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Ideal Analogues of Some Variants of the Hurewicz Property

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Abstract. In this paper, we continue the study on the ideal analogues of several variations of the Hurewicz property introduced by Das et al. [4, 6, 7] for example, the $\mathcal{I}$-Hurewicz ($\mathcal{I}$H), the star $\mathcal{I}$-Hurewicz (S$\mathcal{I}$H), the weakly $\mathcal{I}$-Hurewicz (W$\mathcal{I}$H) and the weakly star-$\mathcal{I}$-Hurewicz (WS$\mathcal{I}$H). It is shown that several implications in the relationship diagram of their concepts are reversible under certain conditions, for instance; (1) If a paracompact Hausdorff space has the WS$\mathcal{I}$H property, then it has the W$\mathcal{I}$H property. (2) If the complement of dense set has the $\mathcal{I}$H property, then the W$\mathcal{I}$H property implies the $\mathcal{I}$H property and (3) If the complement of dense set has the S$\mathcal{I}$H property, then the WS$\mathcal{I}$H property implies the S$\mathcal{I}$H property. In addition, we introduce the ideal analogues of some new variations of the Hurewicz property called the mildly $\mathcal{I}$-Hurewicz and the star K-$\mathcal{I}$-Hurewicz properties and explore their relationships with other variants of the $\mathcal{I}$-Hurewicz property. We also study the preservation properties under certain mappings.

1. Introduction

In [23] Scheepers began the systematic study of selection principles in topology and their relations to game theory and the Ramsey theory (see also [14]). Over the period it becomes the most active areas of set theoretic topology. In [8] Di Maio and Kočinac introduced the statistical analogues of certain types of open covers and selection principles using the ideal of asymptotic density zero sets of natural numbers. Das, Kočinac and Chandra extended this idea to study the similar investigation to arbitrary ideals of natural numbers (see [4, 5, 7]). It should be noted that the classical selection principles described in the next section have been used to define and characterize various types of covering properties. Hurewicz [12, 13] defined a covering property called the Hurewicz which lies between the σ-compactness and the Lindelöffness. Kočinac and Scheepers [19] studied the Hurewicz property in details and found its relations with function spaces, game theory and the Ramsey theory. As a generalization of the Hurewicz property, several authors studied other variations of the Hurewicz property (see [1, 2, 15, 16, 18, 22, 26, 28–30]). Das et al. [4, 7] studied a version of the Hurewicz property (called $\mathcal{I}$-Hurewicz, $\mathcal{I}$H) using an arbitrary ideal $\mathcal{I}$ of natural numbers and in [6] introduced the ideal analogues of several variations of the Hurewicz property such

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as the star $\mathcal{I}$-Hurewicz (S$\mathcal{I}$H), the strongly star $\mathcal{I}$-Hurewicz (SS$\mathcal{I}$H), the weakly $\mathcal{I}$-Hurewicz (W$\mathcal{I}$H), the weakly star-$\mathcal{I}$-Hurewicz (WS$\mathcal{I}$H) and the mildly star-$\mathcal{I}$-Hurewicz (MS$\mathcal{I}$H) properties.

The purpose of this paper is to extend the results of some variations of the $\mathcal{I}$-Hurewicz property. We introduce the ideal analogues of some new variations of the Hurewicz property called the mildly $\mathcal{I}$-Hurewicz (M$\mathcal{I}$H) and the star K-$\mathcal{I}$-Hurewicz (SK$\mathcal{I}$H) and explore their relationships with the above variants. We also study their preservations under several types of mappings.

This paper is organized as follows. Section 2, develops the necessary preliminaries. In Section 3, we study the W$\mathcal{I}$H property and prove that for a space X has the $\mathcal{I}$H property if and only if the Alexandorff duplicate of X has the $\mathcal{I}$H property if and only if the Alexandorff duplicate of X has the W$\mathcal{I}$H property. In Section 4, we introduce the mildly $\mathcal{I}$-Hurewicz (M$\mathcal{I}$H) property and study its preservation properties under certain mappings. In Section 5, we introduce the star K-$\mathcal{I}$-Hurewicz property (SK$\mathcal{I}$H). It is proved that in a paracompact Hausdorff space, $\mathcal{I}$H, S$\mathcal{I}$H, SK$\mathcal{I}$H and SS$\mathcal{I}$H properties become equivalent. In Section 6, it is shown that the several implications in the relationship diagram [6] are reversible under certain conditions.

2. Preliminaries

Throughout the paper (X, τ) stands for a Hausdorff topological space.

Let A be a subset of X and $\mathcal{P}$ be a collection of subsets of X, then St(A,$\mathcal{P}$)=$\bigcup\{U \in \mathcal{P} : U \cap A \neq \phi\}$. We usually write St($x$,$\mathcal{P}$)=St($\{x\}$,$\mathcal{P}$).

For two nonempty classes of sets $\mathcal{A}$ and $\mathcal{B}$ of an infinite set X, following [15, 17]; we define

$\mathsf{S}_1^*(\mathcal{A}, \mathcal{B})$: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{St(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{SS}_1^*(\mathcal{A}, \mathcal{B})$: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ of one-point subset of X such that $\{St(\mathcal{K}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{SS}_{\mathsf{fin}}^*(\mathcal{A}, \mathcal{B})$: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ of finite subset of X such that $\{St(\mathcal{K}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{U}_{\mathsf{fin}}^*(\mathcal{A}, \mathcal{B})$: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\{St(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

We consider basic definitions below.

A space X is said to have the Hurewicz property [12] (in short, H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n.

