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Positive Strongly Decreasing Solutions of Emden-Fowler Type Second-Order Difference Equations with Regularly Varying Coefficients

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Abstract. Positive decreasing solutions of the nonlinear difference equation

$$\Delta(p_n|\Delta x_n|^{\alpha-1}\Delta x_n) = q_n|x_{n+1}|^{\beta-1}x_{n+1}, \quad n \geq 1, \quad \alpha > \beta > 0,$$

are studied under the assumption that p, q are regularly varying sequences. Necessary and sufficient conditions are established for the existence of regularly varying strongly decreasing solutions and it is shown that the asymptotic behavior of all such solutions is governed by a unique formula.

1. Introduction

Consider the nonlinear difference equation of second order

$$\Delta(p_n|\Delta x_n|^{\alpha-1}\Delta x_n) = q_n|x_{n+1}|^{\beta-1}x_{n+1}, \quad n \geq 1, \tag{E}$$

where α and β are positive constants such that $\alpha > \beta$, $p = \{p_n\}$, $q = \{q_n\}$ are positive real sequences and Δ is forward difference operator defined as $\Delta x_n = x_{n+1} - x_n$. In our case, when $\alpha > \beta$, equation (E) is said to be sub-half-linear, while otherwise, for $\alpha = \beta$ or $\alpha < \beta$ equation (E) is called half-linear or super-half-linear, respectively.

By a solution of (E) we mean a not trivial real sequence $x = \{x_n\}$ satisfying (E). A solution x of the equation (E) is called oscillatory if for every $M \in \mathbb{N}$ there exist $m, n \in \mathbb{N}$, $M \leq m < n$ such that $x_m x_n < 0$, otherwise, it is called nonoscillatory. In other words, a solution x is called nonoscillatory if it is eventually positive or eventually negative. It is known that every solution of (E) is nonoscillatory. If $x = \{x_n\}$ is a solution of (E), then clearly $-x = \{-x_n\}$ is also a solution. Thus, in studying nonoscillatory solutions of (E), for the sake of simplicity, we restrict ourself to solutions which are eventually positive. Any such solution $\{x_n\}$ is eventually strongly monotone and belongs to one of the two classes listed below (see [6, Lemma 1]):

$$\begin{aligned} \mathbb{M}^+ &= \{x \text{ solution of } (E) \,|\, \exists\, n_0 \geq 1 \,:\, x_n > 0,\ \Delta x_n > 0,\ \text{for } n \geq n_0\}, \\ \mathbb{M}^- &= \{x \text{ solution of } (E) \,|\, x_n > 0,\ \Delta x_n < 0,\ \text{for } n \geq 1\}. \end{aligned}$$

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It is well-known that the differential equation

$$\left(p(t)|x'|^{\alpha-1}\,x'\right) = q(t)|x|^{\beta-1}\,x, \quad \alpha,\beta > 0\,, \tag{1.1}$$

where p, q are continuous positive functions on $[a,\infty)$, may have a nontrivial solution x, with the property that there exists $T_x < \infty$, such that $x(t) \equiv 0$ on $[T_x,\infty)$. Such a solution is said to be *extinct singular solution* or *singular solution of the first kind*. On the contrary, such solutions of difference equation (E) do not exists. One more difference between differential and difference equations is that for the differential equation (1.1) classes $\mathbb{M}^+$ and $\mathbb{M}^-$ can be empty, while for the difference equation (E), this case cannot occur (see [6] and [1, Section 5.3]). The asymptotic behaviour of nonoscillatory solutions for nonlinear second-order difference equations has been studied in many papers, see, e.g. [2, 3], [6]-[12], [27], [38], the monograph [1] and references therein.

For any solution x of (E) denote by $x^{[1]} = \left\{x_n^{[1]}\right\}$ its quasi-difference $x_n^{[1]} = p_n|\Delta x_n|^{\alpha-1}\Delta x_n$. Thus, under our assumptions, the classes $\mathbb{M}^+$ and $\mathbb{M}^-$ can be a-priori divided into the following subclasses:

$$\begin{aligned}
\mathbb{M}^+_{\infty,\infty} &= \{x \in \mathbb{M}^+ : \lim_n x_n = \infty,\ \lim_n x_n^{[1]} = \infty,\}\,,\\
\mathbb{M}^+_{\infty,l} &= \{x \in \mathbb{M}^+ : \lim_n x_n = \infty,\ \lim_n x_n^{[1]} = l,\ 0 < l < \infty\}\,,\\
\mathbb{M}^+_{k,\infty} &= \{x \in \mathbb{M}^+ : \lim_n x_n = k,\ 0 < k < \infty,\ \lim_n x_n^{[1]} = \infty\}\,,\\
\mathbb{M}^+_{k,l} &= \{x \in \mathbb{M}^+ : \lim_n x_n = k,\ 0 < k < \infty,\ \lim_n x_n^{[1]} = l,\ 0 < l < \infty\}\,,\\
\mathbb{M}^-_{k,l} &= \{x \in \mathbb{M}^- : \lim_n x_n = k,\ 0 < k < \infty,\ \lim_n x_n^{[1]} = -l,\ 0 < l < \infty\}\,,\\
\mathbb{M}^-_{0,l} &= \{x \in \mathbb{M}^- : \lim_n x_n = 0,\ \lim_n x_n^{[1]} = -l,\ 0 < l < \infty\}\\
\mathbb{M}^-_{k,0} &= \{x \in \mathbb{M}^- : \lim_n x_n = k,\ 0 < k < \infty,\ \lim_n x_n^{[1]} = 0\}\,,\\
\mathbb{M}^-_{0,0} &= \{x \in \mathbb{M}^- : \lim_n x_n = 0,\ \lim_n x_n^{[1]} = 0\}\,.
\end{aligned}$$

A solution $x \in \mathbb{M}^+_{\infty,\infty}$ is said to be *strongly increasing* and a solution $x \in \mathbb{M}^-_{0,0}$ is said to be *strongly decreasing* or *strongly decaying.* For solutions which tends to some constant we use $\mathbb{M}^-_B = \mathbb{M}^-_{k,0} \cup \mathbb{M}^-_{k,l}$, $\mathbb{M}^+_B = \mathbb{M}^+_{k,\infty} \cup \mathbb{M}^+_{k,l}$ and for decreasing solutions which tends to zero we use $\mathbb{M}^-_0 = \mathbb{M}^-_{0,l} \cup \mathbb{M}^-_{0,0}$.

Let

$$S = \sum_{n=1}^{\infty} \frac{1}{p_n^{1/\alpha}}.$$

Depending on whether $S = \infty$ or $S < \infty$ some of the above classes may be empty.

(i) If $S = \infty$ then

$$\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \cup \mathbb{M}^+_{\infty,l} \quad \text{and} \quad \mathbb{M}^- = \mathbb{M}^-_{k,0} \cup \mathbb{M}^-_{0,0}, \quad \text{i.e.} \quad \mathbb{M}^+_B = \emptyset\,,\ \mathbb{M}^-_{0,l} \cup \mathbb{M}^-_{k,l} = \emptyset.$$

(ii) If $S < \infty$ then

$$\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \cup \mathbb{M}^+_B \quad \text{and} \quad \mathbb{M}^- = \mathbb{M}^-_0 \cup \mathbb{M}^-_B, \quad \text{i.e.} \quad \mathbb{M}^+_{\infty,l} = \emptyset.$$

In this paper, we consider only positive decreasing solutions, i.e. solutions in $\mathbb{M}^-$. Concerning the existence of solutions in the classes $\mathbb{M}^-_B$ and $\mathbb{M}^-_{0,l}$, the following holds.

Theorem 1.1. *(i) Equation (E) has solutions in $\mathbb{M}^-_B$ if and only if*

$$I = \sum_{n=1}^{\infty} \left(\frac{1}{p_n} \sum_{k=n}^{\infty} q_k\right)^{\frac{1}{\alpha}} < \infty.$$

(ii) Equation (E) has solutions in $\mathbb{M}^-_{0,l}$ *if and only if*

$$J = \sum_{n=1}^{\infty} q_n \left(\sum_{k=n}^{\infty} \frac{1}{p_{k+1}^{1/\alpha}} \right)^{\beta} < \infty.$$

The assertion *(i)* follows from [6, Theorem 2 and Theorem 5-(a)], while the assertion *(ii)* follows from [8, Theorem 2.2 and Theorem 3.1] and [27, Theorem 9].

As regards to the existence of strongly decreasing solutions, it is an open problem. The existence of strongly decreasing solutions in the continuous case, that is for the differential equation (1.1), can be proved as in [37] with the help of fixed point theory by proving that the operator

$$(\mathcal{F}x)(t) = \int_t^{\infty} \left(\frac{1}{p(s)} \int_s^{\infty} q(r)x(r)^{\beta}\, dr \right)^{\frac{1}{\alpha}} ds$$

has a nonzero fixed point. To this end the operator $\mathcal{F}$ acts on the set

$$\Omega = \{x \in C[t_0, \infty] : z(t) \le x(t) \le z(t_0),\ t \ge t_0\},$$

where z is a singular solution of the first kind of (1.1). The second approach, due to [34], is to construct the sequence $\{x_n\}$ of asymptotically constant solutions of differential equation (1.1), having the limit function x, and it gives rise to a positive strongly decreasing solution of (1.1). This approach, however, requires lower bound for such a sequence of solutions, which is again given by a singular solution of the first kind of (1.1). Clearly, due to the nonexistence of singular solutions in the discrete case, neither of these two approaches work.

