}^{q,t}$ is $\sigma$-subadditive but not increasing and the function $\overline{\mathcal{P}}_{\mu,\nu}^{q,t}$ is increasing but not $\sigma$-subadditive. For this reason, Cole defined the generalized Hausdorff and packing measures relatively to tow measures $\mathcal{H}_{\mu,\nu}^{q,t}$ and $\mathcal{P}_{\mu,\nu}^{q,t}$, by

$$\mathcal{H}_{\mu,\nu}^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_{\mu,\nu}^{q,t}(F) \quad \text{and} \quad \mathcal{P}_{\mu,\nu}^{q,t}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu,\nu}^{q,t}(E_i).$$

These functions are metric outer measures and thus measures on the Borel family of subsets of $\mathbb{R}^n$. An important feature of the Hausdorff and packing measures is that $\mathcal{P}_{\mu,\nu}^{q,t} \le \overline{\mathcal{P}}_{\mu,\nu}^{q,t}$, and there exists an integer $\xi \in \mathbb{N}$, such that $\mathcal{H}_{\mu,\nu}^{q,t} \le \xi \mathcal{P}_{\mu,\nu}^{q,t}$.

The measures $\mathcal{H}_{\mu,\nu}^{q,t}$ and $\mathcal{P}_{\mu,\nu}^{q,t}$ and the pre-measure $\overline{\mathcal{P}}_{\mu,\nu}^{q,t}$ assign in the usual way a dimension to each subset $E$ of $\mathbb{R}^n$. They are respectively denoted by $\dim_{\mu,\nu}^q(E)$, $\mathrm{Dim}_{\mu,\nu}^q(E)$ and $\Delta_{\mu,\nu}^q(E)$.

**Proposition 2.1.** *[17]*

1. *There exists a unique number $\dim_{\mu,\nu}^q(E) \in [-\infty, +\infty]$ such that*

$$\mathcal{H}_{\mu,\nu}^{q,t}(E) = \begin{cases} \infty & if \quad t < \dim_{\mu,\nu}^q(E), \\ 0 & if \quad \dim_{\mu,\nu}^q(E) < t. \end{cases}$$

2. *There exists a unique number* $\text{Dim}_{\mu,\nu}^{q}(E) \in [-\infty, +\infty]$ *such that*

$$\mathcal{P}_{\mu,\nu}^{q,t}(E) = \begin{cases} \infty & if \quad t < Dim_{\mu,\nu}^{q}(E), \\ 0 & if \quad Dim_{\mu,\nu}^{q}(E) < t. \end{cases}$$

3. *There exists a unique number* $\Delta_{\mu,\nu}^{q}(E) \in [-\infty, +\infty]$ *such that*

$$\overline{\mathcal{P}}_{\mu,\nu}^{q,t}(E) = \begin{cases} \infty & if \quad t < \Delta_{\mu,\nu}^{q}(E), \\ 0 & if \quad \Delta_{\mu,\nu}^{q}(E) < t. \end{cases}$$

*In addition we have*

$$\dim_{\mu,\nu}^{q}(E) \leq \text{Dim}_{\mu,\nu}^{q}(E) \leq \Delta_{\mu,\nu}^{q}(E). \tag{2.1}$$

**Remark 2.2.** *If $q = 0$, we can see obviously, for $t > 0$, that the functions $\overline{\mathcal{H}}_{\mu,\nu}^{0,t}, \mathcal{H}_{\mu,\nu}^{0,t}, \mathcal{P}_{\mu,\nu}^{0,t}$ and $\overline{\mathcal{P}}_{\mu,\nu}^{0,t}$ do not depend on $\mu$, and they will be denoted respectively, by $\overline{\mathcal{H}}_{\nu}^{t}, \mathcal{H}_{\nu}^{t}, \mathcal{P}_{\nu}^{t}$ and $\overline{\mathcal{P}}_{\nu}^{t}$. Hence, we denote $\nu$-Hausdorff, $\nu$-packing and $\nu$-pre-packing dimension by $\dim_{\nu}, \text{Dim}_{\nu}$ and $\Delta_{\nu}$ respectively. Then, for $E \subset \text{supp}\,\mu \cap \text{supp}\,\nu$, we have*

$$\dim_{\nu}(E) = \dim_{\mu,\nu}^{0}(E), \quad \text{Dim}_{\nu}(E) = \text{Dim}_{\mu,\nu}^{0}(E), \quad \Delta_{\nu}(E) = \dim_{\mu,\nu}^{0}(E).$$

If $E \subseteq \text{supp}\,\mu \cap \text{supp}\,\nu$ and $q, t \in \mathbb{R}$. We define the functions $b_{\mu,\nu}, B_{\mu,\nu}$ and $\Delta_{\mu,\nu}$ by

$$b_{\mu,\nu}^{q}(E) = \dim_{\mu,\nu}^{q}(E), \quad B_{\mu,\nu}^{q}(E) = Dim_{\mu,\nu}^{q}(E) \quad \text{and} \quad \Lambda_{\mu,\nu}^{q}(E) = \Delta_{\mu,\nu}^{q}(E).$$

The special case, where $E = \text{supp}\,\mu \cap \text{supp}\,\nu$, $b_{\mu,\nu}^{q}(E), B_{\mu,\nu}^{q}(E)$ and $\Delta_{\mu,\nu}^{q}(E)$ will be denoted, respectively, by $b_{\mu,\nu}(q), B_{\mu,\nu}(q)$ and $\Lambda_{\mu,\nu}(q)$. It is well known that the functions $b_{\mu,\nu}, B_{\mu,\nu}$ and $\Lambda_{\mu,\nu}$ are decreasing, $B_{\mu,\nu}, \Lambda_{\mu,\nu}$ are convex and $b_{\mu,\nu}$ is pre-convex (see [24]) and satisfying

$$b_{\mu,\nu} \quad \leq \quad B_{\mu,\nu} \leq \Lambda_{\mu,\nu}. \tag{2.2}$$

In addition, we have the following result.

**Proposition 2.3.** *[17] Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^{n})$. Then, we have*

1. *For $q < 1$, $0 \leq b_{\mu,\nu}(q) \leq B_{\mu,\nu}(q) \leq \Lambda_{\mu,\nu}(q)$.*
2. *$b_{\mu,\nu}(1) = B_{\mu,\nu}(1) = \Lambda_{\mu,\nu}(1) = 0$.*
3. *For $q > 1$, $b_{\mu,\nu}(q) \leq B_{\mu,\nu}(q) \leq \Lambda_{\mu,\nu}(q) \leq 0$.*

For $\mu \in \mathcal{P}(\mathbb{R}^{n})$ and $a > 1$, we write

$$T_{a}(E) = \limsup_{r \searrow 0} \left( \sup_{x \in E} \frac{\mu\big(B(x, ar)\big)}{\mu\big(B(x, r)\big)} \right).$$

Now, we will say that the measure $\mu$ satisfies the doubling condition if there exists $a > 1$ such that $T_{a}(E) < \infty$. It is easily seen that the exact value of the parameter $a$ is unimportant: $T_{a}(E) < \infty$, for some $a > 1$ if and only if $T_{a}(E) < \infty$, for all $a > 1$. Also, we will write $\mathcal{P}_{1}(E)$ for the family of Borel probability measures on $E$ which satisfy the doubling condition. We can cite as classical examples of doubling measures, the self-similar measures and the self-conformal ones [44]. We also write

$$\mathcal{P}_{0}(E) = \Big\{ \mu \in \mathcal{P}(\mathbb{R}^{n}) : \ \exists\, a > 1, \forall x \in \text{supp}\,\mu, \ T_{a}(\{x\}) < \infty \Big\}.$$

We present the following technical lemma, which will be used in the proof of our main results.

