On k-Type Spacelike Slant Helices
Lying on Lightlike Surfaces

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Abstract. In this paper, we define k-type spacelike slant helices lying on a lightlike surface in Minkowski space $\mathbb{E}_1^3$ according to their Darboux frame for $k \in \{0, 1, 2\}$. We obtain the necessary and the sufficient conditions for spacelike curves with non-null and null principal normal lying on lightlike surface to be the k-type spacelike slant helices in terms of their geodesic curvature, normal curvature and geodesic torsion. Additionally, we determine their axes and show that the Darboux frame of a spacelike curve lying on a lightlike surface coincides with its Bishop frame if and only if it has zero geodesic torsion. Finally, we give some examples.

1. Introduction

The geometry of Minkowski space is important for both differential geometry and physics, particularly in the theory of general relativity. Geometric properties of non-degenerate (spacelike and timelike) submanifolds in Minkowski spaces are studied in analogy with the Riemannian submanifolds. However, lightlike (degenerate) submanifolds in those spaces have a quite new and sometimes unexpected properties which have no Riemannian analogues ([2]). The most important difference between degenerate and non-degenerate submanifolds is that the normal bundle of degenerate submanifolds intersects their tangent bundle. For example, the normal vector of a lightlike plane belongs to that plane, which is not the case if the plane is a spacelike or a timelike. It is known that lightlike surfaces are the models of event, Cauchy's and Kruskal horizonts, which are studied in the relativity theory.

In classical differential geometry, the *general helices* are defined as a regular space curves whose tangent makes a constant angle with a fixed direction. They are geodesics of cylinders shaped over a plane curve [11]. *Slant helices* are the successor curves of the general helices [12]. In particular, they are geodesics of the helix surfaces [11]. For the recent characterizations of the general and the slant helices, we refer to [1, 6–8, 15, 17, 19, 20].

Isophotic curves in Euclidean 3-space are regular curves lying on a surface and having a property that the surface normals along those curves make a constant angle with a fixed direction. Isophotic curves in $\mathbb{E}^3$ are related with general and slant helices as follows - geodesic isophotic curves are slant helices and asymptotic isophotic curves are general helices [4]. Isophotic curves are also studied in Minkowski space $\mathbb{E}^3_1$. Null Cartan isophotic curves in $\mathbb{E}^3_1$ are characterized in [13], and spacelike and timelike isophotic curves lying on a timelike surface in $\mathbb{E}^3_1$ are studied in [3].

To the best of authors' knowledge, there are no references related with the spacelike isophotic curves lying on a lightlike surface in Minkowski space $\mathbb{E}^3_1$ yet. Accordingly, we can ask the following question: "Can we determine the conditions under which a spacelike curve lying on a lightlike surface in $\mathbb{E}^3_1$ is an isophotic curve by using its Darboux frame?" In order to answer to this question, we introduce k-type spacelike slant helices as the spacelike curves lying on a lightlike surface and having a property that the scalar product of their Darboux's frame vector field V_k and a fixed direction is constant for $k \in \{0, 1, 2\}$. According to the introduced definition, 0-type spacelike slant helices correspond to spacelike general helices, and 2-type spacelike slant helices correspond to the spacelike isophotic curves. We define the Darboux frame of a spacelike curve lying on a lightlike surface in $\mathbb{E}^3_1$ as positively oriented pseudo orthonormal frame consisting of two null and one spacelike vector field. We show that the Darboux frame of a spacelike curve lying on a lightlike surface coincides with its Bishop frame if and only if it has zero geodesic torsion. We give the necessary and sufficient conditions for spacelike curves with non-null and null principal normal to be the k-type spacelike slant helices for $k \in \{0, 1, 2\}$ in terms of their geodesic curvature, normal curvature and geodesic torsion. We prove that every pseudo null curve lying on a lightlike surface with geodesic torsion $\tau_g \neq 0$ is 0-type and 1-type pseudo null slant helix. We also show that pseudo null curve lying on a lightlike surface with geodesic torsion $\tau_g = 0$ is 2-type pseudo null slant helix. Finally, we give some examples. The obtained characterizations of 2-type spacelike slant helices (with non-null and null principal normal) represent a new contribution to the geometry of spacelike isophotic curves.

2. Preliminaries

Minkowski space $\mathbb{E}^3_1$ is the real vector space $\mathbb{E}^3$ equipped with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$ given by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}^3_1$. Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, an arbitrary vector $x \in \mathbb{E}^3_1$ can have one of three causal characters: it can be a spacelike, a timelike, or a null (lightlike), if $\langle x, x \rangle > 0, \langle x, x \rangle < 0$, or $\langle x, x \rangle = 0$ and $x \neq 0$ respectively. In particular, the vector $x = 0$ is said to be a spacelike. The norm (length) of a vector $x \in \mathbb{E}^3_1$ is given by $\|x\| = \sqrt{|\langle x, x \rangle|}$. If $\|x\| = 1$, x is called a *unit vector*.

The *vector product* of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}^3_1$ is defined by

$$u \times v = (u_3 v_2 - u_2 v_3,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1).$$

An arbitrary curve $\alpha : I \to \mathbb{E}^3_1$ can be *spacelike, timelike*, or *null (lightlike)*, if all of its velocity vectors α' are spacelike, timelike or null, respectively [14].

Denote by $\{T, N, B\}$ the moving Frenet frame along a non-null curve α parameterized by an arc-length parameter s. Then T, N and B are the tangent, the principal normal and the binormal vector field of α, respectively.

If α is a spacelike curve with a non-null principal normal N, the Frenet formulae of α read ([9])

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \epsilon\kappa(s) & 0 \\ -\kappa(s) & 0 & -\epsilon\tau(s) \\ 0 & -\epsilon\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{1}$$

where $\kappa(s)$ is the *first curvature* (curvature) and $\tau(s)$ is the *second curvature* (torsion). In particular, the following relations hold

$$\langle T, T \rangle = 1, \quad \langle N, N \rangle = \epsilon = \pm 1, \quad \langle B, B \rangle = -\epsilon, \tag{2}$$

$$T \times N = -\epsilon B, \quad N \times B = T, \quad B \times T = \epsilon N, \tag{3}$$

A spacelike curve in $\mathbb{E}_1^3$ is called a *pseudo null curve*, if its principal normal vector $N(s)$ and the binormal vector $B(s)$ are linearly independent null vectors. The Frenet formulae of a pseudo null curve have the form ([21])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{4}$$

where $\kappa(s) = 1$ is the *curvature* and $\tau(s)$ is the *torsion*. The Frenet's frame vectors of a pseudo null curve satisfy the relations

$$\langle T, T \rangle = 1, \ \langle N, N \rangle = \langle B, B \rangle = 0, \\ \langle N, B \rangle = 1, \ \langle T, N \rangle = \langle T, B \rangle = 0, \tag{5}$$

$$T \times N = N, \ N \times B = T, \ B \times T = B. \tag{6}$$

Definition 2.1. *A surface M in Minkowski space $\mathbb{E}_1^3$ is called a lightlike (null, degenerate), if the induced metric on the surface is degenerate.*

Throughout the next sections, let $\mathbb{R}_0$ denote $\mathbb{R} \backslash \{0\}$.

