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On Concomitants of Ordered Random Variables under General Forms of Morgenstern Family

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Abstract. Based on the extensions of Morgenstern family (Huang and Bairamov extensions), the concomitants of different types of generalized order statistics (*gos*) and dual generalized order statistics (*dgos*) are obtained. Moreover, a unified approach to such models is derived. Information properties such as Shannon entropy and Kullback-Leibler divergence for Huang and Kotz extension are obtained.

1. Introduction

The Farlie-Gumbel-Morgenstern family (Morgenstern family) is an important class of bivariate distributions, it was originally introduced by Morgenstern [17] for Cauchy marginal. Morgenstern distributions are important and efficient in applications for multivariate distributions with given marginal. Johnson and Kotz [11] studied the multivariate case and provided a detailed analysis of probabilistic and statistical characteristics. Huang and Kotz [9] extended Morgenstern family to increase the dependence between the underlying variables by introducing an additional parameter. The generalizations of Morgenstern family of bivariate distributions received a great deal of attention of many researchers. A polynomial type single parameter extension of Morgenstern distribution was considered by Huang and Kotz [10], which is specified by the cumulative distribution function (cdf) and probability density function (pdf), respectively, as follows:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F_X^p(x))(1-F_Y^p(y))], \tag{1}$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1+\alpha((1+p)F_X^p(x)-1)((1+p)F_Y^p(y)-1)], p \geq 1, \tag{2}$$

where the admissible range of the associated parameter α is $-\max(1,p)^{-2} \leq \alpha \leq p^{-1}$, and since $p \geq 1$, this admissible becomes $p^{-2} \leq \alpha \leq p^{-1}$. Furthermore, the conditional pdf of Y given X is given by:

$$f_{Y|X}(y \mid x) = f_Y(y)[1+\alpha((1+p)F_X^p(x)-1)((1+p)F_Y^p(y)-1)]. \tag{3}$$

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Bairamov et al. [2] presented a general form of the model described above, as follows:

$$F^{BA}_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F^{p_1}_X(x))^{q_1}(1-F^{p_2}_Y(y))^{q_2}], \tag{4}$$

$$\begin{aligned} f^{BA}_{X,Y}(x,y) = {} & f_X(x)f_Y(y)[1+\alpha(F^{p_1}_X(x)-1)^{q_1-1}(1-(1+p_1q_1)F^{p_1}_X(x))(F^{p_2}_Y(y)-1)^{q_2-1} \\ & \times(1-(1+p_2q_2)F^{p_2}_Y(y))], p_1, p_2 \geq 1, q_1, q_2 > 1, \end{aligned} \tag{5}$$

where the admissible range of the associated parameter α is

$$\begin{aligned} & -\min\left\{1, \frac{1}{p_1p_2}\left(\frac{1+p_1q_1}{p_1(q_1-1)}\right)^{q_1-1}\left(\frac{1+p_2q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\} \leq \alpha \\ & \leq \min\left\{\frac{1}{p_1}\left(\frac{1+p_1q_1}{p_1(q_1-1)}\right)^{q_1-1}, \frac{1}{p_2}\left(\frac{1+p_2q_2}{p_2(q_2-1)}\right)^{q_2-1}\right\}. \end{aligned} \tag{6}$$

Moreover, the conditional pdf of Y given X is given by:

$$\begin{aligned} f^{BA}_{Y|X}(y \mid x) = {} & f_Y(y)[1+\alpha(F^{p_1}_X(x)-1)^{q_1-1}(1-(1+p_1q_1)F^{p_1}_X(x))(F^{p_2}_Y(y)-1)^{q_2-1} \\ & \times(1-(1+p_2q_2)F^{p_2}_Y(y))]. \end{aligned} \tag{7}$$

Where $f_X(x)$, $f_Y(y)$, and $F_X(x)$, $F_Y(y)$ are the marginal pdf's and cdf's of the random variables ($R.V.$'s) X and Y respectively.

Originally David et al. [6] studied concomitants of order statistics. From some bivariate population with cdf $F(x,y)$, let (X_i, Y_i), $i = 1, 2, ..., n$, be n pairs of independent $R.V.$'s. Let $X_{(r:n)}$ be the rth order statistics, then Y associated with $X_{(r:n)}$ is called the concomitant of rth order statistics and is denoted by $Y_{[r:n]}$. The pdf and cdf of $Y_{[r:n]}$ are given by:

$$g_{[r:n]}(y) = g_{Y_{[r:n]}}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y \mid x) f_{(r:n)}(x)dx, \tag{8}$$

$$G_{[r:n]}(y) = \int_{-\infty}^{\infty} F_{Y|X}(y \mid x) f_{(r:n)}(x)dx, \tag{9}$$

where $f_{(r:n)}(x)$ is the pdf of $X_{(r:n)}$.