**Lemma 2.4.** *[39] If $\mu \in \mathcal{P}_1(E)$ and $a > 1$. Then there exist constants $r_0, c > 0$, such that*

$$c^{-1} \leq \frac{\mu\big(B(x, ar)\big)}{\mu\big(B(x, r)\big)} \leq c$$

*for all $x \in E$ and $0 < r < r_0$.*

## 3. Relative multifractal box-dimensions and results

We define two types of upper and lower relative multifractal $q$-box-dimensions in $\mathbb{R}^n$, and compare them to both relative multifractal Hausdorff dimension and relative multifractal pre-packing dimension. Let $\mu$ and $\nu$ in $\mathcal{P}(\mathbb{R}^n)$ and $q \in \mathbb{R}$. For $E \subset \mathbb{R}^n$ and $r > 0$, we write

$$\zeta_\nu(r) = \sup_{x \in E \cap \ \text{supp}\, \nu} \nu\big(B(x, r)\big)$$

and

$$S_{\mu,r}^q(E) = \sup\left\{\sum_i \mu(B(x_i, r))^q\right\},$$

where $\big(B(x_i, r)\big)_i$ is a centered packing of $E \cap \text{supp}\, \mu$. The upper, respectively the lower generalized multifractal $q$-box-dimension $\overline{C}_{\mu,\nu}^q$ and $\underline{C}_{\mu,\nu}^q$ of $E$ is defined by

$$\overline{C}_{\mu,\nu}^q(E) = \limsup_{r \to 0} \frac{\log S_{\mu,r}^q(E)}{-\log \zeta_\nu(r)} \quad \text{and} \quad \underline{C}_{\mu,\nu}^q(E) = \liminf_{r \to 0} \frac{\log S_{\mu,r}^q(E)}{-\log \zeta_\nu(r)}.$$

When the limit above exists for all $q$, we speak of the $q$-box-dimension $C_{\mu,\nu}^q$. Another natural way to define $q$-box-dimensions is given by

$$N_{\mu,r}^q(E) = \inf\left\{\sum_i \mu(B(x_i, r))^q\right\},$$

where $\big(B(x_i, r)\big)_i$ is a centered covering of $E \cap \text{supp}\, \mu$. Now, we write

$$\overline{L}_{\mu,\nu}^q(E) = \limsup_{r \to 0} \frac{\log N_{\mu,r}^q(E)}{-\log \zeta_\nu(r)} \quad \text{and} \quad \underline{L}_{\mu,\nu}^q(E) = \liminf_{r \to 0} \frac{\log N_{\mu,r}^q(E)}{-\log \zeta_\nu(r)}.$$

If $\overline{L}_{\mu,\nu}^q = \underline{L}_{\mu,\nu}^q$, their common value at $q$ is denoted by $L_{\mu,\nu}^q$.

In this section, we are interested in studying some relations between the quantities we have just defined (Theorem 3.1) and compare them with $\Delta_{\mu,\nu}^q$ (Proposition 3.2). We will discuss them, naturally, according to the value of $q$. Moreover, it is not possible, without any condition on the measure $\nu$ to prove the existence of $L_{\mu,\nu}^q$ or $C_{\mu,\nu}^q$.

**Theorem 3.1.** *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$.*

1. *For $q \leq 0$, we have*

$$\overline{L}_{\mu,\nu}^q(E) = \overline{C}_{\mu,\nu}^q(E) \quad \text{and} \quad \underline{L}_{\mu,\nu}^q(E) = \underline{C}_{\mu,\nu}^q(E).$$

2. *Suppose that $\mu \in \mathcal{P}_1(E)$, then for $q > 0$ we have*

$$\overline{L}_{\mu,\nu}^q(E) = \overline{C}_{\mu,\nu}^q(E) \quad \text{and} \quad \underline{L}_{\mu,\nu}^q(E) = \underline{C}_{\mu,\nu}^q(E).$$

*Proof.* The proof is similar to the proof of Propositions 2.19 and 2.20 in [44]. □

**Proposition 3.2.** *Let* $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ *and* $E \subset \mathbb{R}^n$, *we have*

1. *for all* $q > 1$, $\overline{C}^q_{\mu,\nu}(E) \leq \Delta^q_{\mu,\nu}(E)$.
2. *For all* $q \leq 0$, $\underline{C}^q_{\mu,\nu}(E) \geq \Delta^q_{\mu,\nu}(E)$.

*Proof.* This is of course obvious when the term $\Delta^q_{\mu,\nu}(E)$ is infinite. So, without loss of the generality, we assume that it is finite.

1. Let $t = \Delta^q_{\mu,\nu}(E) < 0$ and $\varepsilon > 0$ such that $t + \varepsilon < 0$, we may choose $0 < r_\varepsilon < 1$ such that $\overline{\mathcal{P}}^{q,t+\varepsilon}_{\mu,\nu,r}(E) < 1$ for $0 < r < r_\varepsilon$. This is possible because of the fact that

$$\overline{\mathcal{P}}^{q,t+\varepsilon}_{\mu,\nu}(E) = \lim_{r \to 0} \overline{\mathcal{P}}^{q,t+\varepsilon}_{\mu,\nu,r}(E) = 0.$$

Now, fix $0 < r < r_\varepsilon$ and let $\big(B(x_i, r)\big)_i$ be a packing of $E$. We obtain

$$
\begin{aligned}
\sum_i \mu(B(x_i, r))^q &\leq \sup_i \nu(B(x_i, r))^{-(t+\varepsilon)} \sum_i \mu(B(x_i, r))^q \nu(B(x_i, r))^{t+\varepsilon} \\
&\leq \sup_i \nu(B(x_i, r))^{-(t+\varepsilon)} \overline{\mathcal{P}}^{q,t+\varepsilon}_{\mu,\nu,r}(E) \\
&\leq \sup_i \nu(B(x_i, r))^{-(t+\varepsilon)} \\
&\leq \left[\sup_i \nu(B(x_i, r))\right]^{-(t+\varepsilon)} \\
&\leq \big(C\zeta_\nu(r)\big)^{-(t+\varepsilon)}.
\end{aligned}
$$

2. Denote $t = \Delta^q_{\mu,\nu}(E) \geq 0$. Let $\varepsilon > 0$ such that $t - \varepsilon > 0$ and let $0 < r_0 < 1$. It holds that

$$\infty = \overline{\mathcal{P}}^{q,t-\frac{\varepsilon}{2}}_{\mu,\nu}(E) \leq \overline{\mathcal{P}}^{q,t-\frac{\varepsilon}{2}}_{\mu,\nu,r_0}(E).$$

This means that there exist a $r_0$-packing $\big(B(x_i, r_i)\big)_i$ of $E$ such that

$$\sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^{t-\frac{\varepsilon}{2}} \geq 1.$$

Next, for $n$ be a integer, write

$$J_n = \left\{ i : \quad \frac{r_0}{2^{n+1}} \leq r_i < \frac{r_0}{2^n} \right\} \quad \text{and} \quad A_n = \sum_{i \in J_n} \mu(B(x_i, r_i))^q,$$

then

$$
\begin{aligned}
1 &\leq \sum_i \mu\big(B(x_i, r_i)\big)^q \nu\big(B(x_i, r_i)\big)^{t-\frac{\varepsilon}{2}} \\
&\leq \sup_j \sum_i \mu\big(B(x_i, r_i)\big)^q \nu\big(B(x_i, r_i)\big)^{t-\varepsilon} \nu\big(B(x_j, r_i)\big)^{\frac{\varepsilon}{2}} \\
&\leq \sup_j \sum_n A_n \sup_{i \in J_n} \nu\big(B(x_i, \tfrac{r_0}{2^n})\big)^{t-\varepsilon} \nu\big(B(x_j, \tfrac{r_0}{2^n})\big)^{\frac{\varepsilon}{2}} \\
&\leq \sup_m \sup_i A_m \nu\big(B(x_i, \tfrac{r_0}{2^m})\big)^{t-\varepsilon} \sup_j \sum_n \nu\big(B(x_j, \tfrac{r_0}{2^n})\big)^{\frac{\varepsilon}{2}}.
\end{aligned}
$$

It's clear that, for all $j$, $\sum_n \nu\left(B(x_j, \frac{r_0}{2^n})\right)^{\frac{\xi}{2}} < \infty$. Then, there exists $C > 0$ such that

$$1 \le C \sup_m \sup_i A_m \nu\left(B(x_i, \frac{r_0}{2^m})\right)^{t-\varepsilon}.$$