3. Darboux frame of a spacelike curve lying on a lightlike surface

In this section, we define the Darboux frame of a spacelike curve lying on a lightlike surface in $\mathbb{E}_1^3$. We show that the Darboux frame of a spacelike curve lying on a lightlike surface coincides with its Bishop frame if and only if it has zero geodesic torsion. We also derive the corresponding Darboux's frame equations and prove that every pseudo null curve lying on lightlike surface has zero geodesic curvature.

Denote by M a lightlike surface in Minkowski space $\mathbb{E}_1^3$ with parametrization $\varphi(s,t)$. Let $\overline{N} = \varphi_s \times \varphi_t$ be the null normal vector field of M and $\alpha : I \subset \mathbb{R} \rightarrow M$ a spacelike curve lying on M. We define the Darboux frame of α as a positively oriented pseudo orthonormal frame $\{T, \zeta, \eta\}$, consisting of the tangential vector field T, the null transversal vector field ζ and the null normal vector field $\eta = \overline{N}|_\alpha$, satisfying the conditions

$$\langle T, T \rangle = 1, \langle \eta, \eta \rangle = \langle \zeta, \zeta \rangle = 0, \\ \langle T, \zeta \rangle = \langle T, \eta \rangle = 0, \langle \zeta, \eta \rangle = 1, \tag{7}$$

$$T \times \zeta = \zeta, \ \zeta \times \eta = T, \ \eta \times T = \eta. \tag{8}$$

We also define the *normal curvature* k_n, the *geodesic curvature* k_g and the *geodesic torsion* τ_g of α as follows:

$$k_n(s) = \langle T'(s), \eta(s) \rangle, \ k_g(s) = \langle T'(s), \zeta(s) \rangle, \ \tau_g(s) = \langle \zeta'(s), \eta(s) \rangle. \tag{9}$$

Remark 3.1. *Note that the introduced Darboux frame $\{T, \zeta, \eta\}$, the normal curvature k_n, geodesic curvature k_g and geodesic torsion τ_g of α correspond to the Darboux frame $\{T, Y, Z\}$, z-curvature κ_z, y-curvature κ_y and torsion $-\tau_r$ respectively, which are defined in [18] in order to characterize spacelike curves lying on a lightcone $\mathbb{Q}^2$ in $\mathbb{E}_1^3$. Darboux frame of a spacelike curve lying on a lightcone $\mathbb{Q}^2$ in $\mathbb{E}_1^3$ is called asymptotic frame $\{t, \alpha, y\}$ in [10].*

In order to obtain the corresponding Darboux's frame equations, we may distinguish two cases:

Case 1. *N is a non-null vector field.* By using the relations (7) and (9), we find that Darboux's frame equations read

$$
\begin{bmatrix} T' \\ \zeta' \\ \eta' \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ -k_g & \tau_g & 0 \\ -k_n & 0 & -\tau_g \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}. \tag{10}
$$

It can be shown that the Darboux frame of α coincides with its Bishop frame (relatively parallel adapted frame, rotation minimizing frame) if and only if $\tau_g = 0$. For the Bishop frame of a non-null curve with a non-null principal normal, see [16].
By using the relations (1) and (10), we get

$$
T' = \epsilon\kappa N = k_n\zeta + k_g\eta. \tag{11}
$$

The condition $\langle N, N \rangle = \epsilon$ and the relations (7) and (11) give $\epsilon\kappa^2 = 2k_g k_n$. If $k_g = 0$ or $k_n = 0$, α is a spacelike straight line lying on the lightlike surface M. Therefore, we will assume that $k_g \neq 0$ and $k_n \neq 0$. Thus the relationship between the Frenet frame $\{T, N, B\}$ and the Darboux frame $\{T, \zeta, \eta\}$ of α reads

$$
\begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{k_g}{\kappa} & \frac{k_g}{\kappa} \\ 0 & \frac{\kappa}{2\epsilon k_g} & -\frac{\kappa}{2\epsilon k_g} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{12}
$$

In particular, the relations between the curvatures are given by

$$
\kappa^2 = 2\epsilon k_g k_n, \quad \tau = \frac{(\frac{k_g}{\kappa})' - (\frac{k_g}{\kappa})\tau_g}{\epsilon(\frac{k_g}{\kappa})}. \tag{13}
$$

Case 2. *N is a null vector field.* Then α is a pseudo null curve. By using the relations (4), (7) and (9), we get

$$
T' = N = k_n\zeta + k_g\eta. \tag{14}
$$

Consequently, the condition $\langle N, N \rangle = 0$ and the relations (7) and (14) imply $0 = 2k_g k_n$. We will prove that $k_g = 0$. Assume that $k_n = 0$ and $k_g \neq 0$. Then $T' = N = k_g\eta$. The conditions $\langle T, B \rangle = 0$ and $\langle N, B \rangle = 1$ give $B = \frac{1}{k_g}\zeta$. Since $\{T, N, B\}$ is positively oriented Frenet frame, it holds $\langle T \times N, B \rangle = 1$. However, by using (7) and (8) we find $\langle T \times N, B \rangle = \langle T \times (k_g\eta), \frac{1}{k_g}\zeta \rangle = \langle -\eta, \zeta \rangle = -1$, which is a contradiction. This proves the following statement.