The concept of gos was introduced by Kamps [12] and we refer to it as case-I of gos. The use of such connotation has been steadily rising over the years, as it includes important well-known concepts that have been separately treated in statistical literature. Accordingly, many of models of ascendingly ordered $R.V.$'s are contained in it, such as ordinary order statistics, sequential order statistics, record values and Pfeifers record model. Kamps and Cramer [13] derived a second model of gos in which the parameters are pairwise different and we refer to it as case-II of gos. On the other side, the concept of lower gos was given by Pawlas and Szynal [18] and later Burkschat et al. [5] proposed it as $dgos$ to enable a common approach of descending ordered $R.V.$'s like reversed order statistics and lower records models.

A classical measure of uncertainty was launched by Shannon [19] in the information theory literature. The Shannon entropy of a continuous $R.V.$ X measures the average reduction of uncertainty of X. The Shannon entropy for X with pdf $f_X(x)$ is defined as:

$$H(X) = -E(\ln f_X(X)) = -\int_{-\infty}^{\infty} f_X(x)\ln f_X(x)dx. \tag{10}$$

Divergence measures are used to quantify the dissimilarity of two probability distributions. They are equal to zero if and only if the distributions are the same. They are interpreted as distances between probability distributions. Kullback and Leibler [14] considered the Kullback-Leibler divergence (information divergence) for two continuous random variables X_1 and X_2 with pdf's f_1 and f_2, respectively, which is given by:

$$K(X_1, X_2) = \int_{-\infty}^{\infty} f_1(x) \ln\left(\frac{f_1(x)}{f_2(x)}\right) dx, \tag{11}$$

$K(X_1, X_2)$ is non negative, invariant under one-to-one transformation of (X_1, X_2), and it is not symmetric.

Beg and Ahsanullah [4] considered concomitants of generalized order statistics for Morgenstern family and derived the joint distribution of concomitants of two generalized order statistics and obtain their product moments. In this dissertation, we obtain the cdf and pdf of concomitants of ordered $R.V.$'s under Huang and Bairamov extensions. Also, information properties for Huang and Kotz extension are presented. The rest of this article is organized as follows: In Section 2, the pdf and cdf of concomitants for case-I and case-II of gos and $dgos$ under Huang and Bairamov extensions are provided. Section 3, contains information properties such as Shannon entropy and Kullback-Leibler divergence for Huang and Kotz extension. In addition, some examples for some well-known distributions to obtain the entropy are given.

2. Distribution theory for concomitants of ordered *R.V.*'s

In this section, we use case-I and case-II of gos and $dgos$ to obtain the pdf and cdf of concomitants for both Huang and Bairamov extensions. The following theorems deal with this matter. To obtain the cdf of such models, from Equation (1), the conditional cdf of Y given $X = x$, for Huang and Kotz extension, is given by:

$$\begin{aligned} F_{Y|X}(y|x) &= f_X^{-1}(x)\frac{\partial F_{X,Y}(x,y)}{\partial x} \\ &= F_Y(y)[1 + \alpha((1+p)F_X^p(x) - 1)(F_Y^p(y) - 1)]. \end{aligned} \tag{12}$$

From Equation (4), the conditional cdf of Y given $X = x$, for Bairamov extension, is given by:

$$F_{Y|X}^{BA}(y|x) = F_Y(y)[1 + \alpha(1 - F_X^{p_1}(x))^{q_1-1}(1 - (1 + p_1q_1)F_X^{p_1}(x))(1 - F_Y^{p_2}(y))^{q_2}]. \tag{13}$$

We may classify gos and $dgos$ based on $\widetilde{m}$ into the following cases: Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, ..., m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1, 2, ..., n-1\}$, and let $\tilde{m} = (m_1, ..., m_{n-1}) \in \mathbb{R}^{n-1}$.

Case-I of *gos*: If $m_1 = m_2 = \ldots = m_{n-1} = m$, the pdf of rth case-I of gos $X_{(r,n,m,k)}$ can be written as, see Kamps [12]:

$$f_{(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1 - F(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{14}$$

where $c_{r-1} = \prod_{j=1}^{r} \gamma_j$, $g_m(z) = h_m(z) - h_m(0)$, $0 < z < 1$,

$$h_m(z) = \begin{cases} \frac{-(1-z)^{m+1}}{m+1}, & m \neq -1, \\ -\ln(1-z), & m = -1. \end{cases}$$