We may choose $i, N \in \mathbb{N}$ such that

$$CA_N \nu\left(B(x_i, \frac{r_0}{2^N})\right)^{t-\varepsilon} \ge 1. \tag{3.1}$$

Now put $r = \frac{r_0}{2^{N+1}}$. Then $0 \le r < r_0$ and $\left(B(x_i, r)\right)_{i \in J_N}$ is a packing of $E$. It results that

$$
\begin{aligned}
S_{\mu,r}^q(E) &\ge \sum_{i \in J_N} \mu(B(x_i, r))^q \\
&\ge \sum_{i \in J_N} \left[\frac{\mu(B(x_i, \frac{r_0}{2^{N+1}}))}{\mu(B(x_i, \frac{r_0}{2^N}))}\right]^q \mu(B(x_i, r_i))^q \\
&\ge \sum_{i \in J_N} \mu(B(x_i, r_i))^q = A_N.
\end{aligned}
$$

Using (3.1) we get

$$S_{\mu,r}^q(E) \ge C^{-1} \nu\left(B(x_i, \frac{r_0}{2^N})\right)^{-(t-\varepsilon)} \ge C^{-1} \zeta_\nu(r)^{-(t-\varepsilon)}.$$

$\square$

Now, with an additional condition on the measure $\nu$, we will prove the existence of $L_{\mu,\nu}^q$ and $C_{\mu,\nu}^q$. Moreover, they are equal and equal to $\Delta_{\mu,\nu}^q$. This result is true only if $q \le 0$ (Theorem 3.4). The case $q > 0$ treated in Theorem 3.5.

**Definition 3.3.** *A Borel regular measure $\mu$ on $\mathbb{R}^n$ is called uniformly distributed if*

$$0 < \mu(B(x,r)) = \mu(B(y,r)) \quad for \quad x, y \in \operatorname{supp} \mu, \ 0 < r < \infty.$$

Kirchheim and Preiss [37] characterized uniformly distributed measures in $\mathbb{R}$ and gave examples of such measures in $\mathbb{R}^2$. Also, they proved that the support of a uniformly distributed measure on $\mathbb{R}^n$ is a real analytic variety. In fact, there are several authors who characterized uniformly distributed measures, see [23, 54].

**Theorem 3.4.** *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$. Suppose that $\nu$ is a uniformly distributed Borel measure, then for all $q \le 0$, we have*

$$L_{\mu,\nu}^q(E) = C_{\mu,\nu}^q(E) = \Delta_{\mu,\nu}^q(E).$$

*Proof.* Let $t = \Delta_{\mu,\nu}^q(E) \ge 0$ and $\varepsilon > 0$. We may choose $0 < r_\varepsilon < 1$ such that $\overline{\mathcal{P}}_{\mu,\nu,r}^{q,t+\varepsilon}(E) < 1$ for $0 < r < r_\varepsilon$. Now, fix $0 < r < r_\varepsilon$ and let $\left(B(x_i, r)\right)_i$ be a packing of $E$. We obtain, since $\nu$ is a uniformly distributed Borel measure,

$$
\begin{aligned}
\sum_i \mu(B(x_i, r))^q &\le \left[\sup_i \nu(B(x_i, r))\right]^{-(t+\varepsilon)} \sum_i \mu(B(x_i, r))^q \nu(B(x_i, r))^{t+\varepsilon} \\
&\le \left[\sup_i \nu(B(x_i, r))\right]^{-(t+\varepsilon)} \overline{\mathcal{P}}_{\mu,\nu,r}^{q,t+\varepsilon}(E) \\
&\le \left(\zeta_\nu(r)\right)^{-(t+\varepsilon)}. \tag{3.2}
\end{aligned}
$$

We get $S^q_{\mu,r}(E) \le \zeta_\nu(r)^{-(t+\varepsilon)}$. Taking logarithms and letting $r \to 0$ we obtain

$$\overline{C}^q_{\mu,\nu}(E) \le \Delta^q_{\mu,\nu}(E).$$

Finally, we get the desired result from Theorem 3.1 and Proposition 3.2. $\square$

**Theorem 3.5.** *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^n)$ such that $\nu$ is a uniformly distributed Borel measure and $q \in \mathbb{R}_+$. We have*

$$\overline{C}^q_{\mu,\nu}(E) = \Delta^q_{\mu,\nu}(E).$$

*Proof.* This is of course obvious when the term $\Delta^q_{\mu,\nu}(E)$ is infinite. So, without loss of the generality, we assume that it is finite.

**Case 1 :** $q > 1$. Under Proposition 3.2 we only need to prove that

$$\overline{C}^q_{\mu,\nu}(E) \ge \Delta^q_{\mu,\nu}(E). \tag{3.3}$$

Denote $t = \Delta^q_{\mu,\nu}(E) < 0$. From Lemma 2.4, since $\mu \in \mathcal{P}_1(E)$ there exists a constant $c > 0$ and $0 < r_0 < 1$ such that

$$c^{-1}\mu(B(x,r)) \le \mu(B(x,2r)) \le c\mu(B(x,r)) \quad \text{for all} \quad x \in E \quad \text{and } 0 < r < r_0.$$

Let $\varepsilon > 0$ and $1 > r_0 > 0$. It holds that $\infty = \overline{\mathcal{P}}^{q,t-\frac{\varepsilon}{2}}_{\mu,\nu}(E) \le \overline{\mathcal{P}}^{q,t-\frac{\varepsilon}{2}}_{\mu,\nu,r_0}(E)$, This means that there exists a $r_0$-packing $\left(B(x_i, r_i)\right)_i$ of $E$ such that

$$\sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^{t-\frac{\varepsilon}{2}} \ge 1.$$

Next, for $n$ be an integer, we write

$$J_n = \left\{i : \frac{r_0}{2^{n+1}} \le r_i < \frac{r_0}{2^n}\right\} \quad \text{and} \quad A_n = \sum_{i \in J_n} \mu(B(x_i,r_i))^q,$$

then

$$\begin{aligned}
1 &\le \sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^{t-\frac{\varepsilon}{2}} \\
&\le \sup_j \sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^{t-\varepsilon} \nu(B(x_j,r_i))^{\frac{\varepsilon}{2}} \\
&\le \sup_j \sum_n A_n \sup_{i \in J_n} \nu\left(B(x_i, \frac{r_0}{2^{n+1}})\right)^{t-\varepsilon} \nu\left(B(x_j, \frac{r_0}{2^n})\right)^{\frac{\varepsilon}{2}}.
\end{aligned}$$

Since $\nu \in \mathcal{P}_1$, then from Lemma 2.4, there exists $c > 0$ such that, for all $i$,

$$\frac{\nu(B(x_i, \frac{r_0}{2^{n+1}}))}{\nu(B(x_i, \frac{r_0}{2^n}))} \ge c.$$

It follows that

$$\begin{aligned}
1 &\le c^{t-\varepsilon} \sup_j \sum_n A_n \sup_{i \in J_n} \nu\left(B(x_i, \frac{r_0}{2^n})\right)^{t-\varepsilon} \nu\left(B(x_j, \frac{r_0}{2^n})\right)^{\frac{\varepsilon}{2}} \\
&\le c^{t-\varepsilon} \sup_m \sup_i A_m \nu\left(B(x_i, \frac{r_0}{2^m})\right)^{t-\varepsilon} \sup_j \sum_n \nu\left(B(x_j, \frac{r_0}{2^n})\right)^{\frac{\varepsilon}{2}}.
\end{aligned}$$

It's clear that, for all $j$, $\sum_n \nu(B(x_j, \frac{r_0}{2^n}))^{\frac{\varepsilon}{2}} < \infty$. Then, there exists $C_{t,\varepsilon} > 0$ such that

$$1 \le C_{t,\varepsilon} \sup_m \sup_i A_m \nu\left(B(x_i, \frac{r_0}{2^m})\right)^{t-\varepsilon}.$$

We may choose $i, N \in \mathbb{N}$ such that

$$C_{t,\varepsilon} A_N \nu \left( B(x_i, \frac{r_0}{2^N}) \right)^{t-\varepsilon} \geq 1. \tag{3.4}$$