A space X is said to have the star-Hurewicz property [1, 27] (in short, SH) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in St(\cup\mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n.

A space X is said to have the strongly star-Hurewicz property [1, 26] (in short, SSH) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ finite subsets of X such that and for each $x \in X$, $x \in St(A_n, \mathcal{U}_n)$ for all but finitely many n.

A space X is said to have the weakly Hurewicz property [[16] (see also [22, 29])] (in short, WH) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a dense set $Y \subset X$ and sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in Y$, $y \in \bigcup \mathcal{V}_n$ for all but finitely many n.

A space X is said to have the mildly Hurewicz property [18] (in short, MH) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of clopen covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n.

A space X is said to have the star-K-Hurewicz property [28] (in short, SKH) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(K_n : n \in \mathbb{N})$ compact subsets of X such that and for each $x \in X$, $x \in St(K_n, \mathcal{U}_n)$ for all but finitely many n.

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers.

A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$, while an ideal is said to be admissible ideal or free ideal $\mathcal{I}$ of Y if $\{y\} \in \mathcal{I}$ for each $y \in Y$. If $\mathcal{I}$ is a nontrivial proper ideal in Y (that is, $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$), then the family of sets

$\mathcal{F}(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. Throughout the paper $\mathcal{I}$ will stand proper admissible ideal of $\mathbb{N}$. We denote the ideals of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_{fin}$.

In this paper we will continue the study of the Hurewicz, the star Hurewicz, the strongly star Hurewicz and the weakly Hurewicz properties using the ideal respectively called them the $\mathcal{I}$-Hurewicz, the star $\mathcal{I}$-Hurewicz, the strongly star $\mathcal{I}$-Hurewicz and the weakly $\mathcal{I}$-Hurewicz introduced by Silva and Das et al. [6, 24] in the following way.

A space X is said to have the $\mathcal{I}$-Hurewicz property (in short, $\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

A space X is said to have the star-$\mathcal{I}$-Hurewicz property (in short, S$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(\cup\mathcal{V}_n, \mathcal{U}_n)\} \in \mathcal{I}$.

A space X is said to have the strongly star-$\mathcal{I}$-Hurewicz property (in short, SS$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ finite subsets of X such that and for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(A_n, \mathcal{U}_n)\} \in \mathcal{I}$.

A space X is said to have the weakly $\mathcal{I}$-Hurewicz property (in short, W$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a dense set $Y \subset X$ and sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

Consider some more definitions from [6].

A space X is said to have the mildly star-$\mathcal{I}$-Hurewicz property (in short, MS$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of clopen covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{St(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $\mathcal{I} - \gamma$-cover of X.

A space X is said to have the weakly star-$\mathcal{I}$-Hurewicz property (in short, WS$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a dense set $Y \subset X$ and sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin St(\cup\mathcal{V}_n, \mathcal{U}_n)\} \in \mathcal{I}$.

Several types of open covers were studied [3–5, 7, 11] using ideal called $\mathcal{I}$-γ covers, $\mathcal{I}$-γ-covers, $\mathcal{I}$-γ_k-covers. But we will need only the following. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be an $\mathcal{I}$-γ-cover [7] if for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin U_n\} \in \mathcal{I}$. The set of all $\mathcal{I}$-γ-covers will denoted by $\mathcal{I}$-Γ. An open cover $\mathcal{U}$ of a X is said to be $\mathcal{I}$-groupable [4] if it can be expressed in the form $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, where $\mathcal{U}_n$'s are finite, pairwise disjoint and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \cup\mathcal{U}_n\} \in \mathcal{I}$. The set of all $\mathcal{I}$-groupable open covers of X denote by $\mathcal{O}^{\mathcal{I}-gp}$.

3. The Weakly $\mathcal{I}$-Hurewicz Property

Theorem 3.1. ([6]) *The W$\mathcal{I}$H property is an inverse invariant under perfect open mappings.*

It is well known that the product of the weakly Hurewicz space and a compact space is the weakly Hurewicz. For the property W$\mathcal{I}$H, we have similar result by Theorem 3.1.

Corollary 3.2. *If X has the W$\mathcal{I}$H property and Y is compact, then $X \times Y$ has the W$\mathcal{I}$H property.*

Theorem 3.3. *The W$\mathcal{I}$H property is closed under countable unions.*

Proof. Let $X = \bigcup\{X_k : k \in \mathbb{N}\}$, where X_k has the W$\mathcal{I}$H property and $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X . For each $k \in \mathbb{N}$, let us consider the sequence $(\mathcal{U}_n : n \geq k)$. For each $k \in \mathbb{N}$, by the W$\mathcal{I}$H property of X_k, there are a dense subset D_k of X_k and a sequence $(\mathcal{V}_{n,k} : n \geq k)$ such that for each $n \geq k$, $\mathcal{V}_{n,k}$ is a finite subset of $\mathcal{U}_n$ and for each $x \in D_k$, $\{n \in \mathbb{N} : n \geq k, x \notin \bigcup \mathcal{V}_{n,k}\} \in \mathcal{I}$. So $\{n \in \mathbb{N} : n \geq k, x \in \bigcup \mathcal{V}_{n,k}\} \in \mathcal{F}(\mathcal{I})$. Let $D = \bigcup_{k\in\mathbb{N}} D_k$. Then D is a dense subset of X. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \bigcup\{\mathcal{V}_{n,j} : j \leq n\}$. Then each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. Let $x \in D$. Then there is a $k \in \mathbb{N}$ such that $x \in D_k$. Thus for each $n \geq k$, $\mathcal{V}_{n,k}$ is a finite subset of $\mathcal{U}_n$ so that $\{n \in \mathbb{N} : n \geq k, x \in \bigcup \mathcal{V}_{n,k}\} \in \mathcal{F}(\mathcal{I})$. Hence $x \in \mathcal{V}_{n,k}$ for some n as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. So there $V \in \mathcal{V}_{n,k}$ for some $n \geq k$ such that $x \in V$, therefore $V \in \mathcal{V}_n$. Thus $\{n \in \mathbb{N} : x \in \bigcup \mathcal{V}_{n,k}\} \subset \{n \in \mathbb{N} : x \in \bigcup \mathcal{V}_n\}$. Therefore $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$, which completes the proof. $\square$

We have the following result by Corollary 3.2 and Theorem 3.3.