Case-II of *gos*: If $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n$ and $i \neq j$, the pdf of rth case-II of gos $X_{(r,n,\widetilde{m},k)}$ as follows, see Kamps and Cramer [13]:

$$f_{(r,n,\widetilde{m},k)}(x) = c_{r-1} \sum_{i=1}^{r} a_i(r)\,(1 - F(x))^{\gamma_i - 1} f(x), \tag{15}$$

where $a_i(r) = \prod_{j=1, j\neq i}^{r} \frac{1}{\gamma_j - \gamma_i}$, $1 \leq i \leq r \leq n$ and $\gamma_i = k + n - i + M_i > 0$.
Case-I of *dgos*: When $m_1 = m_2 = \ldots = m_{n-1} = m$, the pdf of rth case-I of $dgos$ $X_{d(r,n,m,k)}$ is defined by, see Pawlas and Szynal [18]:

$$f_{d(r,n,m,k)}(x) \;=\; \frac{c_{r-1}}{(r-1)!}(F(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{16}$$

where $c_{r-1} = \prod_{j=1}^{r} \gamma_j$, $g_m(z) = h_m(z) - h_m(1)$, $0 \leq z < 1$,

$$h_m(z) = \begin{cases} \frac{-1}{m+1} z^{m+1}, & m \neq -1, \\ \\ -\ln z, & m = -1. \end{cases}$$

Case-II of *dgos*: When $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n-1$, in this case, the pdf of rth case-II of $dgos$ $X_{d(r,n,\widetilde{m},k)}$ is defined by, see Athar and Faizan [1]:

$$f_{d(r,n,\widetilde{m},k)}(x) = c_{r-1} \sum_{i=1}^{r} a_i(r)\,(F(x))^{\gamma_i - 1} f(x), \tag{17}$$

where $a_i(r) = \prod_{j=1, j\neq i}^{r} \frac{1}{\gamma_j - \gamma_i}$, $1 \leq i \leq r \leq n$ and $\gamma_i = k + n - i + M_i > 0$.

In the following theorems we use the following notations as follows: Based on Huang and Kotz extension the pdf and cdf of the concomitant of rth case-I of gos (case-I of $dgos$) are $g_{1[r;n,m,k]}$ and $G_{1[r;n,m,k]}$ ($g_{d1[r;n,m,k]}$ and $G_{d1[r;n,m,k]}$), respectively. And the pdf and cdf of the concomitant of rth case-II of gos (case-II of $dgos$) are $g_{1[r;n,\widetilde{m},k]}$ and $G_{1[r;n,\widetilde{m},k]}$ ($g_{d1[r;n,\widetilde{m},k]}$ and $G_{d1[r;n,\widetilde{m},k]}$), respectively. Based on Bairamov extension the pdf and cdf of the concomitant of rth case-I of gos (case-I of $dgos$) are $g_{2[r;n,m,k]}$ and $G_{2[r;n,m,k]}$ ($g_{d2[r;n,m,k]}$ and $G_{d2[r;n,m,k]}$), respectively. And the pdf and cdf of the concomitant of rth case-II of gos (case-II of $dgos$) are $g_{2[r;n,\widetilde{m},k]}$ and $G_{2[r;n,\widetilde{m},k]}$ ($g_{d2[r;n,\widetilde{m},k]}$ and $G_{d2[r;n,\widetilde{m},k]}$), respectively.

Theorem 2.1. *Based on Huang and Kotz extension with pdf given by (2) and cdf given by (1), utilizing (3), (14) and (12), the pdf and cdf of the concomitant of rth case-I of gos, $Y_{[r;n,m,k]}$, are given by, $1 \leq r \leq n$, respectively:*

$$g_{1[r;n,m,k]}(y) = f_Y(y)\left[1 + \alpha T_1^*(r;n,m,k)\left((1+p)F_Y^p(y) - 1\right)\right], \tag{18}$$

$$G_{1[r;n,m,k]}(y) = f_Y(y)\left[1 + \alpha T_1^*(r;n,m,k)\left(F_Y^p(y) - 1\right)\right], \tag{19}$$

where

$$T_1^*(r;n,m,k) = (1+p)c_{r-1} \sum_{j=0}^{p} \binom{p}{j} (-1)^j \frac{1}{\prod_{i=1}^{r}(\gamma_i + j)} - 1, \tag{20}$$

$\gamma_r = k + (n-r)(m+1)$, $n \in \mathbb{N}$, $k \geq 1$, $m_1 = \ldots = m_{n-1} = m \in \mathbb{R}$, $c_{r-1} = \prod_{i=1}^{r} \gamma_i$.