Now put $r = \frac{r_0}{2^{N+1}}$. Then $0 \leq r < r_0$ and $\left( B(x_i, r) \right)_{i \in J_N}$ is a packing of $E$. It results that

$$
\begin{aligned}
S_{\mu,r}^q(E) &\geq \sum_{i \in J_N} \mu(B(x_i, r))^q \\
&\geq \sum_{i \in J_N} \left[ \frac{\mu(B(x_i, \frac{r_0}{2^{N+1}}))}{\mu(B(x_i, \frac{r_0}{2^N}))} \right]^q \mu(B(x_i, r_i))^q.
\end{aligned}
$$

Since $\mu \in \mathcal{P}_1$ and $\nu$ is uniformly distributed Borel measure, we get from equation (3.4) that

$$S_{\mu,r}^q(E) \geq C_1 \nu(B(x_i, r))^{-(t-\varepsilon)} \geq C_1 \zeta_\nu(r)^{-(t-\varepsilon)},$$

where $C_1$ is a constant in $\mathbb{R}_+^*$. Taking the logarithms and letting $r \to 0$ and $\varepsilon \to 0$, we get (3.3).
**Case 2 :** $0 \leq q \leq 1$. The proof is similar to Case 1 and the proof of Theorem 3.4. $\quad \square$

It is clear, from Theorem 3.1 and Proposition 3.2 that, for $q \leq 0$ and $E \subset \mathbb{R}$, we have

$$\dim_{\mu,\nu}^q(E) \leq \underline{L}_{\mu,\nu}^q(E).$$

The next result study the case where $q \in \mathbb{R}_+^*$.

**Theorem 3.6.** *Let $E \subset \mathbb{R}^n$ and $\mu, \nu \in \mathcal{P}_0(E)$. Suppose that $\nu$ is a uniformly distributed Borel measure, then*

$$\forall \, q > 0, \qquad \dim_{\mu,\nu}^q(E) \leq \underline{L}_{\mu,\nu}^q(E).$$

*Proof.* For $m \in \mathbb{N}$, we write

$$E_m = \left\{ x \in E : \frac{\mu(B(x, 3r))}{\mu(B(x, r))} < m \quad \text{and} \quad \frac{\nu(B(x, 3r))}{\nu(B(x, r))} < m \quad \text{for} \quad 0 < r < \frac{1}{m} \right\}.$$

Since $E = \bigcup_m E_m$, then $\dim_{\mu,\nu}^q(E) = \sup_m \dim_{\mu,\nu}^q(E_m)$. Hence it is sufficient to prove that

$$\dim_{\mu,\nu}^q(E_m) \leq \underline{L}_{\mu,\nu}^q(E), \qquad \forall m \in \mathbb{N}.$$

Let $t > \underline{L}_{\mu,\nu}^q(E)$. We must now prove that $\mathcal{H}_{\mu,\nu}^{q,t}(E_m) < \infty$.
Next, remark that for $F \subset E_m$, there exists a sequence $(r_n)_n$ such that $r_n \to 0$ and $0 < r_n < 1$, for which

$$t > \frac{\log N_{\mu,r_n}^q(E)}{-\log \zeta_\nu(r_n)} \quad \text{for} \quad n \in \mathbb{N}.$$

Hence, for $n \in \mathbb{N}$ there exists a centered covering $\left( B(x_{n_i}, r_n) \right)_i$ of $E$ satisfying

$$\zeta_\nu(r_n)^{-t} > \sum_i \mu(B(x_{n_i}, r_n))^q. \tag{3.5}$$

Let next $n \in \mathbb{N}$ and put $J = \left\{ i : \ B(x_{n_i}, r_n) \cap F \neq \emptyset \right\}$. For $i \in J$, choose $y_i \in B(x_{n_i}, r_n) \cap F$, then $\left( B(y_i, 2r_n) \right)_i$ is a centered $2r_n$-covering of $F$, whence

$$
\begin{aligned}
\overline{\mathcal{H}}^{q,t}_{\mu,\nu,r_n}(F) \ &\leq \ \sum_i \mu(B(y_i, 2r_n))^q \nu(B(y_i, 2r_n))^t \\
&\leq \ \sum_i \left[ \frac{\mu(B(y_i, 2r_n))}{\mu(B(x_{n_i}, r_n))} \right]^q \mu(B(x_{n_i}, r_n))^q \left[ \frac{\nu(B(y_i, 2r_n))}{\nu(B(x_{n_i}, r_n))} \right]^t \nu(B(x_{n_i}, r_n))^t \\
&\leq \ \sum_i \left[ \frac{\mu(B(x_{n_i}, 3r_n))}{\mu(B(x_{n_i}, r_n))} \right]^q \mu(B(x_{n_i}, r_n))^q \left[ \frac{\nu(B(x_{n_i}, 3r_n))}{\nu(B(x_{n_i}, r_n))} \right]^t \nu(B(x_{n_i}, r_n))^t \\
&\leq \ C(q,t) \sum_i \mu(B(x_{n_i}, r_n))^q \nu(B(x_{n_i}, r_n))^t,
\end{aligned}
$$

where $C(q,t)$ is a positive constant depending only on $q$ and $t$. It follows from (3.5) and since $\nu$ is a uniformly distributed Borel measure, that

$$
\begin{aligned}
\overline{\mathcal{H}}^{q,t}_{\mu,\nu,r_n}(F) \ &\leq \ C(q,t) \ \sup_i \nu(B(x_{n_i}, r_n))^t \sum_i \mu(B(x_{n_i}, r_n))^q \\
&\leq \ C(q,t) \ \zeta_\nu(r_n)^t \zeta_\nu(r_n)^{-t} < \infty.
\end{aligned}
$$

Letting $n \to \infty$ gives $\overline{\mathcal{H}}^{q,t}_{\mu,\nu}(F) < \infty$ for all $F \subset E_m$, whence $\mathcal{H}^{q,t}_{\mu,\nu}(E_m) < \infty$, which yields that $dim^q_{\mu,\nu}(E_m) \leq t$. $\qquad \square$

## 4. Relative Rényi dimensions

Let us introduce the multifrcatal generalization of the $q$-dimensions also called relative Rényi $q$-dimensions based on integral representations. Let $\mu$ and $\nu$ be two probability measures on $\mathbb{R}^n$. For $q \in \mathbb{R} \setminus \{0\}$ and $r > 0$, we write

$$
\underline{T}_{\mu,\nu}(q) = \liminf_{r \to 0} \frac{1}{-q \log \zeta_\nu(r)} \log \int \mu(B(x,r))^q d\mu(x),
$$

and

$$
\overline{T}_{\mu,\nu}(q) = \limsup_{r \to 0} \frac{1}{-q \log \zeta_\nu(r)} \log \int \mu(B(x,r))^q d\mu(x).
$$

Now we define the generalized entropies due to Rényi by,

$$
h^q_r(\mu) = \frac{1}{q-1} \log S^q_{\mu,r}(\text{supp}\,\mu) \quad \text{for} \quad q \neq 1
$$

and

$$
h^1_r(\mu) = \inf \left\{ -\sum_i \mu(E_i) \log \mu(E_i) : \ (E_i)_i \quad \text{is a partition of} \quad \text{supp}\,\mu \right\}.
$$

We define the upper and lower Rényi $q$-dimensions $\overline{D}_{\mu,\nu}(q)$ and $\underline{D}_{\mu,\nu}(q)$ of $\mu$ with respect to $\nu$ by

$$
\overline{D}_{\mu,\nu}(q) = \limsup_{r \to 0} \frac{\log h^q_r(\mu)}{-\log \zeta_\nu(r)} \quad \text{and} \quad \underline{D}_{\mu,\nu}(q) = \liminf_{r \to 0} \frac{\log h^q_r(\mu)}{-\log \zeta_\nu(r)}.
$$

If $\underline{T}_{\mu,\nu}(q) = \overline{T}_{\mu,\nu}(q)$ (respectively $\underline{D}_{\mu,\nu}(q) = \overline{D}_{\mu,\nu}(q)$) we refer to the common value as the relative Rényi $q$-dimension of $\mu$ respect to $\nu$ and denote it $T_{\mu,\nu}(q)$ (respectively $D_{\mu,\nu}(q)$).