Corollary 3.4. *If X has the $W\mathcal{I}H$ property and Y is σ-compact, then $X \times Y$ has the $W\mathcal{I}H$ property.*

Consider the Alexandorff duplicate $A(X) = X \times \{0, 1\}$ of a space X. The basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \bigcup (U \times \{1\} \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X and each point $\langle x, 1\rangle \in X \times \{1\}$ is an isolated point.

The following theorem is proved with the similar line of proof as in [Theorem 2.14, [29]] with necessary modifications.

Theorem 3.5. *The following statements are equivalent for a space X:*

1. *X has the $\mathcal{I}H$ property;*
2. *$A(X)$ has the $\mathcal{I}H$ property;*
3. *$A(X)$ has the $W\mathcal{I}H$ property.*

Proof. (1) $\Rightarrow$ (2) Let X have the $\mathcal{I}$H property and let $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of $A(X)$. For each $n \in \mathbb{N}$ and each $x \in X$, choose an open neighborhood $W_{n_x} = (V_{n_x} \times \{0\}) \bigcup (V_{n_x} \times \{1\} \setminus \{\langle x, 1\rangle\})$ of $\langle x, 0\rangle$ satisfying that there exists some $U_{n_x} \in \mathcal{U}_n$ such that $W_{n_x} \subseteq U_{n_x}$, where V_{n_x} is an open neighborhood of x in X . Then for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{V_{n_x} : x \in X\}$ is an open cover of X. Thus $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X. By the $\mathcal{I}$H property of X, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for each $n \in \mathbb{N}$, $\{V_{n_x} : x \in F_n\}$ is a finite subset of $\mathcal{V}_n$. Let $\mathcal{V}'_n = \{V_{n_x} : x \in F_n\}$. Then for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}'_n\} \in \mathcal{I}$ and consequently $\{n \in \mathbb{N} : x \in \bigcup \mathcal{V}'_n\} \in \mathcal{F}(\mathcal{I})$. For each $n \in \mathbb{N}$ and each $x \in F_1 \cup F_2 \cup ... \cup F_n$, choose $U'_{n_x} \in \mathcal{U}_n$ such that $\langle x, 1\rangle \in U'_{n_x}$. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \{U_{n_x} : x \in F_n\} \bigcup \{U'_{n_x} : x \in F_1 \cup F_2 \cup ... \cup F_n\}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$. If $x \in \bigcup \mathcal{V}'_n$ for some n, then there is $V_{n_x} \in \mathcal{V}'_n$ such that $x \in V_{n_x}$ which implies $\langle x, 0\rangle \in W_{n_x}$ and $W_{n_x} \subseteq U_{n_x}$, $U_{n_x} \in \mathcal{U}'_n$. Thus for each $y \in A(X)$, $\{n \in \mathbb{N} : x \in \bigcup \mathcal{V}'_n\} \subset \{n \in \mathbb{N} : y \in \bigcup \mathcal{U}'_n\}$. Therefore $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{U}'_n\} \in \mathcal{I}$.

(2) $\Rightarrow$ (3) Proof is trivial.

(3) $\Rightarrow$ (1) Let $A(X)$ have the W$\mathcal{I}$H property and for each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{U^n_x : x \in X\}$ be an open cover of X, where U^n_x is an open neighborhood of x. Then $\mathcal{W}_n = \{(U_{n_x} \times \{0, 1\} \setminus \{\langle x, 1\rangle\}) : x \in X\} \bigcup \{\{\langle x, 1\rangle\} : x \in X\}$ is a sequence of open covers of $A(X)$. By the W$\mathcal{I}$H property of $A(X)$, there exists a dense set D in $A(X)$ and sequence $(E_n : n \in \mathbb{N})$ of finite sets in X such that $\{V_{n_x} : x \in E_n\}$ is a finite subset of $\mathcal{W}_n$. Let $\mathcal{W}'_n = \{V_{n_x} : x \in E_n\}$. Then for each $y \in D$, $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{W}'_n\} \in \mathcal{I}$. Consider $\mathcal{V}_n = \{U_{n_x} : x \in E_n\}$. For each $n \in \mathbb{N}$, take a finite subfamily $\mathcal{U}'_n \subseteq \mathcal{U}_n$ satisfying $E_1 \cup E_2 \cup ... \cup E_n \subseteq \bigcup \mathcal{U}'_n$. If $x \in \bigcup_{n\in\mathbb{N}} E_n$, then $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{U}'_n\} \in \mathcal{I}$. If $x \in X \setminus \bigcup_{n\in\mathbb{N}} E_n$ then $\langle x, 1\rangle \in D$, thus $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \subset \{n \in \mathbb{N} : y \notin \bigcup \mathcal{W}'_n\}$. Therefore $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$, which completes the proof. □