Proof. From (3) and (14), the pdf of the concomitant of r-th case-I of gos, $Y_{[r,n,m,k]}$, is given by:

$$\begin{aligned} g_{1[r,n,m,k]}(y) &= \int_{-\infty}^{\infty} f_{Y|X}(y \mid x) f_{(r,n,m,k)}(x)dx \\ &= f_Y(y) - \alpha f_Y(y)((1+p)F_Y^p(y) - 1) \\ &\quad + \frac{\alpha(1+p)c_{r-1}}{(r-1)!(m+1)^{r-1}} f_Y(y)((1+p)F_Y^p(y) - 1) \\ &\quad \times \int_{-\infty}^{\infty} (1-(1-F_X(x)))^p (1-F_X(x))^{\gamma_r - 1} \\ &\quad \times [1-(1-F_X(x))^{m+1}]^{r-1} f_X(x)dx \\ &= f_Y(y) - \alpha f_Y(y)((1+p)F_Y^p(y) - 1) \\ &\quad + \frac{\alpha(1+p)c_{r-1}}{(r-1)!} f_Y(y)((1+p)F_Y^p(y) - 1) \\ &\quad \times \sum_{j=0}^{p} \left(\begin{array}{c} p \\ j \end{array} \right) (-1)^j \int_{-\infty}^{\infty} (1-F_X(x))^{\gamma_r + j - 1} \\ &\quad \times [\frac{1}{m+1}\{1-(1-F_X(x))^{m+1}\}]^{r-1} f_X(x)dx. \end{aligned} \tag{21}$$

From Beg and Ahsanullah [4], we note that

$$\int_{-\infty}^{\infty} (1-F_X(x))^{\gamma_r+j-1} [\frac{1}{m+1}\{1-(1-F_X(x))^{m+1}\}]^{r-1} f_X(x)dx = \frac{(r-1)!}{\prod_{i=1}^{r}(\gamma_i + j)}, \tag{22}$$

and the result follows. By the same manner we can obtain the cdf of case-I of gos.

Theorem 2.2. *Based on Bairamov extension with pdf given by (5) and cdf given by (4) (with $p_1 = p_2 = 1$), utilizing (7), (14) and (13), the pdf and cdf of the concomitant of rth case-I of gos, $Y_{[r;n,m,k]}$, are given by, $1 \le r \le n$, respectively:*

$$g_{2[r;n,m,k]}(y) = f_Y(y) \left[1 + \alpha R_1^*(r;n,m,k)\,(1-(1+q_2)F_Y(y))\,(1-F_Y(y))^{q_2-1}\right], \tag{23}$$

$$G_{2[r;n,m,k]}(y) = f_Y(y) \left[1 + \alpha R_1^*(r;n,m,k)\,(1-F_Y(y))^{q_2}\right], \tag{24}$$

where

$$R_1^*(r;n,m,k) = c_{r-1} \left\{ \frac{(1+q_1)}{\prod_{i=1}^{r}(\gamma_i + q_1)} - \frac{q_1}{\prod_{i=1}^{r}(\gamma_i + q_1 - 1)} \right\}, \tag{25}$$

$\gamma_r = k + (n-r)(m+1)$, $n \in \mathbb{N}$, $k \ge 1$, $m_1 = ... = m_{n-1} = m \in \mathbb{R}$, $c_{r-1} = \prod_{i=1}^{r} \gamma_i$.

Theorem 2.3. *Based on Huang and Kotz extension with pdf given by (2) and cdf given by (1), utilizing (3), (15) and (12), the pdf and cdf of the concomitant of rth case-II of gos, $Y_{[r;n,\widetilde{m},k]}$, are given by, $1 \le r \le n$, respectively:*

$$g_{1[r;n,\widetilde{m},k]}(y) = f_Y(y) \left[1 + \alpha B_1^*(r;n,\widetilde{m},k) \left((1+p)F_Y^p(y) - 1\right)\right], \tag{26}$$

$$G_{1[r;n,\widetilde{m},k]}(y) = f_Y(y) \left[1 + \alpha B_1^*(r;n,\widetilde{m},k) \left(F_Y^p(y) - 1\right)\right], \tag{27}$$

where

$$B_1^*(r;n,\widetilde{m},k) = (1+p)c_{r-1} \sum_{i=0}^{r} a_i(r) \frac{p!(\gamma_i - 1)!}{(\gamma_i + p)!} - 1, \tag{28}$$

$a_i(r) = \prod_{j=1, i \ne j}^{r} \frac{1}{\gamma_j - \gamma_i}, \gamma_j \ne \gamma_i, 1 \le i \le r \le n$, $c_{r-1} = \prod_{j=1}^{r} \gamma_j$.