The following result relates these dimensions to $\Lambda_{\mu,\nu}$.

**Theorem 4.1.** *Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^n)$ such that $\nu$ is a uniformly distributed Borel measure on $\mathbb{R}^n$. Then the following holds*

1. $\Lambda_{\mu,\nu}(q) = \max\left((q-1)\underline{T}_{\mu,\nu}(q-1), (q-1)\overline{T}_{\mu,\nu}(q-1)\right).$
2. $\Lambda_{\mu,\nu}(q) = \max\left((q-1)\overline{D}_{\mu,\nu}(q), (q-1)\underline{D}_{\mu,\nu}(q)\right).$

*Proof.* The proof is similar to the proof of Theorem 2.24 in [44]. □

**Remark 4.2.** *A special case of this theorem, when $\nu$ is the Lebesgue measure on $\mathbb{R}^n$, is treated by Olsen in [44].*

## 5. Relative multifractal spectrum

The functions $b_{\mu,\nu}$ and $B_{\mu,\nu}$ are related to the $\nu$-multifractal spectrum of the measure $\mu$. More precisely, if $f^*(\alpha) = \inf_\beta\left(\alpha\beta + f(\beta)\right)$ denotes the Legendre transform of the function $f$ and let us define, for $\mu$ and $\nu \in \mathcal{P}(\mathbb{R}^n)$

$$\alpha_{min} = \sup_{q>0} -\frac{b_{\mu,\nu}(q)}{q}; \quad \alpha_{max} = \inf_{q<0} -\frac{b_{\mu,\nu}(q)}{q},$$

Cole in [17] to be estimated the upper bound of $\nu$-Hausdorff and $\nu$-packing dimensions of $E(\alpha)$. And he rigorously proved the following result

**Theorem 5.1.** *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \geq 0$.*

1. *If $\alpha \in (\alpha_{min}, \alpha_{max})$, then*

$$\dim_\nu(E(\alpha)) \leq b^*_{\mu,\nu}(\alpha) \quad and \quad \mathrm{Dim}_\nu(E(\alpha)) \leq B^*_{\mu,\nu}(\alpha).$$

2. *If $\alpha \in \mathbb{R}^*_+ \setminus [\alpha_{min}, \alpha_{max}]$, then*

$$\dim_\nu(E(\alpha)) = \mathrm{Dim}_\nu(E(\alpha)) = 0.$$

It is more difficult to obtain a minoration for the $\nu$-dimensions of the sets described in this theorem. Extra conditions are always needed to obtain a minoration for the dimensions of level sets. In general, such a minoration is related to the existence of an auxiliary measure which is supported by the set to be analyzed. In the case where $\nu$ equal to the $n$-dimensional Lebesgue measure, Olsen gave a result in such a way and assumed the existence of a Gibbs measure (see [44]) at state $q$ for the measure $\mu$. In an analogous manner, the next result gives lower bound for $\dim_\nu$ in terms of $\Lambda_{\mu,\nu}$. Then, if $\alpha \geq 0$, let us introduce the fractal sets

$$\underline{E}_\alpha = \left\{x \in \mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu: \quad \underline{\alpha}_{\mu,\nu}(x) \geq \alpha\right\},$$

$$\overline{E}^\alpha = \left\{x \in \mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu: \quad \overline{\alpha}_{\mu,\nu}(x) \leq \alpha\right\}$$

and

$$E(\alpha) = \underline{E}_\alpha \bigcap \overline{E}^\alpha.$$

In the following, we will investigate the validity of the relative multifractal formalism. First of all, we establish the conception for the validity of relative multifractal formalism. Note that, for $q \in \mathbb{R}$

$$\overline{\tau}_{\mu,\nu}(q) = \overline{C}^q_{\mu,\nu}(\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu), \quad \underline{\tau}_{\mu,\nu}(q) = \underline{C}^q_{\mu,\nu}(\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu)$$

and

$$\overline{\tau}_{\mu,\nu,c}(q) = \overline{L}^q_{\mu,\nu}(\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu), \quad \underline{\tau}_{\mu,\nu,c}(q) = \underline{L}^q_{\mu,\nu}(\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu).$$

A relative multifractal formalism is said to be valid if the multifractal dimension functions satisfy

$$b_{\mu,\nu}(q) = B_{\mu,\nu}(q) = \Lambda_{\mu,\nu}(q) = \overline{\tau}_{\mu,\nu}(q).$$

**Theorem 5.2.** *Let $q \in \mathbb{R}$ and suppose that $\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\operatorname{supp}\mu \cap \operatorname{supp}\nu) > 0$. Then,*

$$\dim_\nu \left( \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right) \geq \begin{cases} -\Lambda'_{\mu,\nu-}(q)q + \Lambda_{\mu,\nu}(q), & \text{for} \quad q \leq 0, \\[2mm] -\Lambda'_{\mu,\nu+}(q)q + \Lambda_{\mu,\nu}(q), & \text{for} \quad q \geq 0. \end{cases}$$

*Proof.* It is well known from Theorem 2.7 in [17] that for all $\delta > 0$ and $t \in \mathbb{R}$,

$$\begin{cases} \mathcal{H}_\nu^{-\Lambda'_{\mu,\nu+}(q)q + t - \delta}(E(q)) \geq \mathcal{H}_{\mu,\nu}^{q,t}(E(q)), & \text{for} \quad q \geq 0, \\[2mm] \mathcal{H}_\nu^{-\Lambda'_{\mu,\nu-}(q)q + t - \delta}(E(q)) \geq \mathcal{H}_{\mu,\nu}^{q,t}(E(q)), & \text{for} \quad q \leq 0 \end{cases}$$

where $E(q) = \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)}$. Theorem 5.4 is then an easy consequence of the following lemma. $\square$

**Lemma 5.3.** $\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( \operatorname{supp}\mu \cap \operatorname{supp}\nu \setminus \left( \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right) \right) = 0.$

*Proof.* Let us introduce, for $\alpha$ and $\beta$ in $\mathbb{R}$

$$X_\alpha = \operatorname{supp}\mu \cap \operatorname{supp}\nu \setminus \underline{E}_\alpha \quad \text{and} \quad Y^\beta = \operatorname{supp}\mu \cap \operatorname{supp}\nu \setminus \overline{E}^\beta.$$

We just have to prove that

$$\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( X_\alpha \right) = 0, \quad \text{for all} \quad \alpha < -\Lambda'_{\mu,\nu+}(q) \tag{5.1}$$

and

$$\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( Y^\beta \right) = 0, \quad \text{for all} \quad \beta > -\Lambda'_{\mu,\nu-}(q). \tag{5.2}$$

Indeed

$$\begin{aligned} 0 \quad \leq \quad & \mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( \operatorname{supp}\mu \cap \operatorname{supp}\nu \setminus \left( \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right) \right) \\ \leq \quad & \mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( \operatorname{supp}\mu \cap \operatorname{supp}\nu \setminus \left( \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \right) \right) \\ + \quad & \mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( \operatorname{supp}\mu \cap \operatorname{supp}\nu \setminus \left( \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right) \right) \\ \leq \quad & \mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( \bigcup_{\alpha < -\Lambda'_{\mu,\nu+}(q)} \underline{E}_\alpha \right) + \mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( \bigcup_{\beta > -\Lambda'_{\mu,\nu-}(q)} \overline{E}^\beta \right) \\ \leq \quad & \sum_{\alpha < -\Lambda'_{\mu,\nu+}(q)} \mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( X_\alpha \right) + \sum_{\beta > -\Lambda'_{\mu,\nu-}(q)} \mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}\left( Y^\beta \right) = 0. \end{aligned}$$

We only have to prove that (5.1), the proof of (5.2) is similar.