4. The Mildly $\mathcal{I}$-Hurewicz Property

Definition 4.1. A space X is said to have the mildly $\mathcal{I}$-Hurewicz property (in short, M$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of clopen covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

Lemma 4.2. *For a topological space X and an admissible ideal $\mathcal{I}$, then the MH property implies the $M\mathcal{I}H$ property.*

Lemma 4.3. *The $M\mathcal{I}H$ property is preserved under continuous mappings.*

Proof. The proof is easy and thus omitted. □

Theorem 4.4. *The $M\mathcal{I}H$ property is preserved under clopen subsets.*

Proof. Let X having the M$\mathcal{I}$H property and F be a clopen subset of X. Consider sequence of clopen covers $(\mathcal{U}_n : n \in \mathbb{N})$ of F. Then the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ is a clopen covers of X, for each n, $\mathcal{W}_n = \mathcal{U}_n \cup \{X \setminus F\}$. By the M$\mathcal{I}$H property of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{W}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$. Let for each n, $\mathcal{V}'_n = \mathcal{V}_n \setminus \{X \setminus F\}$. Then the $(\mathcal{V}'_n : n \in \mathbb{N})$ is a finite subset of $\mathcal{U}_n$ and for each $y \in F$, $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{V}'_n\} \in \mathcal{I}$. □

Recall that a space X is zero-dimensional if it has a base consisting of clopen sets.

Just as the Hurewicz property is equivalent to the mildly Hurewicz property in zero-dimensional space (see [Proposition 3.2, [18]]), we can similarly verify the following result.

Theorem 4.5. *Let X be a zero-dimensional space. Then X has the M$\mathcal{I}$H property if and only if X has the $\mathcal{I}$H property.*

Theorem 4.6. ([6]) *If a space X has the $\mathcal{I}$H property, then X is a Lindelöf space.*

Proof. Let $\mathcal{U}$ be an any open cover of X. Let $\mathcal{U}_n = \mathcal{U}$ for each $n \in \mathbb{N}$. Then $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of X. Apply the $\mathcal{I}$H property of X to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$. Let $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$. Clearly $\mathcal{V}$ is a countable subset of $\mathcal{U}$ also for each $x \in X$, $x \in \bigcup \mathcal{V}$. Thus $\mathcal{V}$ is countable subcover of $\mathcal{U}$. Therefore X is Lindelöf space. □

Since every σ-compact space has the Hurewicz property and hence the Lindelöf, that is, the Hurewicz property lies between σ-compactness and Lindelöfness. Also we know that the $\mathcal{I}$H property is generalization of the Hurewicz property. So from Theorem 4.6 we have the following implications.

σ-compactness $\Rightarrow$ Hurewicz property $\Rightarrow$ $\mathcal{I}$H property $\Rightarrow$ Lindelöf.

Through Theorem 4.6 and [6, Example 3.1], the open problem 3.2, suggested by the referee in [7] is addressed. We give some examples of spaces confirming the existence of spaces not having the $\mathcal{I}$H property.

Example 4.7. The uncountable space with particular point topology does not have the $\mathcal{I}$H property, since it is not the Lindelöf space.

Example 4.8. Let $X = [-1, 1] \subset \mathbb{R}$; define the either-or topology on X [25] by $\tau = \{U : \{0\} \notin U \textit{ or } (-1, 1) \subset U\}$. Then X has the $\mathcal{I}$H property, since it is compact. But the subspace $X \setminus \{0\}$ does not have the $\mathcal{I}$H property, since it is not the Lindelöf space.

Example 4.9. ([25]) Let the first uncountable ordinal be ω_1. The closed ordinal space is $[0, \omega_1] = \{ x : 0 \leq x \leq \omega_1\}$ and open ordinal space is the subspace $[0, \omega_1) = \{ x : 0 \leq x < \omega_1\}$. Sets of the form $(\alpha, \beta + 1) = (\alpha, \beta] = \{x : \alpha < x < \beta + 1\}$ form a basis for this topology.

Since every T_3 Lindelöf space is paracompact, $[0, \omega_1)$ is not the Lindelöf, and thus does not have the $\mathcal{I}$H property. But $[0, \omega_1]$ being compact satisfies the $\mathcal{I}$H property.

It can be noted that from Example 4.8, if the space X has the $\mathcal{I}$H property then the subspace of X need not have the $\mathcal{I}$H property. Also from Example 4.9, it follows that if the subspace of X has the $\mathcal{I}$H property then the space X need not have the $\mathcal{I}$H property.

Theorem 4.10. *The $\mathcal{I}$H property is preserved under closed subsets.*

Proof. Let X be a space having the $\mathcal{I}$H property and F be closed subset of X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of F. Consider $\mathcal{W}_n = \mathcal{U}_n \bigcup\{X \setminus F\}$, then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is open cover of X. Apply the $\mathcal{I}$H property to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$, we have sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{W}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$. Suppose $\mathcal{H}_n = \{U \in \mathcal{U}_n : U \in \mathcal{V}_n\}$. Let $y \in F$. Therefore $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{H}_n\} \subset \{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\}$. Hence F has the $\mathcal{I}$H property. □

Recall that a mapping $f : X \to Y$ is contra-continuous [9] if $f^{-1}(V)$ is closed in X for every open subset V of Y, and precontinuous [21] if $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ for every open subset V of Y.