The recent development of asymptotic analysis of ordinary differential equations by means of regularly varying functions (see [17]-[19],[23]-[26], [29], [33]-[35] and monograph [28] for results up to 2000.), suggests to investigate the discrete problem of the existence of strongly decreasing solutions in the framework of regularly varying sequences. The aim of this paper is twofold. We will determine conditions for the existence of strongly decreasing solutions and give an explicit asymptotic formula for those solutions.

The theory of regularly varying sequences, sometimes called Karamata sequences, was initiated in 1930 by Karamata [22] and further developed in the seventies by Galambos, Seneta and Bojanić in [5, 16] and recently in [14, 15]. However, until the papers of Matucci and Rehak [30, 31], the relation between regularly varying sequences and difference equations has never been discussed. In these two papers, as well as in succeeding papers [32, 36], the theory of regularly varying sequences has been further developed and applied in the asymptotic analysis of second-order linear and half-linear difference equations, providing necessary and sufficient conditions for the existence of regularly varying solutions of these equations. Afterward, further development of discrete regularly varying theory and application to second-order nonlinear difference equations of Emden-Fowler type was done by Agarwal and Manojlović in [3], Kapešić and Manojlović in [21] and Kapešić in [20]. Actually, in [21], Kapešić and Manojlović gave necessary and sufficient conditions for the existence of strongly increasing regularly varying solutions of (E) and obtained a precise asymptotic representation of such solutions. Thus, the purpose of this paper is to proceed further in this direction and to establish results which can be considered as a discrete analog of results in the continuous case (see e.g.[17, 25, 29]).

Throughout this paper, symbol $\sim$ is used to denote the asymptotic equivalence of two positive sequences, i.e.

$$x_n \sim y_n,\ n \to \infty \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{y_n}{x_n} = 1.$$

Our main tools are, besides the theory of regularly varying sequences presented in Section 2, the fixed point technique and Stolz-Cezaro theorem. Thus, we recall two variants of Stolz-Cezaro theorem as well as Knaster-Tarski fixed point theorem [1, Theorem 5.2.1].

Lemma 1.2. *If $f = \{f_n\}$ is a strictly increasing sequence of positive real numbers, such that $\lim_{n\to\infty} f_n = \infty$, then for any sequence $g = \{g_n\}$ of positive real numbers one has the inequalities:*

$$\liminf_{n\to\infty} \frac{\Delta f_n}{\Delta g_n} \le \liminf_{n\to\infty} \frac{f_n}{g_n} \le \limsup_{n\to\infty} \frac{f_n}{g_n} \le \limsup_{n\to\infty} \frac{\Delta f_n}{\Delta g_n}\,.$$

In particular, if the sequence $\{\Delta f_n/\Delta g_n\}$ has a limit, then

$$\lim_{n\to\infty} \frac{f_n}{g_n} = \lim_{n\to\infty} \frac{\Delta f_n}{\Delta g_n}\,. \tag{1.2}$$

Lemma 1.3. *Let $f = \{f_n\}$, $g = \{g_n\}$ be sequences of positive real numbers, such that*
(i) $\lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n = 0$;
(ii) the sequence g is strictly monotone;
(iii) the sequence $\{\Delta f_n/\Delta g_n\}$ has a limit.
Then, a sequence $\{f_n/g_n\}$ is convergent and (1.2) *holds.*

Lemma 1.4. (Knaster-Tarski fixed point theorem) *Let X be a partially ordered Banach space with ordering $\le$. Let M be a subset of X with the following properties: The infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $\mathcal{F} : M \to M$ be an increasing mapping, i.e. $x \ge y$ implies $\mathcal{F}x \ge \mathcal{F}y$. Then $\mathcal{F}$ has a fixed point in M.*

2. Regularly Varying Sequences

We state here definitions and some basic properties of regularly varying sequences which will be essential in establishing our main results on the asymptotic behavior of nonoscillatory solutions stated and proved in the next section. For a comprehensive treatise on regular variation, the reader is referred to Bingham et al. [4].

Two main approaches are known in the basic theory of regularly varying sequences: the approach due to Karamata [22], based on a definition that can be understood as a direct discrete counterpart of elegant and straightforward continuous definition (see Definition 2.2), and the approach due to Galambos and Seneta, based on purely sequential definition.

Definition 2.1. (Karamata [22]) A positive sequence $y = \{y_k\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{k\to\infty} \frac{y_{[\lambda k]}}{y_k} = \lambda^\rho \quad \textit{for} \quad \forall \lambda > 0,$$

where $[u]$ denotes the integer part of u.

Definition 2.2. A measurable function $f : (a,\infty) \to (0,\infty)$ for some $a > 0$ is said to be *regularly varying at infinity of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \textit{for all } \lambda > 0.$$

Definition 2.3. (Galambos and Seneta [16]) A positive sequence $y = \{y_k\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if there exists a positive sequence $\{\alpha_k\}$ satisfying

$$\lim_{k\to\infty} \frac{y_k}{\alpha_k} = C,\; 0 < C < \infty \qquad \lim_{k\to\infty} k\,\frac{\Delta\alpha_{k-1}}{\alpha_k} = \rho\,.$$

If $\rho = 0$, then y is said to be *slowly varying*.

The totality of regularly varying sequences of index ρ and slowly varying sequences will be denoted, respectively, by $\mathcal{RV}(\rho)$ and $\mathcal{SV}$.

Bojanić and Seneta have shown in [5] that Definition 2.1 and Definition 2.3 are equivalent.

The concept of normalized regularly varying sequences was introduced by Matucci and Rehak in [30]:

Definition 2.4. A positive sequence $y = \{y_k\}, k \in \mathbb{N}$ is said to be *normalized regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{k\to\infty} \frac{k\Delta y_k}{y_k} = \rho.$$

If $\rho = 0$, then y is called a *normalized slowly varying sequence.*

In what follows, $\mathcal{NRV}(\rho)$ and $\mathcal{NSV}$ will be used to denote the set of all normalized regularly varying sequences of index ρ and the set of all normalized slowly varying sequences. Typical examples are:

$$\{\log k\} \in \mathcal{NSV}, \quad \{k^\rho \log k\} \in \mathcal{NRV}(\rho), \quad \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}.$$

There exist various necessary and sufficient conditions for a sequence of positive numbers to be regularly varying (see [5, 16, 30, 31]), and consequently, each one of them could be used to define a regularly varying sequence. The one that is the most important is the following Representation theorem (see [5, Theorem 3]), while some other representation formula for regularly varying sequences were established in [31, Lemma 1].

Theorem 2.5. (Representation theorem) *A positive sequence* $\{y_k\}, k \in \mathbb{N}$ *is said to be regularly varying of index* $\rho \in \mathbb{R}$ *if and only if there exist sequences* $\{c_k\}$ *and* $\{\delta_k\}$ *such that*

$$\lim_{k\to\infty} c_k = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{k\to\infty} \delta_k = 0,$$

and

$$y_k = c_k\, k^\rho \, \exp\left(\sum_{i=1}^{k} \frac{\delta_i}{i}\right).$$

In [5] very useful embedding theorem was proved, which gives the possibility of using the continuous theory in developing a theory of regularly varying sequences. However, as noticed in [5], such development is not generally close and sometimes far from a simple imitation of arguments for regularly varying functions.

Theorem 2.6. (Embedding Theorem) *If* $y = \{y_n\}$ *is a regularly varying sequence of index* $\rho \in \mathbb{R}$, *then function* $Y(t)$ *defined on* $[0,\infty)$ *by* $Y(t) = y_{[t]}$ *is a regularly varying function of index* ρ. *Conversely, if* $Y(t)$ *is a regularly varying function on* $[0,\infty)$ *of index* ρ, *then a sequence* $\{y_k\}$, $y_k = Y(k)$, $k \in \mathbb{N}$ *is regularly varying of index* ρ.

Next, we state some important properties of $\mathcal{RV}$ sequences, useful for the development of asymptotic behavior of solutions of (E) in the subsequent section (for more properties and proofs see [5, 30]).

Theorem 2.7. *(i)* $y \in \mathcal{RV}(\rho)$ *if and only if* $y_k = k^\rho\, l_k$, *where* $l = \{l_k\} \in \mathcal{SV}$.

(ii) Let $x \in \mathcal{RV}(\rho_1)$ *and* $y \in \mathcal{RV}(\rho_2)$. *Then,* $xy \in \mathcal{RV}(\rho_1 + \rho_2)$, $x + y \in \mathcal{RV}(\rho)$, $\rho = \max\{\rho_1, \rho_2\}$ *and* $1/x \in \mathcal{RV}(-\rho_1)$.

(iii) If $y \in \mathcal{RV}(\rho)$, *then* $\lim_{k\to\infty} \dfrac{y_{k+1}}{y_k} = 1$.