Let $\alpha < -\Lambda'_{\mu,\nu+}(q)$ and take $t > 0$ such that $\Lambda_{\mu,\nu}(q + t) < \Lambda_{\mu,\nu}(q) - \alpha t$, which implies that

$$\overline{\mathcal{P}}_{\mu,\nu}^{q+t,\Lambda_{\mu,\nu}(q)-\alpha t}\left( \operatorname{supp}\mu \cap \operatorname{supp}\nu \right) = 0.$$

If $x \in X_\alpha$, let $\delta > 0$ we can find $0 < r_x < \delta$ such that

$$\mu(B(x, r_x)) > \nu(B(x, r_x))^\alpha.$$

The family $\left(B(x, r_x)\right)_{x \in X_\alpha}$ is then a centered $\delta$-covering of $X_\alpha$. Using Besicovitch's covering theorem, we can construct $\xi$ finite or countable sub-families $\left(B(x_{1j}, r_{1j})\right)_j, ...., \left(B(x_{\xi j}, r_{\xi j})\right)_j$ such that each $X_\alpha \subseteq \bigcup\limits_{i=1}^{\xi} \bigcup\limits_{j} B(x_{ij}, r_{ij})$ and $\left(B(x_{ij}, r_{ij})\right)_j$ is a $\delta$-packing of $X_\alpha$. Observing that

$$\mu(B(x_{ij}, r_{ij}))^q \nu(B(x_{ij}, r_{ij}))^{\Lambda_{\mu,\nu}(q)} \leq \mu(B(x_{ij}, r_{ij}))^{q+t} \nu(B(x_{ij}, r_{ij}))^{\Lambda_{\mu,\nu}(q)-\alpha t}.$$

So, we give

$$\overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,\Lambda_{\mu,\nu}(q)}(X_\alpha) \leq \xi \, \overline{\mathcal{P}}_{\mu,\nu,\delta}^{q+t,\Lambda_{\mu,\nu}(q)-\alpha t}(X_\alpha).$$

Letting $\delta \to 0$, we obtain

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(X_\alpha) \leq \xi \, \overline{\mathcal{P}}_{\mu,\nu}^{q+t,\Lambda_{\mu,\nu}(q)-\alpha t}(X_\alpha).$$

We can replace $X_\alpha$ by any arbitrary subset of $X_\alpha$. Then, we can finally conclude that

$$\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(X_\alpha) \leq \xi \, \overline{\mathcal{P}}_{\mu,\nu}^{q+t,\Lambda_{\mu,\nu}(q)-\alpha t}(\operatorname{supp} \mu \cap \operatorname{supp} \nu) = 0.$$

$\square$

Using Theorems 3.4 and 3.5 we get the following consequence.

**Corollary 5.4.** *Let $q \in \mathbb{R}$ and suppose that the following hypotheses hold,*

1. *$\mu$ is doubling on $\mathbb{R}^n$.*
2. *$\nu$ is a uniformly distributed and doubling Borel measure on $\mathbb{R}^n$.*
3. *$\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\operatorname{supp} \mu \cap \operatorname{supp} \nu) > 0$.*

*Then,*

$$\dim_\nu \left( \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right) \geq \begin{cases} -\Lambda'_{\mu,\nu-}(q)q + \overline{\tau}_{\mu,\nu}(q), & \text{for} \quad q \leq 0, \\ -\Lambda'_{\mu,\nu+}(q)q + \overline{\tau}_{\mu,\nu}(q), & \text{for} \quad q \geq 0. \end{cases}$$

The following result proves that the condition $\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\operatorname{supp} \mu \cap \operatorname{supp} \nu) > 0$ is very close to being a necessary and sufficient condition for the validity of the multifractal formalism.

**Theorem 5.5.** *Let $q \in \mathbb{R}$ and $\mu, \nu$ be two compactly supported Borel probability measures on $\mathbb{R}^n$ such that $\mu$ is doubling and $\nu$ is a uniformly distributed and doubling Borel measure. Now, suppose that one of the following hypotheses is satisfied,*

1. *$\dim_\nu \left( \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right) \geq -\Lambda'_{\mu,\nu+}(q)q + \overline{\tau}_{\mu,\nu}(q), \quad \text{for} \quad q \leq 0,$*
2. *$\dim_\nu \left( \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right) \geq -\Lambda'_{\mu,\nu-}(q)q + \overline{\tau}_{\mu,\nu}(q), \quad \text{for} \quad q \geq 0.$*

*Then,*

$$b_{\mu,\nu}(q) = B_{\mu,\nu}(q) = \Lambda_{\mu,\nu}(q) = \overline{\tau}_{\mu,\nu}(q).$$

*Proof.* We have, for $q \geq 0$

$$\underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \subset \overline{E}^{-\Lambda'_{\mu,\nu-}(q)},$$

so that,

$$-\Lambda'_{\mu,\nu+}(q)q + \overline{\tau}_{\mu,\nu}(q) \leq \dim_\nu \left( \underline{E}_{-\Lambda'_{\mu,\nu+}(q)} \cap \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right) \leq \dim_\nu \left( \overline{E}^{-\Lambda'_{\mu,\nu-}(q)} \right).$$

Suppose that $\alpha = -\Lambda'_{\mu,\nu-}(q)$. We only prove the case where $q \geq 0$. The other one is similar. Then

$$\dim_\nu \left( \overline{E}^\alpha \right) \geq \alpha q + \overline{\tau}_{\mu,\nu}(q).$$

For this, from Theorem 3.5 and (2.2), we have that

$$b_{\mu,\nu}(q) \leq B_{\mu,\nu}(q) \leq \Lambda_{\mu,\nu}(q) = \overline{\tau}_{\mu,\nu}(q).$$

It is then sufficient to prove $b_{\mu,\nu}(q) \geq \overline{\tau}_{\mu,\nu}(q)$. Let $t < \overline{\tau}_{\mu,\nu}(q)$ and choose $\beta$ such that $\beta < \alpha$. Then $\beta q + t < \alpha q + \overline{\tau}_{\mu,\nu}(q)$. For $p \in \mathbb{N}$ we consider the set

$$F_p = \left\{ x \in \overline{E}^\alpha : \ \mu(B(x,r)) \geq \nu(B(x,r))^\beta, \ 0 < r < \frac{1}{p} \right\}.$$

It is clear that $F_p \nearrow \overline{E}^\alpha$ as $p \to \infty$. It follows that there exists $p > 0$, such that

$$\dim_\nu(F_p) > \beta q + t \Rightarrow \mathcal{H}_\nu^{\beta q + t}(F_p) > 0.$$

Let $0 < \delta < \frac{1}{p}$ and $\left( B(x_i, r_i) \right)_i$ is a centered $\delta$-covering of $F_p$. Then,

$$\sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t \geq \sum_i \nu(B(x_i, r_i))^{\beta q + t}.$$

Hence, by Theorem 2.7 in [17],

$$\mathcal{H}_{\mu,\nu}^{q,t}(\text{supp } \mu \cap \text{supp } \nu) \geq \mathcal{H}_{\mu,\nu}^{q,t}(\overline{E}^\alpha) \geq \mathcal{H}_{\mu,\nu}^{q,t}(F_p) \geq \mathcal{H}_\nu^{\beta q + t}(F_p) > 0.$$

It follows that $t \leq b_{\mu,\nu}(q)$. Finally, we get

$$b_{\mu,\nu}(q) = B_{\mu,\nu}(q) = \Lambda_{\mu,\nu}(q) = \overline{\tau}_{\mu,\nu}(q).$$

$\square$

**Corollary 5.6.** *Assume that the hypotheses of Corollary 5.4 hold for all $q \in \mathbb{R}$ and that $\Lambda_{\mu,\nu}$ is differentiable at $q$. Let $\alpha = -\Lambda'_{\mu,\nu}(q)$, there holds*

$$\text{Dim}_\nu \left( E(\alpha) \right) = \dim_\nu \left( E(\alpha) \right) = b^*_{\mu,\nu}(\alpha) = B^*_{\mu,\nu}(\alpha) = \Lambda^*_{\mu,\nu}(\alpha) = \overline{\tau}^*_{\mu,\nu}(\alpha).$$