Theorem 2.4. *Based on Bairamov extension with pdf given by (5) and cdf given by (4) (with $p_1 = p_2 = 1$), utilizing (7), (15) and (13), the pdf and cdf of the concomitant of rth case-II of gos, $Y_{[r;n,\widetilde{m},k]}$, are given by, $1 \leq r \leq n$, respectively:*

$$g_{2[r;n,\widetilde{m},k]}(y) = f_Y(y)\left[1 + \alpha Q_1^*(r;n,\widetilde{m},k)\,(1-(1+q_2)F_Y(y))\,(1-F_Y(y))^{q_2-1}\right], \tag{29}$$

$$G_{2[r;n,\widetilde{m},k]}(y) = f_Y(y)\left[1 + \alpha Q_1^*(r;n,\widetilde{m},k)\,(1-F_Y(y))^{q_2}\right], \tag{30}$$

where

$$Q_1^*(r;n,\widetilde{m},k) = c_{r-1}\sum_{i=0}^{r} a_i(r)\left\{\frac{(1+q_1)}{(\gamma_i+q_1)} - \frac{q_1}{(\gamma_i+q_1-1)}\right\}, \tag{31}$$

$a_i(r) = \prod_{j=1,i\neq j}^{r} \frac{1}{\gamma_j-\gamma_i}, \gamma_j \neq \gamma_i, 1 \leq i \leq r \leq n,\ c_{r-1} = \prod_{j=1}^{r}\gamma_j.$

Theorem 2.5. *Based on Huang and Kotz extension with pdf given by (2) and cdf given by (1), utilizing (3), (16) and (12), the pdf and cdf of the concomitant of rth case-I of dgos, $Y_{d[r;n,m,k]}$, are given by, $1 \leq r \leq n$, respectively:*

$$g_{d1[r;n,m,k]}(y) = f_Y(y)\left[1 + \alpha T_2^*(r;n,m,k)\left((1+p)F_Y^p(y) - 1\right)\right], \tag{32}$$

$$G_{d1[r;n,m,k]}(y) = f_Y(y)\left[1 + \alpha T_2^*(r;n,m,k)\left(F_Y^p(y) - 1\right)\right], \tag{33}$$

where

$$T_2^*(r;n,m,k) = (1+p)c_{r-1}\frac{1}{\prod_{i=1}^{r}(\gamma_i+p)} - 1, \tag{34}$$

$\gamma_r = k + (n-r)(m+1),\ n \in \mathbb{N},\ k \geq 1,\ m_1 = \ldots = m_{n-1} = m \in \mathbb{R},\ c_{r-1} = \prod_{i=1}^{r}\gamma_i.$

Theorem 2.6. *Based on Bairamov extension with pdf given by (5) and cdf given by (4) (with $p_1 = p_2 = 1$), utilizing (7), (16) and (13), the pdf and cdf of the concomitant of rth case-I of dgos, $Y_{d[r;n,m,k]}$, are given by, $1 \leq r \leq n$, respectively:*

$$g_{d2[r;n,m,k]}(y) = f_Y(y)\left[1 + \alpha R_2^*(r;n,m,k)\,(1-(1+q_2)F_Y(y))\,(1-F_Y(y))^{q_2-1}\right], \tag{35}$$

$$G_{d2[r;n,m,k]}(y) = f_Y(y)\left[1 + \alpha R_2^*(r;n,m,k)\,(1-F_Y(y))^{q_2}\right], \tag{36}$$

where

$$R_2^*(r;n,m,k) = c_{r-1}\left\{(1+q_1)\sum_{j=0}^{q_1}\binom{q_1}{j}\frac{(-1)^j}{\prod_{i=1}^{r}(\gamma_i+j)} - q_1\sum_{j=0}^{q_1-1}\binom{q_1-1}{j}\frac{(-1)^j}{\prod_{i=1}^{r}(\gamma_i+j)}\right\}, \tag{37}$$

$\gamma_r = k + (n-r)(m+1),\ n \in \mathbb{N},\ k \geq 1,\ m_1 = \ldots = m_{n-1} = m \in \mathbb{R},\ c_{r-1} = \prod_{i=1}^{r}\gamma_i.$

Theorem 2.7. *Based on Huang and Kotz extension with pdf given by (2) and cdf given by (1), utilizing (3), (17) and (12), the pdf and cdf of the concomitant of rth case-II of dgos, $Y_{d[r;n,\widetilde{m},k]}$, are given by, $1 \leq r \leq n$, respectively:*

$$g_{d1[r;n,\widetilde{m},k]}(y) = f_Y(y)\left[1 + \alpha B_2^*(r;n,\widetilde{m},k)\left((1+p)F_Y^p(y) - 1\right)\right], \tag{38}$$

$$G_{d1[r;n,\widetilde{m},k]}(y) = f_Y(y)\left[1 + \alpha B_2^*(r;n,\widetilde{m},k)\left(F_Y^p(y) - 1\right)\right], \tag{39}$$

where

$$B_2^*(r;n,\widetilde{m},k) = (1+p)c_{r-1}\sum_{i=0}^{r}\frac{a_i(r)}{\gamma_i+p} - 1, \tag{40}$$

$a_i(r) = \prod_{j=1,i\neq j}^{r} \frac{1}{\gamma_j-\gamma_i}, \gamma_j \neq \gamma_i, 1 \leq i \leq r \leq n,\ c_{r-1} = \prod_{j=1}^{r}\gamma_j.$