(iv) If $l \in \mathcal{SV}$ *and* $l_k \sim L_k$, $k \to \infty$, *then* $L \in \mathcal{SV}$.

(v) If $y \in \mathcal{RV}(\rho)$, *then* $\{n^{-\sigma} y_n\}$ *is eventually increasing for each* $\sigma < \rho$ *and* $\{n^{-\mu} y_n\}$ *is eventually decreasing for each* $\mu > \rho$.

(vi) Let $l \in \mathcal{SV}$. Then, $\lim_{n\to\infty} n^\rho\, l_n = 0$ *if* $\rho < 0$ *and* $\lim_{n\to\infty} n^\rho\, l_n = \infty$ *if* $\rho > 0$.

In view of the statement (*i*) of the previous theorem, if for $y \in \mathcal{RV}(\rho)$

$$\lim_{k\to\infty} \frac{y_k}{k^\rho} = \lim_{k\to\infty} l_k = \text{const} > 0,$$

then $y = \{y_n\}$ is said to be a *trivial regularly varying sequence of index* ρ and is denoted by $y \in tr - \mathcal{RV}(\rho)$. Otherwise y is said to be a *nontrivial regularly varying sequence of index* ρ, denoted by $y \in ntr - \mathcal{RV}(\rho)$.

Next theorem can be found in [3] for normalized regularly varying sequences, but it apparently holds for all regularly varying sequences.

Theorem 2.8. *If $f = \{f_n\} \in \mathcal{RV}$ is a strictly decreasing sequence, such that* $\lim_{n\to\infty} f_n = 0$, *then for each* $\gamma \in \mathbb{R}$

$$\lim_{n\to\infty} f_n^{-\gamma} \sum_{k=n}^{\infty} f_k^{\gamma-1}\Big(-\Delta f_k\Big) = \frac{1}{\gamma}.$$

If $g = \{g_n\} \in \mathcal{RV}$ is a strictly increasing sequence such that $\lim_{n\to\infty} g_n = \infty$, *then*

$$\lim_{n\to\infty} g_n^{-\gamma} \sum_{k=1}^{n-1} g_k^{\gamma-1} \Delta g_k = \frac{1}{\gamma}.$$

The following theorem can be seen as *the discrete analog of the Karamata's integration theorem* and plays a central role in the proof of our main results in Section 3. The proof of this theorem can be found in [5], [21] and [36].

Theorem 2.9. *Let $l = \{l_n\} \in \mathcal{SV}$.*

$$(i)\quad \text{If } \alpha > -1, \quad \text{then} \quad \lim_{n\to\infty} \frac{1}{n^{\alpha+1} l_n} \sum_{k=1}^{n} k^\alpha l_k = \frac{1}{1+\alpha};$$

$$(ii)\quad \text{If } \alpha < -1, \quad \text{then} \quad \lim_{n\to\infty} \frac{1}{n^{\alpha+1} l_n} \sum_{k=n}^{\infty} k^\alpha l_k = -\frac{1}{1+\alpha};$$

$$(iii)\quad \text{If } \sum_{k=1}^{\infty} \frac{l_k}{k} < \infty, \quad \text{then} \quad \sum_{k=n}^{\infty} \frac{l_k}{k} \in \mathcal{SV} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{l_n} \sum_{k=n}^{\infty} \frac{l_k}{k} = \infty;$$

$$(iv)\quad \text{If } \sum_{k=1}^{\infty} \frac{l_k}{k} = \infty, \quad \text{then} \quad \sum_{k=1}^{n} \frac{l_k}{k} \in \mathcal{SV} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{l_n} \sum_{k=1}^{n} \frac{l_k}{k} = \infty\,.$$

Remark 2.10. In view of Theorem 2.7-(iii) and Theorem 2.9-(i), it is easy to see, that for $l \in \mathcal{SV}$, if $\alpha > -1$, then we have

$$\sum_{k=1}^{n-1} k^\alpha l_k \sim \frac{(n-1)^{\alpha+1} l_{n-1}}{\alpha+1} \sim \frac{n^{\alpha+1} l_n}{\alpha+1} \sim \sum_{k=1}^{n} k^\alpha l_k, \qquad n \to \infty.$$

3. Main results

In this section we assume that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and use expressions

$$p_n = n^\eta \xi_n \quad q_n = n^\sigma \omega_n, \quad \xi = \{\xi_n\},\ \omega = \{\omega_n\} \in \mathcal{SV}\,, \tag{3.1}$$

considering strongly decreasing $\mathcal{RV}$−solutions expressed as

$$x_n = n^\rho l_n, \quad l = \{l_n\} \in \mathcal{SV}\,. \tag{3.2}$$

Moreover, we assume that $\eta \neq \alpha$ and distinguish two mutually exclusive cases:

(i) $\eta < \alpha$ implying that $S = \infty$; and (ii) $\eta > \alpha$ implying that $S < \infty$.

Case (i): It is clear that for any strongly decreasing solution of (E) it holds that $x_n \leq c$, for large n. Thus, we have that the index of regularity ρ of strongly decreasing $\mathcal{RV}-$solution x must satisfy $\rho \leq 0$. If $\rho = 0$ then $l_n = x_n \to 0$, so x is a member of $ntr - \mathcal{SV}$.

Case (ii): Using (3.1) and Theorem 2.9 we have

$$\pi_n = \sum_{k=n}^{\infty} \frac{1}{p_k^{1/\alpha}} = \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}} \xi_k^{-\frac{1}{\alpha}} \sim \frac{\alpha}{\eta - \alpha} n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}} = \frac{\alpha}{\eta - \alpha} \cdot \frac{n}{p_n^{1/\alpha}}, \quad n \to \infty, \tag{3.3}$$

so that $\{\pi_n\} \in \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. For any strongly decreasing solution x of (E), by application of Lemma 1.3, we have that

$$\lim_{n\to\infty} \frac{x_n}{\pi_n} = \lim_{n\to\infty} \frac{\Delta x_n}{-\frac{1}{p_n^{\frac{1}{\alpha}}}} = \lim_{n\to\infty} (x_n{}^{[1]})^{\frac{1}{\alpha}} = 0\,,$$

implying that the index of regularity ρ of strongly decreasing solutions must satisfy $\rho \leq \frac{\alpha-\eta}{\alpha}$.

If $\eta < \alpha$, the totality of strongly decreasing $\mathcal{RV}-$solutions will be divided into the following two classes

$$ntr - \mathcal{SV} \quad \text{or} \quad \mathcal{RV}(\rho) \;\; \text{with} \;\; \rho < 0,$$

while, if $\eta > \alpha$, the totality of strongly decreasing $\mathcal{RV}-$solutions of (E) will be divided into the following two subclasses:

$$\mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right) \quad \text{or} \quad \mathcal{RV}(\rho) \;\; \text{with} \;\; \rho < \frac{\alpha-\eta}{\alpha}.$$

Our purpose is to show that all solutions in each of this four subclasses of strongly decreasing $\mathcal{RV}-$solutions of (E) enjoy one and the same asymptotic behavior as $n \to \infty$, whereby the regularity index of such a solution is uniquely determined by α, β and the regularity indices η, σ of coefficients p, q. Moreover, necessary and sufficient conditions for the existence of solutions belonging to these four subclasses of strongly decreasing $\mathcal{RV}-$solutions will be established.

3.1. *Existence of strongly decreasing solutions*

Conditions for the existence of a strongly decreasing solution of differential equation (1.1) is given by the following theorem:

Theorem 3.1. (i) *Let* $\int_a^\infty p(t)^{-\frac{1}{\alpha}}\, dt = \infty,\ a \geq 0$. *If*

$$\int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty q(s)ds\right)^{\frac{1}{\alpha}} dt < \infty,$$

then equation (1.1) *has a strongly decreasing solution.*

(ii) *Let* $\int_a^\infty p(t)^{-\frac{1}{\alpha}}\, dt < \infty,\ a \geq 0$. *If*

$$\int_a^\infty q(t) \left(\int_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}}}\right)^{\beta} dt < \infty,$$

then equation (1.1) *has a strongly decreasing solution.*

The proof of the previous theorem can be found in [13, 34] for (*i*) and in [37] for (*ii*). It is expected that for the discrete version of Theorem 3.1, the existence of strongly decreasing solution is characterized by the assumption $I < \infty$ if $S = \infty$ and by the assumption $J < \infty$ if $S < \infty$. In fact, we prove

Theorem 3.2. *Suppose that* $p \in \mathcal{RV}(\eta)$ *and* $q \in \mathcal{RV}(\sigma)$.
(i) Let $\eta < \alpha$. *If* $I < \infty$, *then* $\mathbb{M}^-_{0,0} \neq \emptyset$.
(ii) Let $\eta > \alpha$. *If* $J < \infty$, *then* $\mathbb{M}^-_{0,0} \neq \emptyset$.