Proposition 3.2. *A pseudo null curve α lying on a lightlike surface in $\mathbb{E}_1^3$ with the Darboux frame $\{T, \zeta, \eta\}$ which satisfies the conditions (7) and (8), has the geodesic curvature $k_g = 0$ and the normal curvature $k_n \neq 0$.*

Relations (7), (9) and the Proposition 3.2 imply that the Darboux's frame equations read

$$
\begin{bmatrix} T' \\ \zeta' \\ \eta' \end{bmatrix} = \begin{bmatrix} 0 & k_n & 0 \\ 0 & \tau_g & 0 \\ -k_n & 0 & -\tau_g \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}. \tag{15}
$$

It can be verified that the Darboux frame of α coincides with its Bishop frame if and only if $\tau_g = 0$. For the Bishop frame of a pseudo null curve, we refer to [5].
Since $k_g = 0$, according to (4) and (15) we have $T' = N = k_n\zeta$. By using the conditions $\langle T, B \rangle = 0$ and $\langle N, B \rangle = 1$, we obtain $B = \frac{1}{k_n}\eta$. Therefore, the Frenet frame and the Darboux frame of α are related by

$$
\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k_n & 0 \\ 0 & 0 & \frac{1}{k_n} \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}, \tag{16}
$$

where $k_n \neq 0$. Differentiating the relation $B = \frac{1}{k_n}\eta$ and using (4), we get that relations between the curvature functions read

$$\kappa = 1, \quad \tau = \frac{k_n'}{k_n} + \tau_g. \tag{17}$$

4. k-type spacelike slant helix lying on a lightlike surface

In this section, we define k-type spacelike slant helices lying on a lightlike surface in $\mathbb{E}_1^3$ according to their Darboux frame for $k \in \{0, 1, 2\}$. We obtain the necessary and sufficient conditions for spacelike curves with non-null and null principal normal lying on the lightlike surface to be the k-type spacelike slant helices in terms of their geodesic curvature, normal curvature and geodesic torsion. Additionally, we determine their axes and prove that every pseudo null curve lying on a lightlike surface with $\tau_g \neq 0$ is 0-type and 1-type pseudo null slant helix. We also prove that every pseudo null curve lying on a lightlike surface with $\tau_g = 0$ is 2-type pseudo null slant helix.

If $\{T, \zeta, \eta\}$ is the Darboux frame of the spacelike curve lying on the lightlike surface, let us set

$$T = V_0, \quad \zeta = V_1, \quad \eta = V_2.$$

Definition 4.1. *A spacelike curve α lying on a lightlike surface M in $\mathbb{E}_1^3$ with the Darboux frame $\{V_0, V_1, V_2\}$ is called a k-type spacelike slant helix for $k \in \{0, 1, 2\}$, if there exists a non-zero fixed direction $U \in \mathbb{E}_1^3$ such that holds*

$$\langle V_k, U \rangle = c,$$

where $c \in \mathbb{R}$.

The fixed direction U spans an *axis* of the helix. It can be a non-null or a null direction. We will exclude the case when the Darboux's frame vector V_k for $k \in \{0, 1, 2\}$ is constant, since the relation $\langle V_k, U \rangle = constant$ is always satisfied. According to the Definition 4.1, 0-type spacelike slant helices correspond to the spacelike general helices, and 2-type spacelike slant helices correspond to the spacelike isophotic curves.

In the sequel, we will consider two cases: **(A)** α is a k-type spacelike slant helix with non-null principal normal N; **(B)** α is a k-type spacelike slant helix with null principal normal N, i.e. the k-type pseudo null slant helix.

(A) α **is a k-type spacelike slant helix with non-null principal normal** N. In this case, we will consider the next three subcases (A.0), (A.1) and (A.2).

(A.0) α **is a 0-type spacelike slant helix with non-null N**.
According to the Definition 4.1, there exists a non-zero fixed direction $U \in \mathbb{E}_1^3$ such that holds

$$\langle V_0, U \rangle = \langle T, U \rangle = c, \quad c \in \mathbb{R}.$$

Assume that $c \neq 0$. With respect to the Darboux frame $\{T, \zeta, \eta\}$, the fixed direction U can be decomposed as

$$U = cT + u_2\zeta + u_3\eta,$$

where u_2 and u_3 are some differentiable functions of s. Differentiating the previous equation with respect to s and using the equations (10), we obtain the following system of differential equations

$$\begin{cases} -u_2 k_g - u_3 k_n = 0, \\ u_2' + u_2 \tau_g + c k_n = 0, \\ u_3' - u_3 \tau_g + c k_g = 0. \end{cases} \tag{18}$$

Since $k_n \neq 0$ and $k_g \neq 0$, from the second and the third equation of (18), we get

$$\begin{cases} u_2 = -ce^{-\int \tau_g ds} \left(\int k_n e^{\int \tau_g ds} ds \right), \\ u_3 = -ce^{\int \tau_g ds} \left(\int k_g e^{-\int \tau_g ds} ds \right). \end{cases} \tag{19}$$

Substituting (19) in the first equation of (18), we obtain that the curvature functions of α satisfy the relation

$$k_g e^{-\int \tau_g ds} \left(\int k_n e^{\int \tau_g ds} ds \right) + k_n e^{\int \tau_g ds} \left(\int k_g e^{-\int \tau_g ds} ds \right) = 0. \tag{20}$$

Conversely, assume that relation (20) holds. Consider the vector U given by

$$U = cT - ce^{-\int \tau_g ds} \left(\int k_n e^{\int \tau_g ds} ds \right) \zeta - ce^{\int \tau_g ds} \left(\int k_g e^{-\int \tau_g ds} ds \right) \eta,$$

where $k_n \neq 0$, $k_g \neq 0$ and $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using (10), we find $U' = 0$. Hence U is a fixed direction. It can be easily checked that

$$\langle T, U \rangle = c, \quad c \in \mathbb{R}_0.$$

According to the Definition 4.1, α is a 0-type spacelike slant helix whose axis is spanned by vector U. Therefore, we can give the following theorem and corollary.

Theorem 4.2. *Let α be a spacelike curve with non-null principal normal N lying on a lightlike surface in $\mathbb{E}_1^3$. Then α is a 0-type spacelike slant helix whose axis is not orthogonal to T if and only if its curvature functions $k_n \neq 0$, $k_g \neq 0$ and $\tau_g \neq 0$ satisfy the relation*

$$k_g e^{-\int \tau_g ds} \left(\int k_n e^{\int \tau_g ds} ds \right) + k_n e^{\int \tau_g ds} \left(\int k_g e^{-\int \tau_g ds} ds \right) = 0. \tag{21}$$

Corollary 4.3. *If an axis of the 0-type spacelike slant helix in $\mathbb{E}_1^3$ is not orthogonal to T, then it is spanned by*

$$U = cT - ce^{-\int \tau_g ds} \left(\int k_n e^{\int \tau_g ds} ds \right) \zeta - ce^{\int \tau_g ds} \left(\int k_g e^{-\int \tau_g ds} ds \right) \eta,$$

where $k_n \neq 0$, $k_g \neq 0$, $\tau_g \neq 0$ and $c \in \mathbb{R}_0$.