Theorem 4.11. *Let X be a space having the M$\mathcal{I}$H property and $f : X \to Y$ be contra-continuous and precontinuous. Then $f(X)$ has the $\mathcal{I}$H property.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of $f(X)$. Since f is contra-continuous, for each $n \in \mathbb{N}$ and for each open set $V \in \mathcal{V}_n$, $f^{-1}(V)$ is closed in X. Also since f is precontinuous, so for each open set $V \in \mathcal{V}_n$, $f^{-1}(V) \subset Int(Cl(f^{-1}(V)))$ and so $f^{-1}(V) = Int(f^{-1}(V))$. Therefore, for each n the set $\mathcal{U}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$ is a clopen cover of X. By the M$\mathcal{I}$H property of X, there exists a sequence $(\mathcal{H}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{H}_n\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : x \in \bigcup \mathcal{H}_n\} \in \mathcal{F}(\mathcal{I})$. For each n let $\mathcal{W}_n = \{V \in \mathcal{V}_n : f^{-1}(V) \in \mathcal{H}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset $\mathcal{V}_n$. Choose $y \in f(X)$. Thus $y = f(x)$ for some $x \in X$. Then $x \in \bigcup \mathcal{H}_n$ for some n as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. Choose $f^{-1}(V) \in \mathcal{H}_n$, $V \in \mathcal{V}_n$ such that $x \in f^{-1}(V)$, $y = f(x) \in V$ that is $y \in \bigcup \mathcal{W}_n$. Therefore $\{n \in \mathbb{N} : x \in \bigcup \mathcal{H}_n\} \subset \{n \in \mathbb{N} : y \in \bigcup \mathcal{W}_n\}$. This shows that $f(X)$ has the $\mathcal{I}$H property. □

Recall that a mapping $f : X \to Y$ is called weakly continuous [20] if for each $x \in X$ and each open set V in Y containing $f(x)$ there is an open set U in X containing x such that $f(U) \subset Cl(V)$.

Theorem 4.12. *If $f : X \to Y$ is a weakly continuous mapping and X has the $\mathcal{I}$H property, then $f(X)$ has the M$\mathcal{I}$H property.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of clopen covers of $f(X)$. For each $x \in X$ and each $n \in \mathbb{N}$, there is $V_{n,x} \in \mathcal{V}_n$ such that $f(x) \in V_{n,x}$. By weakly continuity of f, there is an open set $U_{n,x}$ containing x such that $f(U_{n,x}) \subset Cl(V_{n,x}) = V_{n,x}$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{U_{n,x} : x \in X\}$. Then $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of X. Apply the $\mathcal{I}$H property of X, there is a sequence $(\mathcal{G}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{G}_n\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : x \in \bigcup \mathcal{G}_n\} \in \mathcal{F}(\mathcal{I})$. For each $n \in \mathbb{N}$, let $\mathcal{W}_n = \{V_{n,x} : f(G) \subset V_{n,x}, G \in \mathcal{G}_n\}$. Clearly $\mathcal{W}_n$ is a finite subsets of $\mathcal{V}_n$. Let $y = f(x)$ for some $x \in X$. Then $x \in \bigcup \mathcal{G}_n$ for some n as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. Choose $G \in \mathcal{G}_n$, such that $x \in G$, $y = f(x) \in f(G) \subset V_{n,x}$ that is $y \in \bigcup \mathcal{W}_n$. Therefore $\{n \in \mathbb{N} : x \in \bigcup \mathcal{G}_n\} \subset \{n \in \mathbb{N} : y \in \bigcup \mathcal{W}_n\}$. This shows that $f(X)$ has the M$\mathcal{I}$H property. □

From Theorem 4.5, Theorem 4.11 and Theorem 4.12, we have the following corollary.

Corollary 4.13. *(1) If X is a zero-dimensional space, then the $\mathcal{I}$H property is preserved by contra-continuous and precontinuous mappings.*

(2) If Y is a zero-dimensional space, then the M$\mathcal{I}$H property is preserved by contra-continuous and precontinuous mappings.

(3) If X is a zero-dimensional space, then the M$\mathcal{I}$H property is preserved by weakly continuous mappings.

(4) If Y is a zero-dimensional space, then the $\mathcal{I}$H property is preserved by weakly continuous mappings.

Theorem 4.14. *If $f : X \to Y$ is an open, perfect mapping from a space X onto a space Y having the $\mathcal{I}$H property, then X has the M$\mathcal{I}$H property.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of clopen covers of X. For each $y \in Y$ the set $F_y = f^{-1}\{y\}$ is compact so that for each $n \in \mathbb{N}$ there is a finite set cover $\mathcal{V}_{y,n}$ of F_y such that $\mathcal{V}_{n,y} \subset \mathcal{U}_n$. Let $V_{y,n} = \bigcup \mathcal{V}_{y,n}$. Since f is a open mapping, for each $y \in Y$ and each $n \in \mathbb{N}$ there is an open set $W_{y,n}$ in Y such that $y \in W_{y,n}$ and $f^{-1}(W_{y,n}) \subset V_{y,n}$. For each $n \in \mathbb{N}$ let $\mathcal{W}_n = \{W_{y,n} : y \in Y\}$. Then $(\mathcal{W}_n : n \in \mathbb{N})$ is a sequence of open cover of Y. Apply the $\mathcal{I}$H property to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$, there is a sequence $(\mathcal{H}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{W}_n$, and each $y \in Y$, $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{H}_n\} \in \mathcal{I}$. So $\{n \in \mathbb{N} : y \in \bigcup \mathcal{H}_n\} \in \mathcal{F}(\mathcal{I})$. For each n, and each $H \in \mathcal{H}_n$ there is a finite $\mathcal{U}_{H,n} \subset \mathcal{U}$ with $f^{-1}(H) \subset \bigcup \mathcal{U}_{H,n}$. If $\mathcal{G}_n = \{U \in \mathcal{U}_n : U \in \mathcal{U}_{H,n}, H \in \mathcal{H}_n\}$. Then for each n, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$. Let $x \in X$ and $y = f(x)$. Let $x \in \bigcup \mathcal{H}_n$ for some n. Choose $H \in \mathcal{H}_n$ such that $x \in H$, then there is a finite set $\mathcal{U}_{H,n}$ such that $f^{-1}(H) \subset \bigcup \mathcal{U}_{H,n}$. Then $y \in \bigcup \mathcal{U}_{H,n}$, choose $U \in \mathcal{U}_{H,n}$ such that $y \in U$. Therefore $\{n \in \mathbb{N} : y \in \bigcup \mathcal{H}_n\} \subset \{n \in \mathbb{N} : x \in \bigcup \mathcal{G}_n\}$. This shows that X has the M$\mathcal{I}$H property. □