First of all, let's notice that if $\eta < \alpha$, then $\sigma < -1$ is a necessary condition for $I < \infty$. Then, using discrete Karamata theorem, (3.1) and (3.3), we have

$$\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}} \sim \frac{1}{(-(\sigma+1))^{\frac{1}{\alpha}}}\left(\frac{k^{\sigma+1-\eta}\omega_k}{\xi_k}\right)^{\frac{1}{\alpha}}, \quad k \to \infty .$$

On the other hand, if $\eta > \alpha$ application of discrete Karamata theorem gives

$$q_k\left(\sum_{j=k}^{\infty} \frac{1}{p_j^{1/\alpha}}\right)^{\beta} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta} k^{\sigma+\beta-\frac{\beta}{\alpha}\eta}\,\frac{\omega_k}{\xi_k^{\beta/\alpha}}, \ \ k \to \infty.$$

Consequently,

(i) for $\eta < \alpha$, $I < \infty$ if and only if

$$\sigma < \eta - \alpha - 1 \tag{3.4}$$

or

$$\sigma = \eta - \alpha - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1}\left(\frac{\omega_k}{\xi_k}\right)^{\frac{1}{\alpha}} < \infty ; \tag{3.5}$$

(ii) for $\eta > \alpha$, $J < \infty$ if and only if

$$\sigma < \frac{\beta\eta}{\alpha} - \beta - 1 \tag{3.6}$$

or

$$\sigma = \frac{\beta\eta}{\alpha} - \beta - 1 \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-1}\frac{\omega_k}{\xi_k^{\beta/\alpha}} < \infty . \tag{3.7}$$

Taking into account the previous consideration, Theorem 3.2 will be proved by considering the above four cases.

Theorem 3.3. *Suppose that* $p \in \mathcal{RV}(\eta)$ *and* $q \in \mathcal{RV}(\sigma)$.
(i) Let $\eta < \alpha$. *If* (3.4) *holds, then equation* (*E*) *possesses a solution* $x \in \mathbb{M}^-_{0,0}$.
(ii) Let $\eta > \alpha$. *If* (3.6) *holds, then equation* (*E*) *possesses a solution* $x \in \mathbb{M}^-_{0,0}$.

Proof. Suppose either $\eta < \alpha$ and (3.4) holds or $\eta > \alpha$ and (3.6) holds. Denote

$$X_n = \left[\frac{n^{\alpha+1}p_n^{-1}q_n}{(-\rho)^{\alpha}(\alpha-\eta-\rho\alpha)}\right]^{\frac{1}{\alpha-\beta}}, \qquad n \geq 1, \tag{3.8}$$

and $\lambda = (-\rho)^{\alpha}(\alpha-\eta-\rho\alpha)$, where ρ is given by

$$\rho = \frac{\sigma+\alpha+1-\eta}{\alpha-\beta}. \tag{3.9}$$

Clearly, $X = \{X_n\} \in \mathcal{RV}(\rho)$ and it may be expressed in the form

$$X_n = \lambda^{-\frac{1}{\alpha-\beta}} n^\rho \left(\frac{\omega_n}{\xi_n}\right)^{\frac{1}{\alpha-\beta}} . \tag{3.10}$$

Notice that (3.4) and (3.9) imply that $\rho < 0$, while (3.6) and (3.9) imply that $\rho < \frac{\alpha-\eta}{\alpha}$, so that by Theorem 2.7-(v),(vi), $X_n \to 0$ as $n \to \infty$ and $\{X_n\}$ is eventually decreasing, in both cases (*i*) and (*ii*).

Let us first prove that the sequence X satisfies the asymptotic relation

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} \sim X_n, \qquad n \to \infty. \tag{3.11}$$

Using (3.1), by application of Theorem 2.9-(ii) and Theorem 2.7-(iii), we get

$$\begin{aligned} \sum_{k=n}^{\infty} q_k X_{k+1}^{\beta} &\sim \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \xi_k^{-\frac{\beta}{\alpha-\beta}} \omega_k^{\frac{\alpha}{\alpha-\beta}} = \lambda^{-\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{\alpha(\rho-1)+\eta-1} \xi_k^{-\frac{\beta}{\alpha-\beta}} \omega_k^{\frac{\alpha}{\alpha-\beta}} \\ &\sim \lambda^{-\frac{\beta}{\alpha-\beta}} \frac{n^{\alpha(\rho-1)+\eta} \xi_n^{-\frac{\beta}{\alpha-\beta}} \omega_n^{\frac{\alpha}{\alpha-\beta}}}{-(\alpha(\rho-1)+\eta)}, \quad n \to \infty. \end{aligned} \tag{3.12}$$

Notice that $\alpha(\rho-1)+\eta < 0$ in both cases (*i*) and (*ii*). From (3.12), applying Theorem 2.9-(ii), we obtain the desired asymptotic relation for X:

$$\begin{aligned} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} &\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} (\alpha(1-\rho)-\eta)^{-\frac{1}{\alpha}} \sum_{k=n}^{\infty} k^{\rho-1} \xi_k^{-\frac{1}{\alpha-\beta}} \omega_k^{\frac{1}{\alpha-\beta}} \\ &\sim \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} (\alpha(1-\rho)-\eta)^{-\frac{1}{\alpha}} \frac{n^\rho \xi_n^{-\frac{1}{\alpha-\beta}} \omega_n^{\frac{1}{\alpha-\beta}}}{-\rho} \\ &= \lambda^{-\frac{\beta}{\alpha(\alpha-\beta)}} \cdot \lambda^{-\frac{1}{\alpha}} n^\rho \xi_n^{-\frac{1}{\alpha-\beta}} \omega_n^{\frac{1}{\alpha-\beta}} = X_n, \qquad n \to \infty . \end{aligned}$$

Thus, there exists $n_0 > 1$ such that

$$X_{n+1} \le X_n \ \text{ and } \ \frac{1}{2} X_n \le \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} \le 2X_n, \quad \text{for } n \ge n_0 . \tag{3.13}$$

Let such n_0 be fixed. We choose constants $\kappa \in (0,1)$ and $K > 1$ such that

$$\kappa^{1-\frac{\beta}{\alpha}} \le \frac{1}{2} \qquad \text{and} \quad K^{1-\frac{\beta}{\alpha}} \ge 2 . \tag{3.14}$$

Consider the space Υ_{n_0} of all real sequences $x = \{x_n\}_{n=n_0}^{\infty}$ such that x_n/X_n is bounded for $n \ge n_0$. Then, Υ_{n_0} is a Banach space, endowed with the norm

$$\|x\| = \sup_{n \ge n_0} \frac{x_n}{X_n} .$$

Further, Υ_{n_0} is partially ordered, with the usual pointwise ordering $\le$: for $x, y \in \Upsilon_{n_0}$, $x \le y$ means $x_n \le y_n$ for all $n \ge n_0$. Define the subset $\mathcal{X} \subset \Upsilon_{n_0}$ by

$$\mathcal{X} = \{x \in \Upsilon_{n_0} : \kappa X_n \le x_n \le K X_n,\ n \ge n_0 \}. \tag{3.15}$$

For any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. Next, define the operator $\mathcal{F} : \mathcal{X} \to \Upsilon_{n_0}$ by

$$\left(\mathcal{F} x\right)_n = \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j x_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} , \quad n \ge n_0, \tag{3.16}$$

and show that $\mathcal{F}$ has a fixed point by using Lemma 1.4. Namely, the operator $\mathcal{F}$ has the following properties:

(i) *Operator $\mathcal{F}$ maps $\mathcal{X}$ into itself*: Let $x \in \mathcal{X}$. Using (3.13), (3.14), (3.15) and (3.16), we get

$$(\mathcal{F}x)_n \leq K^{\frac{\beta}{\alpha}} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} \leq 2K^{\frac{\beta}{\alpha}} X_n \leq K\, X_n, \quad n \geq n_0\,.$$

and

$$(\mathcal{F}x)_n \geq \kappa^{\frac{\beta}{\alpha}} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j X_{j+1}^{\beta} \right)^{\frac{1}{\alpha}} \geq \kappa^{\frac{\beta}{\alpha}} \frac{X_n}{2} \geq \kappa X_n, \quad n \geq n_0\,.$$

This shows that $(\mathcal{F}x)_n \in \mathcal{X}$, for all $n \geq n_0$, that is, $\mathcal{F}(\mathcal{X}) \subset \mathcal{X}$.

(ii) *Operator $\mathcal{F}$ is increasing*, i.e. for any $x, y \in \mathcal{X}$, $x \leq y$ implies $\mathcal{F}x \leq \mathcal{F}y$.