Theorem 2.8. *Based on Bairamov extension with pdf given by (5) and cdf given by (4) (with $p_1 = p_2 = 1$), utilizing (7), (17) and (13), the pdf and cdf of the concomitant of rth case-II of dgos, $Y_{d[r;n,\widetilde{m},k]}$, are given by, $1 \le r \le n$, respectively:*

$$g_{d2[r;n,\widetilde{m},k]}(y) = f_Y(y)\left[1 + \alpha Q_2^*(r;n,\widetilde{m},k)\,(1-(1+q_2)F_Y(y))\,(1-F_Y(y))^{q_2-1}\right], \tag{41}$$

$$G_{d2[r;n,\widetilde{m},k]}(y) = f_Y(y)\left[1 + \alpha Q_2^*(r;n,\widetilde{m},k)\,(1-F_Y(y))^{q_2}\right], \tag{42}$$

where

$$Q_2^*(r;n,\widetilde{m},k) = c_{r-1}\sum_{i=0}^{r} a_i(r)\left\{(1+q_1)\sum_{j=0}^{q_1}\binom{q_1}{j}\frac{(-1)^j}{\gamma_i+j} - q_1\sum_{j=0}^{q_1-1}\binom{q_1-1}{j}\frac{(-1)^j}{\gamma_i+j}\right\}, \tag{43}$$

$a_i(r) = \prod_{j=1,i\neq j}^{r}\frac{1}{\gamma_j-\gamma_i}, \gamma_j \neq \gamma_i, 1 \le i \le r \le n, c_{r-1} = \prod_{j=1}^{r}\gamma_j$.

Remark 2.1. *By substituting $p = 1$ in Huang and Kotz extension, and $p_1 = p_2 = q_1 = q_2 = 1$ in Bairamov extension, Equations (18) to (43) reduces to the ordinary Morgenstern family, and we obtain the same results that mentioned in Mohie El-Din et al. [16].*

Now, we can generalize the previous models in unified models as follows: based on Huang and Kotz extension, Equations (18), (26), (32) and (38), and Equations (19), (27), (33) and (39) can be combined, respectively, as follows:

$$g^{(1)}_{Y_r^*}(y) = f_Y(y)\left[1 + \alpha M_r^*\left((1+p)F_Y^p(y) - 1\right)\right], \tag{44}$$

$$G^{(1)}_{Y_r^*}(y) = f_Y(y)\left[1 + \alpha M_r^*\left(F_Y^p(y) - 1\right)\right], \tag{45}$$

where

$$Y_r^* = \begin{cases} Y_{[r;n,m,k]}, & for\ case-I\ of\ gos \\ Y_{d[r;n,m,k]}, & for\ case-I\ of\ dgos \\ Y_{[r;n,\widetilde{m},k]}, & for\ case-II\ of\ gos \\ Y_{d[r;n,\widetilde{m},k]}, & for\ case-II\ of\ dgos, \end{cases} \tag{46}$$

$$M_r^* = \begin{cases} T_1^*(r;n,m,k), & for\ case-I\ of\ gos \\ T_2^*(r;n,m,k), & for\ case-I\ of\ dgos \\ B_1^*(r;n,\widetilde{m},k), & for\ case-II\ of\ gos \\ B_2^*(r;n,\widetilde{m},k), & for\ case-II\ of\ dgos, \end{cases} \tag{47}$$

$$g^{(1)}_{Y_r^*}(y) = \begin{cases} g_{1[r;n,m,k]}(y), & pdf\ of\ concomitant\ of\ rth\ case-I\ of\ gos \\ g_{d1[r;n,m,k]}(y), & pdf\ of\ concomitant\ of\ rth\ case-I\ of\ dgos \\ g_{1[r;n,\widetilde{m},k]}(y), & pdf\ of\ concomitant\ of\ rth\ case-II\ of\ gos \\ g_{d1[r;n,\widetilde{m},k]}(y), & pdf\ of\ concomitant\ of\ rth\ case-II\ of\ dgos, \end{cases} \tag{48}$$

$$G^{(1)}_{Y_r^*}(y) = \begin{cases} G_{1[r;n,m,k]}(y), & cdf\ of\ concomitant\ of\ rth\ case-I\ of\ gos \\ G_{d1[r;n,m,k]}(y), & cdf\ of\ concomitant\ of\ rth\ case-I\ of\ dgos \\ G_{1[r;n,\widetilde{m},k]}(y), & cdf\ of\ concomitant\ of\ rth\ case-II\ of\ gos \\ G_{d1[r;n,\widetilde{m},k]}(y), & cdf\ of\ concomitant\ of\ rth\ case-II\ of\ dgos. \end{cases} \tag{49}$$