Corollary 4.4. *Every 0-type spacelike slant helix whose axis is not orthogonal to T with non-zero constant curvatures k_g, k_n, τ_g is also 1-type and 2-type spacelike slant helix with respect to the same axis.*

In particular, if the axis of α is orthogonal to T, substituting $c = 0$ in the relation (18), we get

$$\begin{cases} -u_2 k_g - u_3 k_n = 0, \\ u_2' + u_2 \tau_g = 0, \\ u_3' - u_3 \tau_g = 0. \end{cases} \tag{22}$$

From the second and third equations of (22), we have

$$\begin{aligned} u_2 &= A_1 e^{-\int \tau_g ds}, \\ u_3 &= A_2 e^{\int \tau_g ds}, \end{aligned} \tag{23}$$

where $A_1, A_2 \in \mathbb{R}^+$. Substituting (23) in the first equation of (22), we obtain

$$A_1 k_g e^{-\int \tau_g ds} + A_2 k_n e^{\int \tau_g ds} = 0.$$

Theorem 4.5. *Let α be a spacelike curve with non-null principal normal N lying on a lightlike surface in $\mathbb{E}_1^3$. Then α is a 0-type spacelike slant helix whose axis is orthogonal to T if and only if its curvature functions $k_n \neq 0$, $k_g \neq 0$ and $\tau_g \neq 0$ satisfy the relation*

$$A_1 k_g e^{-\int \tau_g ds} + A_2 k_n e^{\int \tau_g ds} = 0,$$

where $A_1, A_2 \in \mathbb{R}^+$.

Corollary 4.6. *An axis of the 0-type spacelike slant helix which is orthogonal to T is spanned by*

$$U = A_1 e^{-\int \tau_g ds} \zeta + A_2 e^{\int \tau_g ds} \eta,$$

where $\tau_g \neq 0$ and $A_1, A_2 \in \mathbb{R}^+$.

If the Darboux frame of α makes minimal rotation, i.e. if it coincides with its Bishop frame, we have $\tau_g = 0$. Substituting $\tau_g = 0$ in (18), we get the next theorem.

Theorem 4.7. *Let α be a 0-type spacelike slant helix with the curvatures $k_n \neq 0$ and $k_g \neq 0$ lying on a lightlike surface M in $\mathbb{E}_1^3$. Then α has geodesic torsion $\tau_g = 0$ if and only if its curvature functions k_n and k_g satisfy*

$$k_g \left(\int k_n ds \right) + k_n \left(\int k_g ds \right) = 0.$$

Corollary 4.8. *An axis of the 0-type spacelike slant helix with geodesic torsion $\tau_g = 0$ is spanned by*

$$U = cT - c \left(\int k_n ds \right) \zeta - c \left(\int k_g ds \right) \eta,$$

where $k_n \neq 0$, $k_g \neq 0$ and $c \in \mathbb{R}_0$

Next, let us characterize the 1-type spacelike slant helices.

(A.1) α is a 1-type spacelike slant helix with non-null N.
By the Definition 4.1, there exists a non-zero constant vector $V \in \mathbb{E}_1^3$ such that holds

$$\langle V_1, V \rangle = \langle \zeta, V \rangle = c, \quad c \in \mathbb{R}.$$

Assume that $c \neq 0$. With respect to the Darboux frame $\{T, \zeta, \eta\}$, the fixed direction V can be decomposed as

$$V = v_1 T + v_2 \zeta + c\eta, \tag{24}$$

where v_1 and v_2 are some differentiable functions of the arclength parameter s. Differentiating the equation (24) with respect to s and using the equations (10), we obtain the system of equations

$$\begin{cases} v_1' - v_2 k_g - ck_n = 0, \\ v_2' + v_2 \tau_g + v_1 k_n = 0, \\ v_1 k_g - c\tau_g = 0. \end{cases} \tag{25}$$

Since $k_g \neq 0$ and $k_n \neq 0$, from the second and the third equation of (25), we get

$$\begin{aligned} v_1 &= c\frac{\tau_g}{k_g}, \\ v_2 &= -ce^{-\int \tau_g ds} \left(\int \frac{\tau_g k_n}{k_g} e^{\int \tau_g ds} ds \right), \end{aligned} \tag{26}$$

where $c \in \mathbb{R}_0$. Substituting (26) in the first equation of (25), we obtain

$$\left(\frac{\tau_g}{k_g} \right)' + k_g e^{-\int \tau_g ds} \left(\int \frac{\tau_g k_n}{k_g} e^{\int \tau_g ds} ds \right) - k_n = 0. \tag{27}$$

Conversely, assume that relation (27) holds. Consider the vector V given by

$$V = c\left(\frac{\tau_g}{k_g}\right)T - ce^{-\int \tau_g ds}\left(\int \frac{\tau_g k_n}{k_g}e^{\int \tau_g ds}ds\right)\zeta + c\eta,$$

where $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using the equations (10), we find $V' = 0$. Hence V is a fixed direction. It can be easily checked that

$$\langle \zeta, V \rangle = c, \quad c \in \mathbb{R}_0.$$

According to the Definition 4.1, α is a 1-type spacelike slant helix whose axis is spanned by vector V.

Theorem 4.9. *Let α be a spacelike curve with non-null principal normal N lying on a lightlike surface in $\mathbb{E}_1^3$. Then α is a 1-type spacelike slant helix whose axis is not orthogonal to ζ if and only if its curvature functions $k_n \neq 0$, $k_g \neq 0$ and $\tau_g \neq 0$ satisfy the relation*

$$\left(\frac{\tau_g}{k_g}\right)' + k_g e^{-\int \tau_g ds}\left(\int \frac{\tau_g k_n}{k_g}e^{\int \tau_g ds}ds\right) - k_n = 0. \tag{28}$$

Corollary 4.10. *If the axis of the 1-type spacelike slant helix α is not orthogonal to ζ, then it is spanned by*

$$V = c\left(\frac{\tau_g}{k_g}\right)T - ce^{-\int \tau_g ds}\left(\int \frac{\tau_g k_n}{k_g}e^{\int \tau_g ds}ds\right)\zeta + c\eta,$$

where $k_n \neq 0$, $k_g \neq 0$, $\tau_g \neq 0$ and $c \in \mathbb{R}_0$

In particular, if the axis of α is orthogonal to ζ, substituting $c = 0$ in the relation (25) we obtain $v_1 = v_2 = 0$. Hence the next corollary holds.