Thus all the hypotheses of Lemma 1.4 are fulfilled implying the existence of a fixed point $x \in \mathcal{X}$ of $\mathcal{F}$, satisfying

$$x_n = \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j x_{j+1}^{\beta} \right)^{\frac{1}{\alpha}}, \qquad n \geq n_0\,. \tag{3.17}$$

It is clear in view of (3.15) and the fact that $X_n \to 0$, $n \to \infty$, that x is a positive solution of (E) which satisfies $x_n \to 0$, $n \to \infty$. Moreover, due to (3.10), (3.13) and (3.15), we have

$$p_n(-\Delta x_n)^{\alpha} \leq K^{\beta} \sum_{k=n}^{\infty} q_k X_{k+1}^{\beta} \leq m \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} f_k, \tag{3.18}$$

where

$$f_k = \left(\frac{{\omega_k}^{\alpha}}{{\xi_k}^{\beta}} \right)^{\frac{1}{\alpha-\beta}}, \quad f = \{f_k\} \in \mathcal{SV} \quad \text{and} \quad m = K^{\beta} \lambda^{-\frac{\beta}{\alpha-\beta}}.$$

Since, $\eta < \alpha$ and (3.4) as well as $\eta > \alpha$ and (3.6) imply that $\sigma + \rho\beta < -1$, from (3.18) we conclude that $x_n^{[1]} \to 0$, $n \to \infty$, that is $x \in \mathbb{M}_{0,0}^{-}$. □

Theorem 3.4. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. If* (3.5) *holds, then there exists $x \in \mathbb{M}_{0,0}^{-}$.*

Proof. Suppose (3.5) holds. Define sequences $T = \{T_n\}$ and $G = \{G_n\}$ by

$$G_n = \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}}, \quad T_n = \left(\frac{\alpha-\beta}{\alpha} \sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j \right)^{\frac{1}{\alpha}} \right)^{\frac{\alpha}{\alpha-\beta}}, \quad n \geq 1\,. \tag{3.19}$$

Since the first condition from (3.5) implies $\sigma < -1$, application of Theorem 2.9 gives

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j \right)^{\frac{1}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}} \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}}, \quad n \to \infty,$$

so that

$$T_n \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha} \right)^{\frac{\alpha}{\alpha-\beta}} G_n^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty$$

Clearly, $G \in \mathcal{SV}$ and $T \in \mathcal{SV}$. Applying Theorem 2.9-(ii) and using the first condition from (3.5) we get

$$\sum_{k=n}^{\infty} q_k T_{k+1}^{\beta} \sim \frac{1}{(\alpha-\eta)^{\frac{\alpha}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} n^{\eta-\alpha} \omega_n G_n^{\frac{\alpha\beta}{\alpha-\beta}},\ n \to \infty.$$

Thus, by Theorem 2.8, the previous relation gives

$$\begin{aligned}
\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j T_{j+1}^{\beta}\right)^{\frac{1}{\alpha}} &\sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}} G_k^{\frac{\beta}{\alpha-\beta}} \\
&\sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} (-\Delta G_k) \cdot G_k^{\frac{\beta}{\alpha-\beta}} \\
&\sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} G_n^{\frac{\alpha}{\alpha-\beta}} \sim T_n, \quad n \to \infty.
\end{aligned}$$

Consequently, we conclude that T satisfies the asymptotic relation (3.11).

The rest of the proof is the same as the proof of Theorem 3.3 where X_n is replaced with T_n. Then, a solution x of the equation (E) satisfying $\kappa T_n \le x_n \le K T_n$, for large n, is obtained by the application of Knaster-Tarski fixed point theorem and belongs to the class $\mathbb{M}_{0,0}^{-}$. □

Theorem 3.5. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. If (3.7) holds, then there exists $x \in \mathbb{M}_{0,0}^{-}$.*

Proof. Suppose (3.7) holds. Using (3.1) and the assumption (3.7), we have that

$$\sum_{k=1}^{\infty} q_k \left(\sum_{j=k}^{\infty} \frac{1}{p_k^{1/\alpha}}\right)^{\beta} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta} \sum_{k=1}^{\infty} k^{\beta} q_k p_k^{-\frac{\beta}{\alpha}} = \left(\frac{\alpha}{\eta-\alpha}\right)^{\beta} \sum_{k=1}^{\infty} k^{-1} \omega_k \xi_k^{-\frac{\beta}{\alpha}},\ \ n \to \infty\,.$$

Define sequences $Y = \{Y_n\}$ and $W = \{W_n\}$ by

$$W_n = \sum_{k=n}^{\infty} k^{-1} \omega_k \xi_k^{-\frac{\beta}{\alpha}}, \quad Y_n = \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} n\, p_n^{-\frac{1}{\alpha}} W_n^{\frac{1}{\alpha-\beta}},\ \ n \ge 1\,. \tag{3.20}$$

Note that $W \in \mathcal{SV}$ and since $n\,p_n^{-\frac{1}{\alpha}} = n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}}$, we see that $Y \in \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. Thus, application of Theorem 2.8 gives

$$\begin{aligned}
\sum_{k=n}^{\infty} q_k Y_{k+1}^{\beta} &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{-1} \omega_k \xi_k^{-\frac{\beta}{\alpha}} W_k^{\frac{\beta}{\alpha-\beta}} \\
&\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \sum_{k=n}^{\infty} (-\Delta W_k) \cdot W_k^{\frac{\beta}{\alpha-\beta}} \\
&\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} W_n^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty,
\end{aligned}$$

which yields with the help of Theorem 2.9-(ii)

$$\begin{aligned}
\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j Y_{j+1}^{\beta}\right)^{\frac{1}{\alpha}} &\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}} \xi_k^{-\frac{1}{\alpha}} W_k^{\frac{1}{\alpha-\beta}} \\
&\sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{1}{\alpha-\beta}} n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}} W_n^{\frac{1}{\alpha-\beta}} = Y_n, \quad n \to \infty.
\end{aligned}$$

Therefore, $Y = \{Y_n\}$ satisfies the asymptotic relation (3.11). Then, proceeding exactly as in the proof of Theorem 3.3, replacing X_n with Y_n, a solution x satisfying $\kappa Y_n \le x_n \le K Y_n$, for large n, is obtained by the application of Knaster-Tarski fixed point theorem, and belongs to a class $\mathbb{M}^-_{0,0}$. □

Proof of Theorem 3.2:

(i) Follows from Theorem 3.3-(i) and Theorem 3.4.

(ii) Follows from Theorem 3.3-(ii) and Theorem 3.5. □

3.2. *Asymptotic representation of strongly decreasing $\mathcal{RV}$-solutions*

To simplify the "only if" part of the proof of main results we prove the next two lemmas.

Lemma 3.6. *Let $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. For any $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le 0$ only one of the following two statements holds:*

(i) $\rho = 0$ *and*

$$x_n \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}} \sum_{k=n}^{\infty} k^{-1} \xi_k^{-\frac{1}{\alpha}} \omega_k^{\frac{1}{\alpha}} l_k^{\frac{\beta}{\alpha}}, \quad n \to \infty. \tag{3.21}$$

Then, it is $\sigma = \eta - \alpha - 1 < -1$.

(ii) ρ *is given by* (3.9) *and*

$$x_n \sim \left[\frac{n^{\alpha+1} p_n^{-1} q_n}{(-\rho)^{\alpha}(\alpha - \eta - \rho\alpha)} \right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty. \tag{3.22}$$

Then, it is $\sigma < \eta - \alpha - 1$.

Proof. Suppose that (E) has a solution $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le 0$, satisfying $x_n > 0$, $\Delta x_n < 0$ for $n \ge n_0 + 1 \ge 2$ and expressed with (3.2). Summing (E) for $k \ge n \ge n_0$, we get

$$p_n(-\Delta x_n)^{\alpha} = \sum_{k=n}^{\infty} q_k x_{k+1}^{\beta},$$

which yields, using (3.1) and (3.2)

$$p_n(-\Delta x_n)^{\alpha} \sim \sum_{k=n}^{\infty} q_k x_k^{\beta} = \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \omega_k l_k^{\beta}, \quad n \to \infty. \tag{3.23}$$

The fact that $x_n^{[1]} = p_n(-\Delta x_n)^{\alpha} \to 0$ as $n \to \infty$ implies

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} k^{\sigma+\rho\beta} \omega_k l_k^{\beta} = 0,$$

so it must be $\sigma + \rho\beta \le -1$. We first consider the case $\sigma + \rho\beta = -1$. Then,

$$p_n(-\Delta x_n)^{\alpha} \sim \sum_{k=n}^{\infty} k^{-1} \omega_k l_k^{\beta} = \Omega_n, \quad n \to \infty, \tag{3.24}$$

where $\Omega = \{\Omega_n\} \in \mathcal{SV}$ and $\Omega_n \to 0$, $n \to \infty$. Consequently

$$-\Delta x_n \sim \left(\frac{\Omega_n}{p_n}\right)^{\frac{1}{\alpha}} = n^{-\frac{\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}} \Omega_n^{\frac{1}{\alpha}}, \quad n \to \infty.$$

Since $\lim_{n\to\infty} x_n = 0$, summing previous relation from n to ∞, we get

$$x_n \sim \sum_{k=n}^{\infty} k^{-\frac{\eta}{\alpha}} \left(\frac{\Omega_k}{\xi_k}\right)^{\frac{1}{\alpha}}, \quad n \to \infty, \tag{3.25}$$

implying that $1 - \frac{\eta}{\alpha} \leq 0$ i.e. $\eta \geq \alpha$ which is contradiction, so this case is impossible.