Based on Bairamov extension, Equations (23), (29), (35) and (41), and Equations (24), (30), (36) and (42) can be combined, respectively, as follows:

$$g^{(2)}_{Y_r^*}(y) = f_Y(y)\left[1 + \alpha Z_r^*\,(1-(1+q_2)F_Y(y))\,(1-F_Y(y))^{q_2-1}\right], \tag{50}$$

$$G^{(2)}_{Y^*_r}(y) = f_Y(y)\left[1 + \alpha Z^*_r(1 - F_Y(y))^{q_2}\right], \tag{51}$$

where

$$Y^*_r = \begin{cases} Y_{[r;n,m,k]}, & for\ case-I\ of\ gos \\ Y_{d[r;n,m,k]}, & for\ case-I\ of\ dgos \\ Y_{[r;n,\widetilde{m},k]}, & for\ case-II\ of\ gos \\ Y_{d[r;n,\widetilde{m},k]}, & for\ case-II\ of\ dgos, \end{cases} \tag{52}$$

$$Z^*_r = \begin{cases} R^*_1(r;n,m,k), & for\ case-I\ of\ gos \\ R^*_2(r;n,m,k), & for\ case-I\ of\ dgos \\ Q^*_1(r;n,\widetilde{m},k), & for\ case-II\ of\ gos \\ Q^*_2(r;n,\widetilde{m},k), & for\ case-II\ of\ dgos. \end{cases} \tag{53}$$

$$g^{(2)}_{Y^*_r}(y) = \begin{cases} g_{2[r;n,m,k]}(y), & pdf\ of\ concomitant\ of\ rth\ case-I\ of\ gos \\ g_{d2[r;n,m,k]}(y), & pdf\ of\ concomitant\ of\ rth\ case-I\ of\ dgos \\ g_{2[r;n,\widetilde{m},k]}(y), & pdf\ of\ concomitant\ of\ rth\ case-II\ of\ gos \\ g_{d2[r;n,\widetilde{m},k]}(y), & pdf\ of\ concomitant\ of\ rth\ case-II\ of\ dgos, \end{cases} \tag{54}$$

$$G^{(2)}_{Y^*_r}(y) = \begin{cases} G_{2[r;n,m,k]}(y), & cdf\ of\ concomitant\ of\ rth\ case-I\ of\ gos \\ G_{d2[r;n,m,k]}(y), & cdf\ of\ concomitant\ of\ rth\ case-I\ of\ dgos \\ G_{2[r;n,\widetilde{m},k]}(y), & cdf\ of\ concomitant\ of\ rth\ case-II\ of\ gos \\ G_{d2[r;n,\widetilde{m},k]}(y), & cdf\ of\ concomitant\ of\ rth\ case-II\ of\ dgos. \end{cases} \tag{55}$$

3. Information properties for concomitants in Huang and Kotz extension

In this section, we derive an analytical expression of entropy and Kullback-Leibler divergence for Y^*_r in Huang and Kotz extension. Also, applying the entropy for some well-known distributions of this model.

Theorem 3.1. *For any absolutely continuous R.V. Y^*_r, which is the concomitant of rth ordered R.V. of Huang and Kotz extension defined in Equation (46), $1 \le r \le n$. From Equations (10) and (44), Y^*_r has entropy $H(Y^*_r)$ iff*

$$H(Y^*_r) = H(Y)[1 - \alpha M^*_r] + W(r,\alpha) - \alpha(1+p)M^*_r \int_{-\infty}^{\infty} F^p_Y(y) f_Y(y) \ln f_Y(y) dy, \tag{56}$$

where

$$W(r,\alpha) = -\ln(1 - \alpha M^*_r) + (1 - \alpha M^*_r)\, {}_2F_1^{(0,1,0,0)}\left(\frac{1}{p}, -1; 1 + \frac{1}{p}; \frac{\alpha M^*_r(1+p)}{\alpha M^*_r - 1}\right), \tag{57}$$

${}_2F_1^{(0,1,0,0)}(a,b;c;z)$ is the derivative of the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ with respect to b, and

$${}_2F_1^{(0,1,0,0)}(a,b;c;z) = \frac{z\,a}{c} F^{2\,1\,2}_{2\,0\,1}\left(\begin{matrix} a+1, b+1; 1; 1, b; \\ 2, c+1; -; b+1; \end{matrix}\ z, z\right), \tag{58}$$

and the general form of it is known as Kampé de Fériet's series (Srivastava and Karlsson [20]).