Corollary 4.11. *There are no 1-type spacelike slant helices with the curvatures $k_n \neq 0$ and $k_g \neq 0$ lying on a lightlike surface in $\mathbb{E}_1^3$ whose the Darboux's frame vector ζ is orthogonal to their axes.*

If the Darboux frame of α makes minimal rotation, i.e. if it coincides with the Bishop frame of α, then $\tau_g = 0$. Substituting $\tau_g = 0$ in (25), we get the next theorem.

Theorem 4.12. *Let α be a 1-type spacelike slant helix lying on a lightlike surface in $\mathbb{E}_1^3$, with the curvatures $k_g \neq 0$ and $k_n \neq 0$. Then α has geodesic torsion $\tau_g = 0$ if and only if its curvature functions satisfy*

$$\frac{k_g}{k_n} = constant \neq 0,$$

and its axis is spanned by

$$V = -c\left(\frac{k_n}{k_g}\right)\zeta + c\eta,$$

where $c \in \mathbb{R}_0$.

(A.2) α is a 2-type spacelike slant helix with non-null N.
The following theorems and corollaries can be proved analogously as in the cases (A.0) and (A.1), so we omit their proofs.

Theorem 4.13. *Let α be a spacelike curve with non-null principal normal N lying on a lightlike surface in $\mathbb{E}_1^3$. Then α is a 2-type spacelike slant helix whose axis is not orthogonal to η if and only if its curvature functions $k_n \neq 0$, $k_g \neq 0$ and $\tau_g \neq 0$ satisfy the relation*

$$\left(\frac{\tau_g}{k_n}\right)' + k_n e^{\int \tau_g ds}\left(\int \frac{\tau_g k_g}{k_n}e^{-\int \tau_g ds}ds\right) + k_g = 0. \tag{29}$$

Corollary 4.14. *If an axis of the 2-type spacelike slant helix α is not orthogonal to η, then it is spanned by*

$$W = -c\left(\frac{\tau_g}{k_n}\right)T + c\zeta + ce^{\int \tau_g ds}\left(\int \frac{\tau_g k_g}{k_n}e^{-\int \tau_g ds}ds\right)\eta,$$

where $k_g \neq 0$, $k_n \neq 0$, $\tau_g \neq 0$ and $c \in \mathbb{R}_0$.

Corollary 4.15. *There are no 2-type spacelike slant helices with the curvatures $k_n \neq 0$ and $k_g \neq 0$ lying on a lightlike surface M in $\mathbb{E}_1^3$ whose Darboux's frame vector η is orthogonal to their axes.*

Theorem 4.16. *Let α be a 2-type spacelike slant helix lying on a lightlike surface in $\mathbb{E}_1^3$ with the curvatures $k_g \neq 0$ and $k_n \neq 0$. Then α has geodesic torsion $\tau_g = 0$ if and only if its curvature functions satisfy*

$$\frac{k_g}{k_n} = constant \neq 0,$$

and its axis is spanned by

$$W = c\zeta - c\left(\frac{k_n}{k_g}\right)\eta,$$

where $c \in \mathbb{R}_0$.

(B) α is a k-type pseudo null slant helix. According to the Proposition 3.2, every pseudo null curve lying on a lightlike surface in $\mathbb{E}_1^3$ has the curvature functions $k_g = 0$ and $k_n \neq 0$. In this case, we will consider the next three subcases (B.0), (B.1) and (B.2).

(B.0) α is a 0-type pseudo null slant helix.

According to the Definition 4.1, there exists a non-zero constant vector field $U \in \mathbb{E}_1^3$ such that holds

$$\langle V_0, U \rangle = \langle T, U \rangle = c, \quad c \in \mathbb{R}.$$

Assume that $c \neq 0$. With respect to the Darboux frame $\{T, \zeta, \eta\}$, the fixed direction U can be decomposed as

$$U = cT + u_2\zeta + u_3\eta,$$

where u_2 and u_3 are some differentiable functions in the arclength parameter s. Differentiating the last equation with respect to s and using the relations (15), we obtain the following system of differential equations

$$\begin{cases} -u_3 k_n = 0, \\ u_2' + u_2\tau_g + ck_n = 0, \\ u_3' - u_3\tau_g = 0. \end{cases} \tag{30}$$

Since $k_n \neq 0$, from the first and the second equation of (30) we get

$$\begin{cases} u_2 = -ce^{-\int \tau_g ds}\left(\int k_n e^{\int \tau_g ds}ds\right), \\ u_3 = 0, \end{cases} \tag{31}$$

where $c \in \mathbb{R}_0$.
Conversely, consider the vector U given by

$$U = cT - ce^{-\int \tau_g ds}\left(\int k_n e^{\int \tau_g ds}ds\right)\zeta,$$

where $k_n \neq 0$ and $c \in \mathbb{R}_0$. Differentiating the previous equation with respect to s and using the equations (15), we find $U' = 0$. Hence U is a fixed direction. It can be easily checked that

$$\langle T, U \rangle = c, \quad c \in \mathbb{R}_0.$$

According to the Definition 4.1, α is a 0-type pseudo null slant helix whose axis is spanned by vector U.

In particular, if an axis of α is orthogonal to T, substituting $c = 0$ in the relation (30), we get

$$\begin{cases} -u_3 k_n = 0, \\ u_2' + u_2 \tau_g = 0, \\ u_3' - u_3 \tau_g = 0. \end{cases} \tag{32}$$

Since $k_n \neq 0$, from the first equation and second equation of (32), we have

$$\begin{aligned} u_2 &= C_1 e^{-\int \tau_g ds}, \\ u_3 &= 0, \end{aligned}$$

where $C_1 \in \mathbb{R}^+$. Hence an axis of α is lightlike direction spanned by

$$U = C_1 \left(e^{-\int \tau_g ds} \right) \zeta,$$

where $C_1 \in \mathbb{R}^+$. Conversely, by using the last relation, it can be easily shown that $U' = 0$ and $\langle T, U \rangle = 0$. Therefore, α is 0-type pseudo null slant helix. This proves the following theorem.