Therefore, $\sigma + \rho\beta < -1$. An application of Theorem 2.9-(ii) in (3.23) gives

$$-\Delta x_n = \left(\frac{1}{p_n}\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta}\right)^{\frac{1}{\alpha}} \sim \frac{n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}} \omega_n^{\frac{1}{\alpha}} \xi_n^{-\frac{1}{\alpha}} l_n^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}}, \quad n \to \infty. \tag{3.26}$$

Because $x_n \to 0$, $n \to \infty$, summing (3.26) from n to ∞ we get

$$x_n \sim \sum_{k=n}^{\infty} \frac{k^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}} \omega_k^{\frac{1}{\alpha}} \xi_k^{-\frac{1}{\alpha}} l_k^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}}}, \quad n \to \infty. \tag{3.27}$$

From the last relation we conclude that it must be $(\sigma+\rho\beta+1-\eta)/\alpha \leq -1$, so we distinguish two possibilities:

$$\text{(a)} \quad \frac{\sigma+\rho\beta+1-\eta}{\alpha} = -1, \qquad \text{(b)} \quad \frac{\sigma+\rho\beta+1-\eta}{\alpha} < -1. \tag{3.28}$$

If (a) holds, then $\sigma + \rho\beta + 1 = \eta - \alpha$. From (3.27), we get that (3.21) holds, and according to Theorem 2.9-(iii), $x \in \mathcal{SV}$. Thus, $\rho = 0$ and (a) implies that $\sigma = \eta - \alpha - 1$. On the other hand, if (b) holds, from (3.27), by Theorem 2.9-(ii), we obtain

$$x_n \sim \frac{n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1} \omega_n^{\frac{1}{\alpha}} \xi_n^{-\frac{1}{\alpha}} l_n^{\frac{\beta}{\alpha}}}{(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}} \left(-\frac{\sigma+\rho\beta+1-\eta}{\alpha} - 1\right)}, \quad n \to \infty. \tag{3.29}$$

Thus it must be

$$\rho = \frac{\sigma+\rho\beta+1-\eta}{\alpha} + 1, \tag{3.30}$$

implying that the regularity index of x is given by (3.9). Combined this with the assumption $\rho < 0$, we get that $\sigma < \eta - \alpha - 1$. Moreover, using (3.9) i.e. (3.30), we obtain

$$(-(\sigma+\rho\beta+1))^{\frac{1}{\alpha}} \left(-\frac{\sigma+\rho\beta+1-\eta}{\alpha} - 1\right) = \left((\alpha-\eta-\rho\alpha)(-\rho)^{\alpha}\right)^{\frac{1}{\alpha}}, \tag{3.31}$$

and

$$n^{\frac{\sigma+\rho\beta+1-\eta}{\alpha}+1} \xi_n^{-\frac{1}{\alpha}} \omega_n^{\frac{1}{\alpha}} l_n^{\frac{\beta}{\alpha}} = \left(n^{\alpha+1} p_n^{-1} q_n\right)^{\frac{1}{\alpha}} x_n^{\frac{\beta}{\alpha}}. \tag{3.32}$$

Then, from (3.29) we obtain that the asymptotic representation of x is given by (3.22). □

Lemma 3.7. *Let $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. For any $x \in \mathbb{M}_{0,0}^{-} \cap \mathcal{RV}(\rho)$ with $\rho \leq \frac{\alpha-\eta}{\alpha}$ only one of the following two statements holds:*

(i) $\rho = \dfrac{\alpha-\eta}{\alpha}$ *and*

$$x_n \sim \frac{\alpha}{\eta-\alpha} n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}} \left(\sum_{k=n}^{\infty} k^{-1} \omega_k l_k^{\beta}\right)^{\frac{1}{\alpha}}, \ n \to \infty; \tag{3.33}$$

Then, it is $\sigma = \beta\,\dfrac{\eta-\alpha}{\alpha} - 1$.

(ii) ρ *is given by* (3.9) *and* (3.22) *holds. Then, it is* $\sigma < \beta\,\dfrac{\eta-\alpha}{\alpha} - 1$.

Proof. Suppose that (E) has a solution $x \in \mathbb{M}^{-}_{0,0} \cap \mathcal{RV}(\rho)$ with $\rho \le \frac{\alpha-\eta}{\alpha}$, satisfying $x_n > 0$, $\Delta x_n < 0$ for $n \ge n_0 + 1 \ge 2$ and expressed with (3.2). Using (3.1) and (3.2) we have (3.23). As in the proof of previous lemma, the fact that $x_n^{[1]} = p_n(\Delta x_n)^\alpha \to 0$ as $n \to \infty$ implies that $\sigma + \rho\beta \le -1$.

If $\sigma + \rho\beta = -1$, then as in the proof of previous lemma we get (3.25), where Ω_n is given in (3.24). Using that $\eta > \alpha$, application of Theorem (2.9)-(ii) in (3.25) gives (3.33). Thus, $\rho = \frac{\alpha-\eta}{\alpha}$, implying that $\sigma = \beta\,\frac{\eta-\alpha}{\alpha} - 1$.

Next we consider the case $\sigma+\rho\beta < -1$. An application of Theorem 2.9-(ii) in (3.23) give us (3.27) implying as previously two possibilities (a) or (b) in (3.28). However, the case (a) is not possible, because $\sigma + \rho\beta < -1$ implies

$$-1 = \frac{\sigma + \rho\beta + 1 - \eta}{\alpha} < -\frac{\eta}{\alpha},$$

which is a contradiction with $\eta > \alpha$. Thus, only (b) in (3.28) can be valid and so from (3.27), as previously, we obtain that ρ is given by (3.9) and x satisfies (3.22). Since, $\rho < \frac{\alpha-\eta}{\alpha}$ from (3.9) we conclude that $\sigma < \frac{\beta\eta}{\alpha} - \beta - 1$. □

Now, we are in a position to prove the main results.

Theorem 3.8. *Suppose that $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$.*

(i) *Let $\eta < \alpha$. Equation (E) possesses regularly varying solutions x of index $\rho < 0$ if and only if* (3.4) *holds.*

(ii) *Let $\eta > \alpha$. Equation (E) possesses regularly varying solutions x of index $\rho < \frac{\alpha-\eta}{\alpha}$ if and only if* (3.6).

In both cases ρ is given by (3.9) *and the asymptotic behavior of any such solution x is governed by the unique formula* (3.22).

Proof. **The "only if" part:** Suppose that $\eta < \alpha$ and $x \in \mathcal{RV}(\rho)$ with $\rho < 0$. According to Theorem 2.7-(v) and (vi), $x \in \mathbb{M}^{-}$ and $\lim_{n\to\infty} x_n = 0$. It is easy to prove (see [6, Lemma 3]) that if $S = \infty$, then for any solution in the class $\mathbb{M}^{-}$, it holds $\lim_{n\to\infty} x_n{}^{[1]} = 0$. Thus, $x \in \mathbb{M}^{-}_{0,0}$. Then, it is clear that only the case (ii) of Lemma 3.6 is admissible for x. Thus, the regularity index of x is given by (3.9) and σ satisfies (3.4).

Suppose that $\eta > \alpha$ and $x \in \mathcal{RV}(\rho)$ with $\rho < \frac{\alpha-\eta}{\alpha}$. Since $\rho < 0$ as previously we conclude that $x \in \mathbb{M}^{-}_{0}$. Therewith, in view of (3.3), by Theorem 2.7-(vi) we get

$$\lim_{n\to\infty} \frac{x_n}{\pi_n} = \frac{\eta-\alpha}{\alpha} \lim_{n\to\infty} n^{\rho - \frac{\alpha-\eta}{\alpha}} l_n \xi_n^{\frac{1}{\alpha}} = 0,$$

implying that $x \in \mathbb{M}^{-}_{0,0}$. It is clear that only the case (ii) of Lemma 3.7 is admissible for x, implying that the regularity index of x is given by (3.9) and that (3.6) holds.

From Lemmas 3.6 and 3.7 we obtain that the asymptotic representation of regularly varying solution x of index ρ is given by (3.22) in each of two cases (i) and (ii).

The "if" part: We perform the simultaneous proof for both of the cases. From Theorem 3.3 follows the existence of a solution $x \in \mathbb{M}^{-}_{0,0}$. It remains to prove that x satisfying (3.15) and (3.17) is a regularly varying sequence of index ρ. From (3.15) we have

$$0 < \liminf_{n\to\infty} \frac{x_n}{X_n} \le \limsup_{n\to\infty} \frac{x_n}{X_n} < \infty,$$

where X_n is given by (3.8). Application of Lemma 1.2, using (3.11) and (3.17), yields

$$\begin{aligned}
L &= \limsup_{n\to\infty} \frac{x_n}{X_n} \le \limsup_{n\to\infty} \frac{\Delta x_n}{\Delta X_n} = \limsup_{n\to\infty} \frac{-\left(\frac{1}{p_k}\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta}\right)^{1/\alpha}}{-\left(\frac{1}{p_k}\sum_{k=n}^{\infty} q_k X_{k+1}^{\beta}\right)^{1/\alpha}} \\
&\le \left(\limsup_{n\to\infty} \frac{\sum_{k=n}^{\infty} q_k x_{k+1}^{\beta}}{\sum_{k=n}^{\infty} q_k X_{k+1}^{\beta}}\right)^{1/\alpha} \le \left(\limsup_{n\to\infty} \frac{-q_n x_{n+1}^{\beta}}{-q_n X_{n+1}^{\beta}}\right)^{1/\alpha} \\
&\le \left(\limsup_{n\to\infty} \frac{x_{n+1}}{X_{n+1}}\right)^{\beta/\alpha} = L^{\frac{\beta}{\alpha}}.
\end{aligned}$$