Theorem 4.15. *If $f : X \to Y$ is an open bijective mapping from a space X onto a space Y having the $\mathcal{I}$H property, then X has the M$\mathcal{I}$H property.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of clopen covers of X. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{f(U) : U \in \mathcal{U}_n\}$ covers Y. By the $\mathcal{I}$H property of Y, there is a sequence $(\mathcal{G}_n : n \in \mathbb{N})$ such that for each n $\mathcal{G}_n$ is a finite subset $\mathcal{V}_n$ and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{G}_n\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : y \in \bigcup \mathcal{G}_n\} \in \mathcal{F}(\mathcal{I})$. For each n let $\mathcal{W}_n = \{U : f(U) \in \mathcal{G}_n\}$. Let $y \in Y$. Then $y = f(x)$ for some $x \in X$. Let $y \in \bigcup \mathcal{G}_n$ for some n. Choose $f(U) \in \mathcal{G}_n$, $U \in \mathcal{U}_n$ such that $y = f(x) \in f(U)$ which implies $x \in U$ that is $x \in \bigcup \mathcal{W}_n$ for some n. Therefore $\{n \in \mathbb{N} : y \in \bigcup \mathcal{G}_n\} \subset \{n \in \mathbb{N} : x \in \bigcup \mathcal{W}_n\}$. This shows that X has the M$\mathcal{I}$H property. □

By Theorem 4.5, Theorem 4.14 and Theorem 4.15, we conclude the following corollary.

Corollary 4.16. *(1) If X is a zero-dimensional space, then the $\mathcal{I}$H property is inverse invariant under open, perfect mappings.*

(2) If Y is a zero-dimensional space, then the M$\mathcal{I}$H property is inverse invariant under open, perfect mappings.

(3) If X is a zero-dimensional space, then the $\mathcal{I}$H property is inverse invariant under open bijective mappings.

(4) If Y is a zero-dimensional space, then the M$\mathcal{I}$H property is inverse invariant under open bijective mappings.

By Corollary 4.13 and Corollary 4.16, we conclude the following corollaries.

Corollary 4.17. *Let $f : X \to Y$ be a open, perfect, contra-continuous and precontinuous and X and Y both zero-dimensional spaces. Then:*

(1) X has the $\mathcal{I}$H property if and only if $f(X)$ has the $\mathcal{I}$H property.

(2) X has the M$\mathcal{I}$H property if and only if $f(X)$ has the M$\mathcal{I}$H property.

Corollary 4.18. *Let $f : X \to Y$ be a open, perfect, weakly-continuous and X and Y both zero-dimensional spaces. Then:*

(1) X has the $\mathcal{I}$H property if and only if $f(X)$ has the $\mathcal{I}$H property.

(2) X has the M$\mathcal{I}$H property if and only if $f(X)$ has the M$\mathcal{I}$H property.

Corollary 4.19. *Let $f : X \to Y$ be a open, bijective, contra-continuous and precontinuous and X and Y both zero-dimensional spaces. Then:*

(1) X has the $\mathcal{I}$H property if and only if Y has the $\mathcal{I}$H property.

(2) X has the M$\mathcal{I}$H property if and only if Y has the M$\mathcal{I}$H property.

Corollary 4.20. *Let $f : X \to Y$ be a open, bijective, weakly-continuous and X and Y both zero-dimensional spaces. Then:*

(1) X has the $\mathcal{I}$H property if and only if Y has the $\mathcal{I}$H property.

(2) X has the M$\mathcal{I}$H property if and only if Y has the M$\mathcal{I}$H property.

5. The Star-K-$\mathcal{I}$-Hurewicz Property

Definition 5.1. A space X is said to have the star-K-$\mathcal{I}$-Hurewicz property (in short, SK$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X such that and for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{U}_n)\} \in \mathcal{I}$.

It is clear from the definitions that for a topological space X and an admissible ideal $\mathcal{I}$, $\mathcal{I}$H $\Rightarrow$ SS$\mathcal{I}$H $\Rightarrow$ SK$\mathcal{I}$H $\Rightarrow$ S$\mathcal{I}$H.

We will now investigate some preservations properties of the SK$\mathcal{I}$H property.