Theorem 4.17. *Every pseudo null curve lying on a lightlike surface in $\mathbb{E}_1^3$ with the curvatures $k_n \neq 0$ and $\tau_g \neq 0$, is a 0-type pseudo null slant helix having spacelike and lightlike axes spanned by*

$$U_1 = cT - c e^{-\int \tau_g ds} \left(\int k_n e^{\int \tau_g ds} ds \right) \zeta, \tag{33}$$

and

$$U_2 = C_1 \left(e^{-\int \tau_g ds} \right) \zeta, \tag{34}$$

respectively, where $c \in \mathbb{R}_0$ and $C_1 \in \mathbb{R}^+$.

Corollary 4.18. *Every 0-type pseudo null slant helix lying on a lightlike surface in $\mathbb{E}_1^3$, with the curvatures $k_n \neq 0$ and $\tau_g \neq 0$ is a 1-type pseudo null slant helix with respect to the same axes.*

If the Darboux frame of pseudo null curve α makes a minimal rotation, i.e. if it coincides with its Bishop frame, then $\tau_g = 0$. Substituting $\tau_g = 0$ in (30), we get the next theorem.

Theorem 4.19. *Every pseudo null curve lying on a lightlike surface in $\mathbb{E}_1^3$ with the curvatures $k_n \neq 0$ and $\tau_g = 0$, is a 0-type pseudo null slant helix having spacelike axis spanned by*

$$U = cT - c \left(\int k_n ds \right) \zeta,$$

where $c \in \mathbb{R}_0$.

Next, let us characterize the 1-type pseudo null slant helices.

(B.1) α is a 1-type pseudo null slant helix.

By the Definition 4.1, there exists a non-zero constant vector field $V \in \mathbb{E}_1^3$ such that holds

$$\langle V_1, V \rangle = \langle \zeta, V \rangle = c, \quad c \in \mathbb{R}. \tag{35}$$

Assume that $c \neq 0$. With respect to the Darboux frame $\{T, \zeta, \eta\}$, the fixed direction V can be decomposed as

$$V = v_1 T + v_2 \zeta + c\eta, \tag{36}$$

where v_1 and v_2 are some differentiable functions in the arclength parameter s. Differentiating the equation (36) with respect to s and using the equations (15), we obtain the following system of differential equations

$$\begin{cases} v_1' - ck_n = 0, \\ v_2' + v_2\tau_g + v_1 k_n = 0, \\ \quad -c\tau_g = 0. \end{cases} \tag{37}$$

Since $c \in \mathbb{R}_0$, from the third equation of (37) we get $\tau_g = 0$. Then the relation (15) implies $\zeta' = 0$, so ζ is a constant vector. However, this case we have excluded as the possibility, since then relation $\langle \zeta, V \rangle = constant$ is trivially satisfied.

Theorem 4.20. *There are no 1-type pseudo null slant helices with geodesic torsion $\tau_g = 0$ lying on a lightlike surface in $\mathbb{E}_1^3$.*

Theorem 4.21. *There are no 1-type pseudo null slant helices lying on the lightlike surface in $\mathbb{E}_1^3$ whose axis is not orthogonal to ζ.*

On the other hand, if an axis of α is orthogonal to ζ, we get the next theorem.

Theorem 4.22. *Every pseudo null curve lying on a lightlike surface M in $\mathbb{E}_1^3$ with the curvatures $k_n \neq 0$ and $\tau_g \neq 0$ is a 1-type pseudo null slant helix whose axes are given by (33) and (34).*

Finally, let us characterize the 2-type pseudo null slant helices.

(B.2) α is a 2-type pseudo null slant helix.

The following theorems and corollaries can be proved analogously as in the subcase (B.0), so we omit their proofs.

Theorem 4.23. *Let α be a pseudo null curve lying on a lightlike surface M in $\mathbb{E}_1^3$. Then α is a 2-type pseudo null slant helix whose axis is not orthogonal to Darboux's frame vector η if and only if its curvatures $k_n \neq 0$ and $\tau_g \neq 0$ satisfy the relation*

$$c\left(\frac{\tau_g}{k_n}\right)' = -ak_n e^{\int \tau_g ds}, \tag{38}$$

where $c \in \mathbb{R}_0$ and $a \in \mathbb{R}^+$.

Corollary 4.24. *An axis of the 2-type pseudo null slant helix α, which is not orthogonal to Darboux's frame vector η, is spanned by*

$$W = -c\left(\frac{\tau_g}{k_n}\right)T + c\zeta + a\left(e^{\int \tau_g ds}\right)\eta,$$

where $k_n \neq 0$, $\tau_g \neq 0$, $c \in \mathbb{R}_0$ and $a \in \mathbb{R}^+$.

Corollary 4.25. *There are no 2-type pseudo null slant helices lying on a lightlike surface in $\mathbb{E}_1^3$ whose axis is orthogonal to Darboux's frame vector η.*

Theorem 4.26. *Every pseudo null curve lying on lightlike surface in $\mathbb{E}_1^3$ with geodesic torsion $\tau_g = 0$, is 2-type pseudo null slant helix whose axis is a lightlike direction spanned by*

$$W = c\zeta,$$

where $c \in \mathbb{R}_0$.