Proof. From Equations (10) and (44), the Shannon entropy of Huang and Kotz extension is given by:

$$\begin{aligned} H(Y^*_r) &= \int_{-\infty}^{\infty} g^{(1)}_{Y^*_r}(y) \ln[g^{(1)}_{Y^*_r}(y)] dy \\ &= -\int_{-\infty}^{\infty} \alpha M^*_r(1+p) F^p_Y(y) f_Y(y) \ln[f_Y(y)] dy + H(Y)[1 - \alpha M^*_r] \\ &\quad - E_{g^{(1)}_{Y^*_r}(y)}[\ln[1 + \alpha M^*_r((1+p)F^p_Y(y) - 1)]]. \end{aligned} \tag{59}$$

To evaluate $E_{g^{(1)}_{Y^*_r}(y)}[\ln[1+\alpha M^*_r((1+p)F^p_Y(y)-1)]]$. First, we want to find $E_{g^{(1)}_{Y^*_r}(y)}[1+\alpha M^*_r((1+p)F^p_Y(y)-1)]^t$, let

$$\begin{aligned} u(t) &= E_{g^{(1)}_{Y^*_r}(y)}[1+\alpha M^*_r((1+p)F^p_Y(y)-1)]^t \\ &= \int_{-\infty}^{\infty} f_Y(y)[(1-\alpha M^*_r)+(\alpha M^*_r(1+p)F^p_Y(y))]^{t+1}dy \\ &= \sum_{j=0}^{t+1}\left(\begin{array}{c} t+1 \\ j \end{array}\right)(\alpha M^*_r(1+p))^j(1-\alpha M^*_r)^{t+1-j}\frac{1}{jp+1}, \end{aligned} \tag{60}$$

then

$$\begin{aligned} \acute{u}(0) &= E_{g^{(1)}_{Y^*_r}(y)}[\ln[1+\alpha M^*_r((1+p)F^p_Y(y)-1)]] \\ &= \ln(1-\alpha M^*_r)-(1-\alpha M^*_r)\,{}_2F_1^{(0,1,0,0)}\left(\frac{1}{p},-1;1+\frac{1}{p};\frac{\alpha M^*_r(1+p)}{\alpha M^*_r-1}\right), \end{aligned} \tag{61}$$

and the result follows.

In the following examples, we will choose some subfamilies of Huang and Kotz extension when they are exponential, Pareto and power function, and obtain its entropy as an applications of the last theorem.

Example 3.1. *With the cdf of exponential distribution:*

$$F_Y(y) = \quad 1-e^{-c\,y},\ y\geq 0,\ c>0,$$

from Equation (56), we get

$$H(Y^*_r) = W(r,\alpha)-(1-\alpha M^*_r)\,(\ln(c)-1)-\alpha M^*_r(\ln(c)-B_{[1+p]}),$$

where $B_{[n]}=\psi(n+1)-\psi(1)$ and $\psi(.)$ is the digamma function.

Example 3.2. *With the cdf of Pareto distribution:*

$$F_Y(y) = \quad 1-y^{-c},\ y\geq 1,\ c>0,$$

from Equation (56), we get

$$H(Y^*_r) = W(r,\alpha)-(1-\alpha M^*_r)\left(\ln(c)-\frac{1}{c}-1\right)-\frac{\alpha M^*_r(c\ln(c)-(1+c)B_{[1+p]})}{c}.$$

Example 3.3. *With the cdf of power function distribution:*

$$F_Y(y) = \quad y^c,\ 0\leq y\leq 1,\ c>0,$$

from Equation (56), we get

$$H(Y^*_r) = W(r,\alpha)-(1-\alpha M^*_r)\left(\ln(c)+\frac{1}{c}-1\right)-\frac{\alpha M^*_r(c(1+p)\ln(c)+1-c)}{c(1+p)}.$$

3.1. Kullback-Leibler divergence

In this subsection, we obtain Kullback-Leibler divergence between concomitants of *rth* and *sth* ordered *R.V.*'s of Huang and Kotz extension.

Theorem 3.2. *Let Y_r^* and Y_s^* be the concomitants of rth and sth ordered R.V.'s of Huang and Kotz extension. From Equations (11) and (44), the Kullback-Leibler divergence between Y_r^* and Y_s^* is given by:*

$$K(Y_r^*, Y_s^*) = -W(r,\alpha) - \ln(1-\alpha M_s^*) + \alpha M_r^* \, {}_2F_1^{(0,1,0,0)}\left(1+\frac{1}{p}, 0; 2+\frac{1}{p}; \frac{\alpha M_s^*(1+p)}{\alpha M_s^*-1}\right) - \times (\alpha M_r^* - 1) \, {}_2F_1^{(0,1,0,0)}\left(\frac{1}{p}, 0; 1+\frac{1}{p}; \frac{\alpha M_s^*(1+p)}{\alpha M_s^*-1}\right), \tag{62}$$

where $W(r,\alpha)$ is defined in (57), ${}_2F_1^{(0,1,0,0)}(a,b;c;z)$ is defined in (58).

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