Theorem 5.2. *The SK$\mathcal{I}$H property is preserved under clopen subsets.*

Proof. Let X be a space having the SK$\mathcal{I}$H property and Y be a clopen subset of X. Consider $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \mathcal{U}_n \bigcup \{X \setminus Y\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X and by the SK$\mathcal{I}$H property of X, there exists a sequence $(K_n : n \in \mathbb{N})$ compact subsets of X such that and for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{V}_n)\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : x \in St(K_n, \mathcal{V}_n)\} \in \mathcal{F}(\mathcal{I})$. For each $n \in \mathbb{N}$, let $K_n' = K_n \cap Y$. Since Y is a closed subset of X, thus $(K_n' : n \in \mathbb{N})$ is a sequence of compact subsets of Y. Let $y \in X \cap Y$. If $y \in St(K_n, \mathcal{V}_n)$ for some n. By the construction of $\mathcal{V}_n$ choose $U \in \mathcal{U}_n$ such that $y \in U$ and $U \cap K_n \cap Y \neq \phi$, which implies $U \cap K_n' \neq \phi$. Therefore $\{n \in \mathbb{N} : x \in St(K_n, \mathcal{V}_n)\} \subset \{n \in \mathbb{N} : y \in St(K_n', \mathcal{U}_n)\}$. This completes the proof. □

Theorem 5.3. *The SK$\mathcal{I}$H property is preserves under continuous mappings.*

Proof. The proof is easy and thus omitted. □

Theorem 5.4. *The SK$\mathcal{I}$H property is inverse invariant under perfect open mappings.*

Proof. Let $f : X \to Y$ be a perfect open onto mapping and let Y be a space having the SK$\mathcal{I}$H property. Since $f(X)$ is open and closed in Y, suppose that $f(X) = Y$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Then for each $y \in Y$ and for each $n \in \mathbb{N}$, there is a finite subset $\mathcal{U}_{n_y}$ of $\mathcal{U}$ such that $f^{-1}\{y\} \subset \bigcup \mathcal{U}_{n_y}$ and $U \cap f^{-1}\{y\} \neq \phi$ for each $U \in \mathcal{U}_{n_y}$. Let V_{n_y} is an open neighborhood of y in Y such that $f^{-1}(V_{n_y}) \subset \bigcup\{U : U \in \mathcal{U}_{n_y}\}$, also suppose that $V_{n_y} \subset \bigcap\{f(U) : U \in \mathcal{U}_{n_y}\}$ as f is open. Now for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$ is an open cover of Y. Apply the SK$\mathcal{I}$H property of Y to $(\mathcal{V}_n : n \in \mathbb{N})$, there exist a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of Y such that and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin St(K_n, \mathcal{V}_n)\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : y \in St(K_n, \mathcal{V}_n)\} \in \mathcal{F}(\mathcal{I})$. Since f is perfect, then the sequence $(f^{-1}(K_n) : n \in \mathbb{N})$ is the sequence of compact subsets of X. Let $x \in X$, then there exist $y \in Y$ such that $f(x) = y$. Now let $y \in St(K_n, \mathcal{V}_n)$ for some $n \in \mathbb{N}$. Choose $V_{n_y} \in \mathcal{V}_n$ such that $y \in V_{n_y}$ and $V_{n_y} \cap K_n \neq \phi$. Hence $x \in f^{-1}(V_{n_y}) \subset \bigcup\{U : U \in \mathcal{U}_{n_y}\}$. Choose $U \in \mathcal{U}_{n_y}$ with $x \in U$ such that $V_{n_y} \subset f(U)$ and hence $U \cap f^{-1}(K_n) \neq \phi$. Therefore $\{n \in \mathbb{N} : y \in St(K_n, \mathcal{V}_n)\} \subset \{n \in \mathbb{N} : x \in St(f^{-1}(K_n), \mathcal{U}_n)\}$. Hence X has the SK$\mathcal{I}$H property. □

By the above theorem we have the following corollary.

Corollary 5.5. *If X has the SK$\mathcal{I}$H property and Y is compact, then $X \times Y$ has the SK$\mathcal{I}$H property.*

In [6] it was shown that a paracompact Hausdorff space X has the S$\mathcal{I}$H property if and only if it has the $\mathcal{I}$H property. Thus we have following theorem.

Theorem 5.6. *Let X be a paracompact space. Then the following statements are equivalent:*

1. *X has the $\mathcal{I}$H property;*
2. *X has the SS$\mathcal{I}$H property;*
3. *X has the SK$\mathcal{I}$H property;*
4. *X has the S$\mathcal{I}$H property.*

Theorem 5.7. ([6]) *$\mathcal{I}$H, S$\mathcal{I}$H and MS$\mathcal{I}$H properties are equivalent for a paracompact Hausdorff zero-dimensional space.*

In [6], Das et al. provides some examples of spaces where the properties $\mathcal{I}$H, S$\mathcal{I}$H and SS$\mathcal{I}$H are different.

From Theorem 4.5, Theorem 5.6 and Theorem 5.7, we obtain the following corollary.

Corollary 5.8. *Let X be a paracompact zero-dimensional space. Then the following statements are equivalent:*

1. *X has the $\mathcal{I}$H property;*
2. *X has the SS$\mathcal{I}$H property;*
3. *X has the SK$\mathcal{I}$H property;*
4. *X has the S$\mathcal{I}$H property;*
5. *X has the MS$\mathcal{I}$H property;*
6. *X has the M$\mathcal{I}$H property.*

6. Relationships and Remarks

Das et. al. [6] provided a diagram of relationships among the different variants of the $\mathcal{I}$H property. Based on this diagram some natural questions arise that under which conditions the converse of implications hold. In this section we give answers to these questions. Also this diagram includes the properties, viz. M$\mathcal{I}$H and SK$\mathcal{I}$H as well.