5. Some examples of k-type spacelike slant helices

Example 5.1. *Let us consider a ruled surface M in $\mathbb{E}_1^3$ parameterized by (see Figure 1)*

$$\varphi(s,t) = \alpha(s) + t\left(\cosh\frac{s}{5} - \frac{4}{5}\sinh\frac{s}{5}, \sinh\frac{s}{5} - \frac{4}{5}\cosh\frac{s}{5}, \frac{3}{5}\right)$$

with a spacelike base curve given by

$$\alpha(s) = \left(3\cosh\frac{s}{5}, 3\sinh\frac{s}{5}, \frac{4s}{5}\right).$$

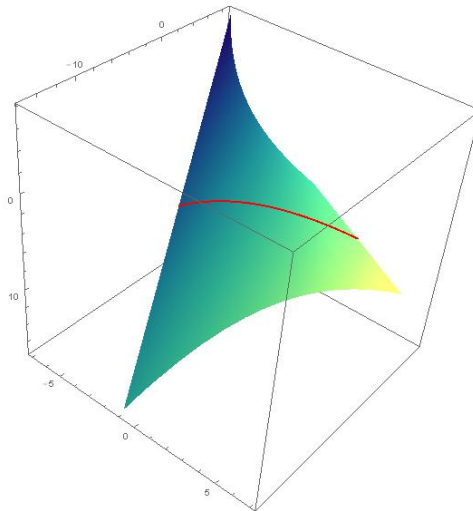Figure 1: The ruled surface M and 0-type, 1-type and 2-type spacelike slant helix α

It can be verified that the normal vector field $\overline{N} = \varphi_s \times \varphi_t$ of M is lightlike. Hence M is a lightlike surface. The Frenet vectors of α have the form

$$T(s) = \left(\frac{3}{5}\sinh\frac{s}{5}, \frac{3}{5}\cosh\frac{s}{5}, \frac{4}{5}\right),$$
$$N(s) = \left(-\cosh\frac{s}{5}, -\sinh\frac{s}{5}, 0\right),$$
$$B(s) = \left(-\frac{4}{5}\sinh\frac{s}{5}, -\frac{4}{5}\cosh\frac{s}{5}, \frac{3}{5}\right),$$

and the Frenet curvatures κ and τ of α read

$$\kappa(s) = \frac{3}{25}, \quad \tau(s) = \frac{4}{25}.$$

The Darboux's frame vectors of α are given by

$$T(s) = \left(\frac{3}{5}\sinh\frac{s}{5}, \frac{3}{5}\cosh\frac{s}{5}, \frac{4}{5}\right),$$
$$\zeta(s) = \frac{1}{2}\left(\cosh\frac{s}{5} + \frac{4}{5}\sinh\frac{s}{5}, \sinh\frac{s}{5} + \frac{4}{5}\cosh\frac{s}{5}, -\frac{3}{5}\right),$$
$$\eta(s) = \left(\frac{4}{5}\sinh\frac{s}{5} - \cosh\frac{s}{5}, \frac{4}{5}\cosh\frac{s}{5} - \sinh\frac{s}{5}, -\frac{3}{5}\right).$$

By using the relation (9) and the previous equations, we obtain that the curvatures k_g, k_n and τ_g of α read

$$k_g(s) = -\frac{3}{50}, \quad k_n(s) = \frac{3}{25}, \quad \tau_g(s) = \frac{4}{25}.$$

Since the curvature functions k_g, k_n and τ_g satisfy the relation (21), the curve α is 0-type spacelike slant helix whose axis is spanned by $U = c\left(0, 0, \frac{5}{4}\right)$, where $c \in \mathbb{R}_0$. Moreover, by the Definition 4.1, the curve α is also 1-type and 2-type spacelike slant helix with respect to the same axis U.

Example 5.2. *Let us consider a ruled surface M in $\mathbb{E}_1^3$ parameterized by (see Figure 2)*

$$\varphi\,(s,t) = \alpha\,(s) + t\left(2s + \frac{1}{2s}, 2s - \frac{1}{2s}, 2\right)$$

with a null rulings and a pseudo null base curve given by

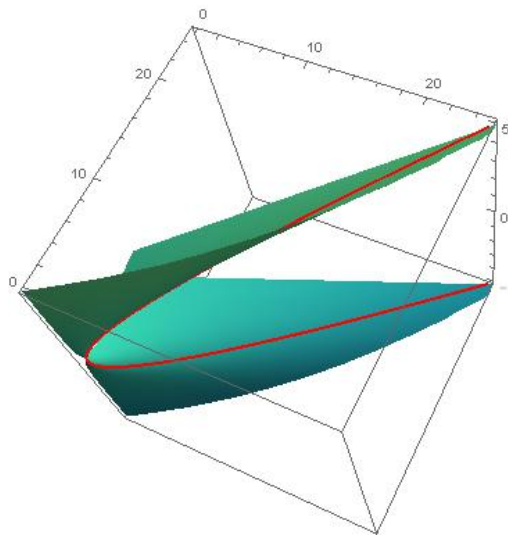$$\alpha(s) = \left(s^2, s^2, s\right).$$

Figure 2: The ruled surface M and 0-type and 1-type pseudo null slant helix α

It can be checked that $\varphi_s \times \varphi_t = 0$, which means that M is a lightlike ruled surface. The Frenet vectors of α have the form

$$T(s) = (2s, 2s, 1),$$
$$N(s) = (2, 2, 0),$$
$$B(s) = \left(-s^2 - \frac{1}{4}, -s^2 + \frac{1}{4}, -s\right),$$

and the Frenet curvatures κ and τ of α read

$$\kappa(s) = 1, \quad \tau(s) = 0.$$

The Darboux's frame vectors of α are given by

$$T\,(s) = (2s, 2s, 1),$$
$$\zeta\,(s) = \tfrac{s}{2}\,(2, 2, 0),$$
$$\eta\,(s) = \left(-2s - \frac{1}{2s}, -2s + \frac{1}{2s}, -2\right).$$

Relation (9) implies that the curvatures k_g, k_n and τ_g of α have the form

$$k_g\,(s) = 0, \quad k_n\,(s) = \frac{2}{s}, \quad \tau_g\,(s) = \frac{1}{s}.$$

According to the Theorem 4.17, the pseudo null curve α is 0-type pseudo null slant helix having two axes spanned by $U_1 = c\,(0, 0, 1)$, $c \in \mathbb{R}_0$ and $U_2 = C_1\,(1, 1, 0)$, $C_1 \in \mathbb{R}^+$. In particular, by the Corollary 4.18, the curve α is also 1-type pseudo null slant helix with respect to the same axes U_1 and U_2.