*Proof.* Denote $\alpha = -\Lambda'_{\mu,\nu}(q)$. Theorems 3.5 and 5.1 imply that

$$\dim_\nu \left( E(\alpha) \right) \leq b^*_{\mu,\nu}(\alpha) \leq B^*_{\mu,\nu}(\alpha) \leq \Lambda^*_{\mu,\nu}(\alpha) = \overline{\tau}^*_{\mu,\nu}(\alpha).$$

On the other hand, Corollary 5.4 yields

$$\dim_\nu \left( E(\alpha) \right) \geq \alpha q + \overline{\tau}_{\mu,\nu}(\alpha), \quad \text{for all} \quad q \in \mathbb{R}.$$

Which implies that

$$\dim_\nu \left( E(\alpha) \right) \geq \overline{\tau}^*_{\mu,\nu}(\alpha) = \Lambda^*_{\mu,\nu}(\alpha) \geq B^*_{\mu,\nu}(\alpha) \geq b^*_{\mu,\nu}(\alpha),$$

which achieves the proof. $\square$

**Corollary 5.7.** *Assume that the hypotheses of Corollary 5.4 hold for all $q \in \mathbb{R}$ and that $\Lambda_{\mu,\nu}$ is differentiable at $q$. Let $\alpha = -\Lambda'_{\mu,\nu}(q)$, there holds*

$$\begin{aligned}
\text{Dim}_\nu \left( E(\alpha) \right) = \dim_\nu \left( E(\alpha) \right) &= b^*_{\mu,\nu}(\alpha) = B^*_{\mu,\nu}(\alpha) = \Lambda^*_{\mu,\nu}(\alpha) = \underline{\tau}^*_{\mu,\nu,c}(\alpha) \\
&= \overline{\tau}^*_{\mu,\nu,c}(\alpha) = \underline{\tau}^*_{\mu,\nu}(\alpha) = \overline{\tau}^*_{\mu,\nu}(\alpha).
\end{aligned}$$

*Proof.* This corollary follows immediately from Corollary 5.6, inequality (2.2) and the following Lemma

**Lemma 5.8.**

1. $b_{\mu,\nu}(q) \leq \underline{\tau}_{\mu,\nu,c}(q) = \underline{\tau}_{\mu,\nu}(q)$.
2. $\overline{\tau}_{\mu,\nu,c}(q) = \overline{\tau}_{\mu,\nu}(q) = \Lambda_{\mu,\nu}(q)$.

□

## 6. Application

Singularity, exponents or spectrum and generalized dimensions are the major components of the multifractal analysis. Recently, the projection behavior of dimensions and multifractal spectra of measures have generated a large interest in the mathematical literature. The study of the behavior of Hausdorff dimension under projection type mappings dates back to the 50's when Marstrand [42] proved a well-known theorem according to which the Hausdorff dimension of a planar set is preserved under typical orthogonal projections. Kaufman [36] proved the same result using potential theoretic methods. Mattila's proof [43] for the general case is also based on the potential theoretic approach that was later generalized to higher dimensions by Hu and Taylor [35] and for the Hausdorff dimension of a measure by Falconer and Mattila [30].

The behavior of the packing dimension under projections is not as straightforward as that of the Hausdorff dimension. While the Hausdorff dimension of a set or a measure is preserved under almost all projections, the packing dimension decrease for almost all of them [29, 30].

As a continuity to these researchs many authors have studied the relationship between multifractal features of a measure $\mu$ on $\mathbb{R}^n$ and those of the projection of the measure onto $m$-dimensional subspaces [4, 25, 26, 49, 58–60, 62, 64]. Other works were carried in this sense for classes of similar measures in Euclidean and symbolic spaces [13, 32, 34, 56, 57]. We briefly recall some basic definitions and facts which will be repeatedly used in subsequent developments. Let $m$ be an integer with $0 < m < n$ and $G_{n,m}$ the Grassmannian manifold of all $m$-dimensional linear subspaces of $\mathbb{R}^n$. Denote by $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$, such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$, define the projection map $\pi_V : \mathbb{R}^n \longrightarrow V$ as the usual orthogonal projection onto $V$. Then, the set $\{\pi_V, V \in G_{n,m}\}$ is compact in the space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and the identification of $V$ with $\pi_V$ induces a compact topology for $G_{n,m}$. Also, for a Borel probability measure $\mu$ with compact support on $\mathbb{R}^n$, denoted by supp $\mu$, and for $V \in G_{n,m}$, define the projection $\mu_V$ of $\mu$ onto $V$, by

$$\mu_V(A) = \mu(\pi_V^{-1}(A)) \quad \forall A \subseteq V.$$

Since $\mu$ is compactly supported and supp $\mu_V = \pi_V(\text{supp } \mu)$ for all $V \in G_{n,m}$, then for any continuous function $f : V \longrightarrow \mathbb{R}$, we have

$$\int_V f d\mu_V = \int f(\pi_V(x)) d\mu(x)$$

whenever these integrals exist.

This section is devoted to the study of the behavior of projections of measures obeying to the relative multifractal formalism. More precisely, we prove that for $q < 0$ if the relative multifractal formalism holds for $\mu$ at $\alpha = -\Lambda'_{\mu,\nu}(q)$, it holds for $\mu_V$ for all $m$-dimensional subspaces $V$. Let us recall the following useful theorem due to Selmi et al. in [24].

**Theorem 6.1.** *Let $\mu$ and $\nu$ be two compactly supported Borel probability measures on $\mathbb{R}^n$ with* supp $\mu \subseteq$ supp $\nu$. *Fix $q \leq 0$ such that $\Lambda_{\mu,\nu}(q) < 1$. Then for all $m$-dimensional subspaces $V$,*

$$\Lambda_{\mu_V,\nu_V}(q) \leq \Lambda_{\mu,\nu}(q), \quad B_{\mu_V,\nu_V}(q) \leq B_{\mu,\nu}(q)$$

*and*

$$b_{\mu_V,\nu_V}(q) \leq b_{\mu,\nu}(q).$$

*In addition, if* $\text{supp}\,\mu = \text{supp}\,\nu$, *we have*

$$b_{\mu_V,\nu_V}(q) = b_{\mu,\nu}(q).$$

**Theorem 6.2.** *Let* $\mu, \nu$ *be two compactly supported Borel probability measures on* $\mathbb{R}^n$ *with* $\text{supp}\,\mu = \text{supp}\,\nu$ *and* $q < 0$. *Suppose that*

$(H_1)$ $\nu$ *is a uniformly distributed Borel measure on* $\mathbb{R}^n$.

$(H_2)$ $\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\text{supp}\,\mu) > 0$,

$(H_3)$ $\Lambda_{\mu,\nu}(q) < 1$,

$(H_4)$ $\Lambda_{\mu,\nu}$ *is differentiable at* $q$.

*Let* $\alpha = -\Lambda'_{\mu,\nu}(q)$, *then for all m-dimensional subspaces V, we have*

$$
\begin{aligned}
\dim_{\nu_V} E_{\mu_V,\nu_V}\big(\alpha\big) &= Dim_{\nu_V} E_{\mu_V,\nu_V}\big(\alpha\big) = \dim_\nu E_{\mu,\nu}\big(\alpha\big) = Dim_\nu E_{\mu,\nu}\big(\alpha\big) \\
&= b_{\mu,\nu}^*(\alpha) = B_{\mu,\nu}^*(\alpha) = \Lambda_{\mu,\nu}^*(\alpha) = \underline{\tau}_{\mu,\nu,c}^*(\alpha) \\
&= \overline{\tau}_{\mu,\nu,c}^*(\alpha) = \underline{\tau}_{\mu,\nu}^*(\alpha) = \overline{\tau}_{\mu,\nu}^*(\alpha).
\end{aligned}
$$

*Proof.* By using Theorem 6.1 and the hypotheses $(H_2)$, $(H_3)$, then for all $m$-dimensional subspaces $V$, we have

$$b_{\mu,\nu}(q) = B_{\mu,\nu}(q) = \Lambda_{\mu,\nu}(q) = b_{\mu_V,\nu_V}(q) = B_{\mu_V,\nu_V}(q) = \Lambda_{\mu_V,\nu_V}(q). \tag{6.1}$$