Remark 6.1. In the definition of the $\mathcal{I}$H property, if we take ideals as finite subsets of $\mathbb{N}$, that is, $\mathcal{I}_{fin}$, then the $\mathcal{I}$H property implies the Hurewicz property. Similarly the SS$\mathcal{I}$H property implies the SSH property, the SK$\mathcal{I}$H property implies the SKH property and the S$\mathcal{I}$H property implies the SH property. That is under $\mathcal{I}_{fin}$ ideals, the $\mathcal{I}$H property is equivalent to the Hurewicz property, the SS$\mathcal{I}$H property is equivalent to the SSH property, the SK$\mathcal{I}$H property is equivalent to the SKH property and the S$\mathcal{I}$H property is equivalent to the SH property.

Theorem 6.2. *If a paracompact space X has the WS$\mathcal{I}$H property then it has the W$\mathcal{I}$H property.*

Proof. Let X has the WS$\mathcal{I}$H property and $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. By the Stone characterization of paracompactness [10] for each $n \in \mathbb{N}$, $\mathcal{U}_n$ has an open star-refinement, say $\mathcal{V}_n$. By the WS$\mathcal{I}$H property of X, there is a dense set $Y \subset X$ and a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin St(\cup\mathcal{W}_n, \mathcal{V}_n)\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : y \in St(\cup\mathcal{W}_n, \mathcal{V}_n)\} \in \mathcal{F}(\mathcal{I})$. For each $W \in \mathcal{W}_n$, let $U_W \in \mathcal{U}_n$ such that $St(W, \mathcal{V}_n) \subset U_W$. Let $\mathcal{H}_n = \{U_W : W \in \mathcal{W}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$. Let $y \in St(\cup\mathcal{W}_n, \mathcal{V}_n)$ for some n, as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. Choose $W \in \mathcal{W}_n$ such that $y \in St(W, \mathcal{V}_n)$ and $St(W, \mathcal{V}_n) \subset U_W$. Thus $y \in \cup\mathcal{H}_n$ for some n. Therefore $\{n \in \mathbb{N} : y \in St(\cup\mathcal{W}_n, \mathcal{V}_n)\} \subset \{n \in \mathbb{N} : y \in \cup\mathcal{H}_n\}$. Hence X has the W$\mathcal{I}$H property. □

Theorem 6.3. *If X has the W$\mathcal{I}$H property and for each dense set $Y \subseteq X$ the subspace $X \setminus Y$ has the $\mathcal{I}$H property, then X has the $\mathcal{I}$H property.*

Proof. Let X has the W$\mathcal{I}$H property and $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. By the W$\mathcal{I}$H property of X, there is a dense set $Y \subset X$ and a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin \bigcup\mathcal{V}_n\} \in \mathcal{I}$. Since Y is dense in X thus $X \setminus Y$ has the $\mathcal{I}$H property. Then for each $n \in \mathbb{N}$ there is a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in X \setminus Y$, $\{n \in \mathbb{N} : y \notin \bigcup\mathcal{V}'_n\} \in \mathcal{I}$. Now let $\mathcal{W}_n = \mathcal{V}_n \bigcup \mathcal{V}'_n$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$ and For each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup\mathcal{W}_n\}$. Therefore X has the $\mathcal{I}$H property. □

If $\mathcal{A} \subset \mathcal{B}$, then $St(\mathcal{A}, \mathcal{U}) \subset St(\mathcal{B}, \mathcal{U})$, using this and by the similar proof of Theorem 6.3, we have the following theorem.

Theorem 6.4. *If X has the WS$\mathcal{I}$H property and for each dense set $Y \subseteq X$ the subspace $X \setminus Y$ has the S$\mathcal{I}$H property, then X has the S$\mathcal{I}$H property.*

The following diagram summarizes the implications between the several notions investigated in this paper. The dotted, reverse arrows indicate conditions under which some implications may be reversed.

In the diagram, (*) means $\mathcal{I}_{fin}$ (case of Remark 6.1),

(**) means paracompact (cases of Theorem 6.2 and [Theorem 4.5, [6]]),

(**1) means metacompact (case of [Theorem 4.6, [6]])

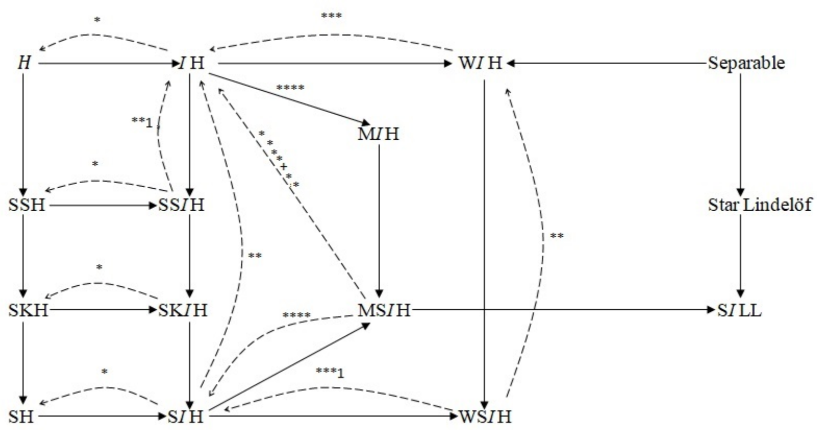
(***) means complement of dense set is $\mathcal{I}$H (case of Theorem 6.3),

(***1) means complement of dense set is S$\mathcal{I}$H (case of Theorem 6.4),

(****) means zero-dimensional (cases of Theorem 4.5 and [Theorem 4.5, [6]]),

(****+**) means zero-dimensional and paracompact (case of [Theorem 5.2, [6]]).

But still there are several open problems of constructing suitable examples to make the diagram complete.



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