Since $\beta < \alpha$, from above we conclude that

$$0 < L \leq 1\,. \tag{3.34}$$

Similarly, we can see that $l = \liminf_{n\to\infty} x_n/X_n$ satisfies

$$1 \leq l < \infty\,. \tag{3.35}$$

From (3.34) and (3.35) we obtain that $l = L = 1$, which means that $x_n \sim X_n$, $n \to \infty$ and ensures that x is a regularly varying solution of (E) with requested regularity index and the asymptotic representation given by (3.22). □

Theorem 3.9. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta < \alpha$ and $q \in \mathcal{RV}(\sigma)$. There exists $x \in \mathbb{M}^-_{0,0} \cap ntr - \mathcal{SV}$ if and only if* (3.5) *holds. All such solutions of (E) enjoy the precise asymptotic formula*

$$x_n \sim \left[\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}}\right]^{\frac{\alpha}{\alpha-\beta}}, \quad n \to \infty. \tag{3.36}$$

Proof. **The "only if" part:** Suppose that $x \in \mathbb{M}^-_{0,0} \cap ntr - \mathcal{SV}$. Then, clearly only the statement (i) of Lemma 3.6 could hold. Therefore, $\rho = 0$, $\sigma = \eta - \alpha - 1$ and x satisfies (3.21). Then, since $\sigma < -1$, application of Theorem 2.9 gives

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha}}}\sum_{k=n}^{\infty} k^{-1}\xi_k^{-\frac{1}{\alpha}}\omega_k^{\frac{1}{\alpha}}, \quad n \to \infty, \tag{3.37}$$

where we used that $\sigma + 1 = \alpha - \eta$. Denote

$$z_n = \sum_{k=n}^{\infty} k^{-1}\xi_k^{-\frac{1}{\alpha}}\omega_k^{\frac{1}{\alpha}} l_k^{\frac{\beta}{\alpha}}. \tag{3.38}$$

From Theorem 2.9-(iii) clearly $z = \{z_n\} \in \mathcal{SV}$ and (3.21) becomes

$$x_n = l_n \sim \frac{z_n}{(\alpha-\eta)^{\frac{1}{\alpha}}}, \quad n \to \infty. \tag{3.39}$$

From (3.38) and (3.39) we obtain the asymptotic relation

$$z_n^{-\frac{\beta}{\alpha}}(-\Delta z_n) \sim \frac{n^{-1}\xi_n^{-\frac{1}{\alpha}}\omega_n^{\frac{1}{\alpha}}}{(\alpha-\eta)^{\frac{\beta}{\alpha^2}}}, \quad n \to \infty. \tag{3.40}$$

By (3.39), we have that $z_n \to 0$, $n \to \infty$ and clearly $\{z_n\}$ is strictly decreasing. Summing (3.40) from n to ∞, using Theorem 2.8 and (3.37), we obtain

$$\frac{\alpha}{\alpha-\beta} z_n^{1-\frac{\beta}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{\beta}{\alpha^2}}}\sum_{k=n}^{\infty} k^{-1}\xi_k^{-\frac{1}{\alpha}}\omega_k^{\frac{1}{\alpha}} \sim \frac{1}{(\alpha-\eta)^{\frac{\beta-\alpha}{\alpha^2}}}\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}}, \; n \to \infty. \tag{3.41}$$

Because $1 - \frac{\beta}{\alpha} > 0$, $z_n^{1-\frac{\beta}{\alpha}} \to 0$, $n \to \infty$, so (3.41) yields that the second condition in (3.5) is satisfied as well as that the asymptotic expression for x is

$$x_n \sim \frac{1}{(\alpha-\eta)^{\frac{1}{\alpha-\beta}}}\left(\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty} k^{-1}\xi_k^{-\frac{1}{\alpha}}\omega_k^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-\beta}} \sim \left(\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty}\left(\frac{1}{p_k}\sum_{j=k}^{\infty} q_j\right)^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{\alpha-\beta}},$$

when $n \to \infty$. This completes the "only if" part of the proof of Theorem 3.9.

The "if" part: From Theorem 3.4 we have the existence of a solution $x \in \mathbb{M}^-_{0,0}$. In the same way as in the proof of Theorem 3.8, replacing X_n with T_n given by (3.19) and with the application of Lemma 1.2 we obtain that $x_n \sim T_n$, $n \to \infty$, implying that such a solution is slowly varying and enjoys the precise asymptotic behavior (3.36). □

Theorem 3.10. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta > \alpha$ and $q \in \mathcal{RV}(\sigma)$. There exists $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$ if and only if (3.7) holds. All such solutions of (E) enjoy the precise asymptotic behaviour*

$$x_n \sim \left(\alpha^{\alpha-1}\frac{\alpha-\beta}{(\eta-\alpha)^\alpha}\right)^{\frac{1}{\alpha-\beta}} n\, p_n^{-\frac{1}{\alpha}} \left[\sum_{k=n}^{\infty} k^\beta q_k p_k^{-\frac{\beta}{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty. \tag{3.42}$$

Proof. **The "only if" part:** Suppose that $x \in \mathbb{M}^-_{0,0} \cap \mathcal{RV}\left(\frac{\alpha-\eta}{\alpha}\right)$. Then, clearly only the statement (i) of Lemma 3.7 could hold. Therefore, $\rho = \frac{\alpha-\eta}{\alpha}$, $\sigma = \frac{\beta}{\alpha}\eta - \beta - 1$ and x satisfies (3.33). From (3.2) and (3.33) we get

$$l_n \sim \frac{\alpha}{\eta-\alpha}\xi_n^{-\frac{1}{\alpha}}\Omega_n^{\frac{1}{\alpha}}, \; n \to \infty, \tag{3.43}$$

where Ω_n is given in (3.24). From (3.24), we conclude that $\Omega \in \mathcal{SV}$, $\Omega_n \to 0$ as $n \to \infty$ and $\{\Omega_n\}$ is strictly decreasing. We transform (3.43) into the asymptotic relation for Ω

$$\Omega_n^{-\frac{\beta}{\alpha}}\Delta\Omega_n \sim -\left(\frac{\alpha}{\eta-\alpha}\right)^\beta n^{-1}\omega_n \xi_n^{-\frac{\beta}{\alpha}} = -\left(\frac{\alpha}{\eta-\alpha}\right)^\beta n^\beta q_n p_n^{-\frac{\beta}{\alpha}}, \quad n \to \infty. \tag{3.44}$$

Summing (3.44) from n to ∞ and using Theorem 2.8 we obtain

$$\frac{\alpha}{\alpha-\beta}\Omega_n^{1-\frac{\beta}{\alpha}} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^\beta \sum_{k=n}^{\infty} k^\beta q_k p_k^{-\frac{\beta}{\alpha}}, \quad n \to \infty. \tag{3.45}$$

Because $\Omega_n^{1-\frac{\beta}{\alpha}} \to 0$ as $n \to \infty$, (3.45) yields that the second condition in (3.7) is satisfied. The asymptotic expression (3.33) for x becomes

$$x_n \sim \frac{\alpha}{\eta-\alpha} n^{\frac{\alpha-\eta}{\alpha}} \xi_n^{-\frac{1}{\alpha}}\Omega_n^{\frac{1}{\alpha}} \sim \left(\frac{\alpha}{\eta-\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} n\, p_n^{-\frac{1}{\alpha}}\left[\frac{\alpha-\beta}{\alpha}\sum_{k=n}^{\infty} k^\beta q_k p_k^{-\frac{\beta}{\alpha}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

This completes the "only if" part of the proof of Theorem 3.10.

The "if" part: From Theorem 3.5 we obtain the existence of a solution $x \in \mathbb{M}^-_{0,0}$, while application of Lemma 1.2 as in the proof of Theorem 3.8, with Y_n instead of X_n, where Y_n is given by (3.20), proves that $x_n \sim Y_n$, $n \to \infty$, so that such a solution is in fact a $\mathcal{RV}$–solution of index $\frac{\alpha-\eta}{\alpha}$, with the precise asymptotic behavior given by (3.42). □

4. Corollaries and examples

In the previous section, we have shown that the existence of strongly decreasing $\mathcal{RV}$–solutions for the equation (E) with $\mathcal{RV}$ coefficients is fully characterized by the assumption $I < \infty$ if $S = \infty$ and by the assumption $J < \infty$ if $S < \infty$. In fact, the following corollary holds.

Corollary 4.1. *Suppose that $p \in \mathcal{RV}(\eta)$, $\eta \neq \alpha$ and $q \in \mathcal{RV}(\sigma)$.*

(i) Let $S = \infty$. Equation (E) has strongly decreasing $\mathcal{RV}$-solutions if and only if $I < \infty$.
(ii) Let $S < \infty$. Equation (E) has strongly decreasing $\mathcal{RV}$-solutions if and only if $J < \infty$.