Now, let us show that

$$0 < \mathcal{H}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(\text{supp}\,\mu_V), \quad \forall\, V \in G_{n,m}.$$

Let $E \subseteq \text{supp}\,\mu$ and $V \in G_{n,m}$. Fix $\delta > 0$ and let $\big(B_i = B(x_i, r_i)\big)_i$ be a $\delta$-centered covering of $E$. Let $E_i$ such that $\pi_V^{-1}(E_i) = E \cap B(x_i, r_i)$. Since $E_i \subset \bigcup_{y \in E_i} B(y, r_i)$ then, the Besicovitch's covering theorem provides a positive integer $K_n$ as well as $K_i \le K_n$ families of pairwise disjoint balls $\mathcal{B}_{i,k} = \big\{B_j^{(i,k)} = B\big(y_j^{(i,k)}, r_{ijk}\big)\big\}$, $1 \le k \le K_i$, extracted from $\big\{B(y, r_i)\big\}_{y \in E_i}$ and such that

$$E_i \subseteq \bigcup_{k=1}^{K_i} \bigcup_j B_j^{(i,k)}.$$

One has,

$$
\begin{aligned}
\sum_i \mu(B_i)^q \nu(B_i)^{\Lambda_{\mu,\nu}(q)} &\le \sum_i \mu_V(B_j^{(i,k)})^q \nu_V\left(\bigcup_{k=1}^{K_i}\bigcup_j B_j^{(i,k)}\right)^{\Lambda_{\mu,\nu}(q)} \\
&\le \sum_{i,j}\sum_{k=1}^{K_i} \mu_V(B_j^{(i,k)})^q \nu_V(B_j^{(i,k)})^{\Lambda_{\mu,\nu}(q)}.
\end{aligned}
$$

From (6.1), we get

$$\overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,\Lambda_{\mu,\nu}(q)}(E) \le \overline{\mathcal{H}}_{\mu_V,\nu_V,\delta}^{q,\Lambda_{\mu_V,\nu_V}(q)}(\pi_V(E)).$$

By tending $\delta \downarrow 0$, we obtain

$$\overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(E) \le \overline{\mathcal{H}}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(\pi_V(E)).$$

Thus, we find that

$$
\begin{aligned}
\overline{\mathcal{H}}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(E) &\le \overline{\mathcal{H}}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(\pi_V(E)) \\
&\le \overline{\mathcal{H}}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(\pi_V(\text{supp}\,\mu)) \\
&= \mathcal{H}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(\text{supp}\,\mu_V).
\end{aligned}
$$

Since $E$ is an arbitrary set then

$$\mathcal{H}_{\mu,\nu}^{q,\Lambda_{\mu,\nu}(q)}(\text{supp } \mu) \leq \mathcal{H}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(\text{supp } \mu_V).$$

Hypothesis $(H_2)$ implies that

$$0 < \mathcal{H}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(\text{supp } \mu_V), \quad \forall \, V \in G_{n,m}.$$

Theorem 6.2 is then an easy consequence of the following lemma.

**Lemma 6.3.** $\mathcal{H}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}\Big( \text{supp } \mu_V \setminus E_{\mu_V,\nu_V}(\alpha)\Big) = 0.$

*Proof.* Let us introduce for $\alpha \in \mathbb{R}$, the sets

$$F_\alpha = \left\{ x \in \text{supp } \mu_V : \ \limsup_{r \to 0} \frac{\log\Big(\mu_V(B(x,r))\Big)}{\log\Big(\nu_V(B(x,r))\Big)} > \alpha\right\}$$

and

$$F_\alpha^1 = \left\{ x \in \text{supp } \nu_V : \ \liminf_{r \to 0} \frac{\log\Big(\mu_V(B(x,r))\Big)}{\log\Big(\nu_V(B(x,r))\Big)} < \alpha\right\}.$$

We have to prove that

$$\mathcal{H}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(F_\alpha) = 0 \quad \text{for every} \quad \alpha > -\Lambda'_{\mu,\nu}(q) \tag{6.2}$$

and

$$\mathcal{H}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(F_\alpha^1) = 0 \quad \text{for every} \quad \alpha < -\Lambda'_{\mu,\nu}(q). \tag{6.3}$$

Let us sketch the proof of assertion (6.2). The proof of (6.3) is similar. Given $\alpha > -\Lambda'_{\mu,\nu}(q)$, we can choose $t > 0$ such that $\Lambda_{\mu,\nu}(q - t) < \Lambda_{\mu,\nu}(q) + \alpha t$. Which implies that,

$$\Lambda_{\mu_V,\nu_V}(q - t) < \Lambda_{\mu_V,\nu_V}(q) + \alpha t$$

and

$$\mathcal{P}_{\mu_V,\nu_V}^{q-t,\Lambda_{\mu_V,\nu_V}(q)+\alpha t}\Big( \text{supp } \mu_V \Big) = 0.$$

Let $\delta > 0$, for each $x \in F_\alpha$ there exists $0 < r_x < \delta$ such that

$$\mu_V(B(x,r_x)) \leq \nu_V(B(x,r_x))^\alpha.$$

The family $\Big(B(x,r_x)\Big)_{x \in F_\alpha}$ is then a centered $\delta$-covering of $F_\alpha$. Using Besicovitch's covering theorem, we can construct $\xi$ finite or countable sub-families $\Big(B(x_{1j},r_{1j})\Big)_j,....,\Big(B(x_{\xi j},r_{\xi j})\Big)_j$ such that each $F_\alpha \subseteq \bigcup_{i=1}^\xi \bigcup_j B(x_{ij},r_{ij})$ and $\Big(B(x_{ij},r_{ij})\Big)_j$ is a $\delta$-packing of $F_\alpha$. Observing that

$$\mu_V(B(x_{ij},r_{ij}))^q \nu_V(B(x_{ij},r_{ij}))^{\Lambda_{\mu_V,\nu_V}(q)} \leq \mu_V(B(x_{ij},r_{ij}))^{q-t} \nu_V(B(x_{ij},r_{ij}))^{\Lambda_{\mu_V,\nu_V}(q)+\alpha t}.$$

We obtain

$$\overline{\mathcal{H}}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(F_\alpha) \leq \ \xi \ \overline{\mathcal{P}}_{\mu_V,\nu_V}^{q-t,\Lambda_{\mu_V,\nu_V}(q)+\alpha t}(F_\alpha).$$

Notice that, in the last inequality, we can replace $F_\alpha$ by any arbitrary subset of $F_\alpha$. Then, we can finally conclude that

$$\mathcal{H}_{\mu_V,\nu_V}^{q,\Lambda_{\mu_V,\nu_V}(q)}(F_\alpha) \leq \ \xi \ \mathcal{P}_{\mu_V,\nu_V}^{q-t,\Lambda_{\mu_V,\nu_V}(q)+\alpha t}(F_\alpha) \leq \ \xi \ \mathcal{P}_{\mu_V,\nu_V}^{q-t,\Lambda_{\mu_V,\nu_V}(q)+\alpha t}\Big( \text{supp } \mu_V \Big) = 0.$$

$\square$

Let us return to the proof of Theorem 6.2.

Theorem 2.10 in [17] along with (6.1) imply that

$$\dim_{\nu_V} E_{\mu_V, \nu_V}\left( - \Lambda'_{\mu,\nu}(q)\right) \geq -q\Lambda'_{\mu,\nu}(q) + \Lambda_{\mu,\nu}(q).$$

So, from Theorem 5.1, the other estimation is satisfied since

$$
\begin{aligned}
\mathrm{Dim}_{\nu_V} E_{\mu_V, \nu_V}\left( - \Lambda'_{\mu,\nu}(q)\right) &\leq -q\Lambda'_{\mu,\nu}(q) + \Lambda_{\mu_V, \nu_V}(q) \\
&= -q\Lambda'_{\mu,\nu}(q) + \Lambda_{\mu,\nu}(q).
\end{aligned}
$$

Finally, Theorem 6.2 follows immediately from (6.1) and Theorem 3.4. $\square$

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