Example 5.3. *Let us consider a lightlike ruled surface in $\mathbb{E}_1^3$ parameterized by (see Figure 3)*

$$\varphi\,(s, t) = \alpha\,(s) + \frac{t}{4\sqrt{2}s}\left(\sqrt{2}\left(s^4 + 16\right), s^4 - 8s^2 - 16, s^4 + 8s^2 - 16\right)$$

with a null rulings and a pseudo null base curve given by

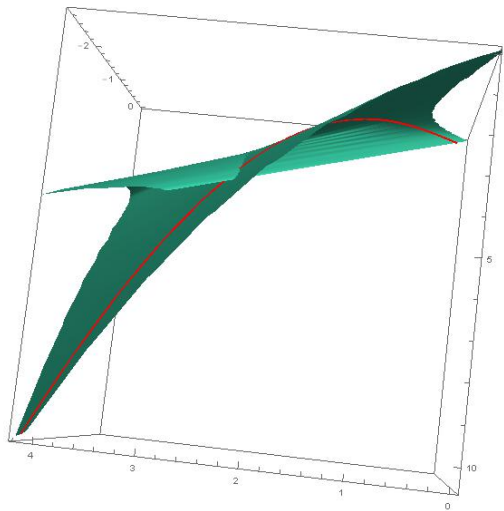$$\alpha(s) = \left(\frac{s^3}{12}, \frac{s^3 + 12s}{12\sqrt{2}}, \frac{s^3 - 12s}{12\sqrt{2}}\right).$$

Figure 3: The ruled surface M and 0-type and 1-type pseudo null slant helix α

The Frenet vectors of α have the form

$$T(s) = \left(\frac{s^2}{4}, \frac{s^2 + 4}{4\sqrt{2}}, \frac{s^2 - 4}{4\sqrt{2}}\right),$$

$$N(s) = \frac{s}{2}\left(1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$$

$$B(s) = \left(-\frac{s^3}{16} - \frac{1}{s}, \frac{s}{2\sqrt{2}} + \frac{1}{\sqrt{2}s} - \frac{s^3}{16\sqrt{2}}, -\frac{s}{2\sqrt{2}} + \frac{1}{\sqrt{2}s} - \frac{s^3}{16\sqrt{2}}\right),$$

and the Frenet curvatures κ and τ of α read

$$\kappa(s) = 1, \quad \tau(s) = \frac{1}{s}.$$

The Darboux's frame vectors of α are given by

$$T\,(s) = \left(\frac{s^2}{4}, \frac{s^2 + 4}{4\sqrt{2}}, \frac{s^2 - 4}{4\sqrt{2}}\right),$$

$$\zeta\,(s) = \left(\frac{s}{8}, \frac{\sqrt{2}}{16}s, \frac{\sqrt{2}}{16}s\right),$$

$$\eta\,(s) = \left(-\frac{s^3}{4} - \frac{4}{s}, -\frac{\sqrt{2}}{8}s^3 + \sqrt{2}s + \frac{2\sqrt{2}}{s}, -\frac{\sqrt{2}}{8}s^3 - \sqrt{2}s + \frac{2\sqrt{2}}{s}\right).$$

By using (9) and the last equations, we find that the curvatures k_g, k_n and τ_g of α read

$$k_g(s) = 0, \quad k_n(s) = 4, \quad \tau_g(s) = \frac{1}{s}.$$

By the Theorem 4.17, the pseudo null curve α is 0-type pseudo null slant helix having two axes spanned by $U_1 = c\left(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $c \in \mathbb{R}_0$ and $U_2 = C_1\left(\frac{1}{8}, \frac{\sqrt{2}}{16}, \frac{\sqrt{2}}{16}\right)$, where $C_1 \in \mathbb{R}^+$. By the Corollary 4.18, the curve α is also 1-type pseudo null slant helix with respect to the same axes U_1 and U_2.

Example 5.4. *Let us consider a lightlike ruled surface M in $\mathbb{E}^3_1$ parameterized by (see Figure 4)*

$$\varphi(s,t) = \alpha(s) + \frac{t}{4\sqrt{2}s}\left(\sqrt{2}\left(s^4 + 16\right), s^4 - 8s^2 - 16, s^4 + 8s^2 - 16\right)$$

with a null rulings and a pseudo null base curve given by

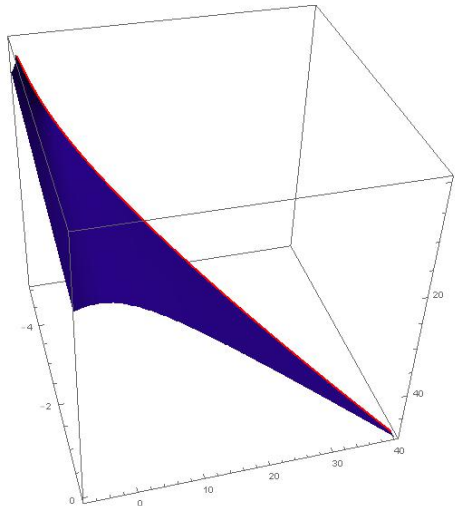$$\alpha(s) = \left(\frac{s^3}{3} + s + 2, \frac{s^3}{3} + s - 7, s - 5\right).$$

Figure 4: The ruled surface M and 0-type and 2-type pseudo null slant helix α

The Frenet vectors of α have the form

$$
\begin{aligned}
T(s) &= \left(s^2 + 1, s^2 + 1, 1\right), \\
N(s) &= (2s, 2s, 0), \\
B(s) &= \left(-\frac{1 + \left(s^2 + 1\right)^2}{4s}, \frac{1 - \left(s^2 + 1\right)^2}{4s}, -\frac{s^2 + 1}{2s}\right),
\end{aligned}
$$

and the Frenet curvatures κ and τ of α read

$$\kappa(s) = 1, \quad \tau(s) = \frac{1}{s}.$$

The Darboux's frame vectors of α are given by

$$T(s) = \left(\frac{s^2}{4}, \frac{s^2 + 4}{4\sqrt{2}}, \frac{s^2 - 4}{4\sqrt{2}} \right),$$
$$\zeta(s) = (2, 2, 0),$$
$$\eta(s) = \left(-\frac{1 + \left(s^2 + 1 \right)^2}{4}, \frac{1 - \left(s^2 + 1 \right)^2}{4}, -\frac{s^2 + 1}{2} \right).$$

According to (9), the curvatures k_g, k_n and τ_g of α read

$$k_g(s) = 0, \quad k_n(s) = s, \quad \tau_g(s) = 0.$$

According to the Theorem 4.19, the pseudo null curve α is a 0-type pseudo null slant helix whose spacelike axis is spanned by $U = c(1, 1, 1)$, $c \in \mathbb{R}_0$. By the Theorem 4.26, α is 2-type pseudo null slant helix with lightlike axis spanned by $W = c\zeta = c(2, 2, 0)$, $c \in \mathbb{R}_0$.

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