Moreover, if $S = \infty$, then $J = \infty$ so by Theorem 1.1 $\mathbb{M}^-_{0,l} = \emptyset$. Otherwise, if $S < \infty$, denoting the series $Q = \sum_{k=1}^{\infty} q_k$, we have two cases:

(a) If $Q = \infty$, then $I = \infty$, so by Theorem 1.1 we have $\mathbb{M}^- = \mathbb{M}^-_0$ i.e. $\mathbb{M}^-_B = \emptyset$.
(b) If $Q < \infty$, then $I < \infty$, so by Theorem 1.1 we have $\mathbb{M}^- = \mathbb{M}^-_0 \cup \mathbb{M}^-_B$.

Using conclusions from the previous corollary and Theorem 1.1, we get the next two corollaries where we will use the following symbols:
* $\mathcal{R}$ denote the set of all regularly varying solutions,
* $\mathcal{R}^-$ denote the set of all decreasing regularly varying solutions,
* $\mathcal{R}^-_0 = \mathcal{R} \cap \mathbb{M}^-_0$.
* $\mathcal{R}^-_{0,0} = \mathcal{R} \cap \mathbb{M}^-_{0,0}$.

Corollary 4.2. *Suppose that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $S = \infty$. Then,*

$$\mathcal{R}^- = ntr - \mathcal{SV} \cup \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}^-_B$$

if and only if $I < \infty$.

Corollary 4.3. *Suppose that $p \in \mathcal{RV}(\eta)$, $q \in \mathcal{RV}(\sigma)$ and $S < \infty$. Then,*

(i) If $\sigma < -1$ or $\sigma = -1$ and $Q < \infty$, then

$$\mathcal{R}^- = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}^-_{0,l} \cup \mathbb{M}^-_B.$$

(ii) If $\sigma = -1$ and $Q = \infty$ or $-1 < \sigma < \frac{\beta\eta}{\alpha} - \beta - 1$, then

$$\mathcal{R}^- = \mathcal{R}^-_0 = \mathcal{RV}\left(\frac{\sigma + \alpha + 1 - \eta}{\alpha - \beta}\right) \cup \mathbb{M}^-_{0,l}.$$

(iii) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J < \infty$, then

$$\mathcal{R}^- = \mathcal{R}^-_0 = \mathcal{RV}\left(\frac{\alpha - \eta}{\alpha}\right) \cup \mathbb{M}^-_{0,l}.$$

(iv) If $\sigma = \frac{\beta\eta}{\alpha} - \beta - 1$ and $J = \infty$ or $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, then

$$\mathcal{R}^- = \emptyset.$$

Example 4.4. Consider the difference equation

$$\Delta\left(\frac{n^\eta}{\log n}(\Delta x_n)^3\right) = \frac{n^{\eta-7}\varphi_n}{\log^5 n}\sqrt{x_{n+1}^3}, \quad n \geq 1, \tag{4.1}$$

where φ_n is a positive real-value sequence such that $\lim_{n\to\infty} \varphi_n = \delta$ and $\eta \neq 3$. In this equation, $\alpha = 3$, $\beta = \dfrac{3}{2}$, $\{p_n\} \in \mathcal{RV}(\eta)$ and $\{q_n\} \in \mathcal{RV}(\sigma)$, where $\sigma = \eta - 7$.

(i) Suppose that $\eta < 3$. In this case

$$\sigma = \eta - 7 < \eta - 4 = \eta - \alpha - 1,$$

so in view of Theorem 3.8-(i) and Theorem 1.1 this equation has a strongly decreasing $\mathcal{RV}$-solution of index $\rho < 0$ as well as a solution in $\mathbb{M}_B^- = tr - \mathcal{SV}$. More precisely, by Theorem 3.8-(i) equation (4.1) has a strongly decreasing solution which belongs to $\mathcal{RV}(-2)$. That solution has asymptotic behavior

$$x_n \sim \left(\frac{\delta}{8(9-\eta)}\right)^{\frac{2}{3}} n^{-2} (\log n)^{-\frac{8}{3}}, \quad n \to \infty. \tag{4.2}$$

If

$$\varphi_n = \frac{n^7 (n+1)^3}{(\log n)^4 (\log(n+1))^5} \left((\log n)^9 \psi_n - (\log(n+1))^9 \left(\frac{n+1}{n}\right)^\eta \psi_{n+1}\right), \tag{4.3}$$

where

$$\psi_n = \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\left(\frac{\log n}{\log(n+1)}\right)^{\frac{8}{3}}\right)^3,$$

then $\delta = 8(9-\eta)$ and the considered equation has an exact solution $n^{-2} (\log n)^{-\frac{8}{3}}$.

(ii) For $\eta \in (3, 9)$ we have that $\eta > \alpha$ and $\sigma = \eta - 7 < \frac{\eta - 5}{2} = \frac{\beta\eta}{\alpha} - \beta - 1$, so in view of Theorem 3.8-(ii) the equation (4.1) has a strongly decreasing solution which belongs to $\mathcal{RV}(-2)$ and satisfies (4.2). This equation also possess a solution which belongs to a class $\mathbb{M}_{0,l}^-$.

(iii) Let $\eta = 9$. Then, $\sigma = 2 = \dfrac{\beta\eta}{\alpha} - \beta - 1$ and $J < \infty$. By Theorem 3.10 the equation (4.1) has a solution $x \in \mathcal{RV}(1 - \frac{\eta}{\alpha}) = \mathcal{RV}(-2)$ and any such solution x has the asymptotic representation

$$\begin{aligned} x_n &\sim \left(\frac{\delta}{16}\right)^{\frac{2}{3}} n^{-2} (\log n)^{\frac{1}{3}} \left(\sum_{k=n}^{\infty} k^{-1} (\log k)^{-\frac{9}{2}}\right)^{\frac{2}{3}} \\ &\sim \left(\frac{\delta}{16}\right)^{\frac{2}{3}} n^{-2} (\log n)^{\frac{1}{3}} \left(\frac{2}{7}\right)^{\frac{2}{3}} (\log n)^{-\frac{7}{3}} = \left(\frac{\delta}{56}\right)^{\frac{2}{3}} (n \log n)^{-2}, \quad n \to \infty, \end{aligned}$$

where we used that

$$\sum_{k=n}^{\infty} k^{-1} (\log k)^{-\frac{9}{2}} \sim \int_n^\infty x^{-1} (\log x)^{-\frac{9}{2}} dx, \quad n \to \infty.$$

If

$$\varphi_n = \frac{(n+1)^3 (\log(n+1))^3 (\log n)^5}{n^2} (\chi_n - \chi_{n+1}),$$

where

$$\chi_n = \frac{n^3}{(\log n)^7} \left(1 - \left(\frac{n \log n}{(n+1)\log(n+1)}\right)^2\right)^3,$$

then $\lim_{n\to\infty} \varphi_n = 56$ and $x_n = (n \log n)^{-2}$ is an exact solution of the equation (4.1).

(iv) If $\eta > 9$, then $\sigma = \eta - 7 > \frac{\eta-5}{2} = \frac{\beta\eta}{\alpha} - \beta - 1$ so $J = \infty$. Therefore, by Corollary 4.1 the equation (4.1) does not have decreasing regularly varying solutions.

Example 4.5. Consider the difference equation

$$\Delta\left(-n^\eta \sqrt{\log n}\, (\Delta x_n)^2\right) = \frac{n^{\eta-3} \varphi_n}{(\log n)^{19/6}} \sqrt[3]{x_{n+1}}, \quad n \geq 1, \tag{4.4}$$

where φ_n is a positive real-value sequence such that $\lim_{n\to\infty} \varphi_n = \delta$ and $\eta \neq 2$. Here, $p_n = n^\eta \sqrt{\log n}$, and $q_n = n^{\eta-3} \varphi_n (\log n)^{-19/6}$, so $p \in \mathcal{RV}(\eta)$ and $q \in \mathcal{RV}(\sigma)$, where $\sigma = \eta - 3 = \eta - \alpha - 1$.

Let $\eta < 2 = \alpha$. Using that

$$\sum_{k=n}^{\infty} \left(\frac{1}{p_k} \sum_{j=k}^{\infty} q_j \right)^{\frac{1}{\alpha}} \sim \sum_{k=n}^{\infty} \sqrt{\frac{\varphi_n}{2-\eta}} \frac{1}{k (\log k)^{11/6}} < \infty, \quad n \to \infty,$$

by Theorem 3.10 the equation (4.4) has a nontrivial slowly varying solution and any such solution x has the asymptotic representation

$$x_n \sim \left(\frac{\delta}{2-\eta} \right)^{\frac{3}{5}} \cdot (\log n)^{-1}, \quad n \to \infty.$$

If

$$\varphi_n = n^3 \left(\frac{\log n}{\log(n+1)} \right)^{\frac{19}{6}} \left[\frac{(\log \frac{n+1}{n})^2}{(\log n)^{\frac{3}{2}} (\log(n+1))^{\frac{1}{2}}} - \left(\frac{n+1}{n} \right)^{\eta} \left(\frac{\log \frac{n+2}{n+1}}{\log(n+2)} \right)^2 \right],$$

then $\delta = 2 - \eta$ and considered equation has an exact solution $x_n = (\log n)^{-1}$, $x \in ntr - \mathcal{SV}$.

Notice that in the case $\eta > 2 = \alpha$, since $\sigma > \frac{\beta\eta}{\alpha} - \beta - 1$, using Corollary 4.3, we conclude that $\mathcal{R}^- = \emptyset$.

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