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On The Almost Everywhere Statistical Convergence of Sequences of Fuzzy Numbers

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Abstract. In this paper, we define the concept of almost everywhere statistical convergence of a sequence of fuzzy numbers and prove that a sequence of fuzzy numbers is almost everywhere statistically convergent if and only if its statistical limit inferior and limit superior are equal. To achieve this result, new representations for statistical limit inferior and limit superior of a sequence of fuzzy numbers are obtained and we show that some properties of statistical limit inferior and limit superior can be easily derived from these representations.

1. Introduction

Fridy and Orhan [13] prove that a sequence (x_n) of real numbers is statistically convergent if and only if its statistical limit inferior and superior are equal. However, in fuzzy analysis this idea is not valid. Until now, two kinds of statistical convergence have been studied for sequences of fuzzy numbers. One of them is statistical convergence with respect to the supremum metric, which is defined by Nuray and Savaş [15]. The other is levelwise statistical convergence, which is defined by Aytar and Pehlivan [6]. Aytar et al. show that a sequence (u_n) of fuzzy numbers may not be statistically convergent while its statistical limit inferior and limit superior are equal. In this case the question that arises here is whether the choice of convergence is true.

In this paper we answer the above question. We define new concept of statistical convergence, called almost everywhere statistical convergence, for sequences of fuzzy numbers. Then we prove that a sequence (u_n) of fuzzy numbers is almost everywhere statistically convergent to fuzzy numbers μ if and only if statistical limit inferior and limit superior are equal to μ.

To accomplish this objective we give new representations for statistical limit inferior and limit superior by means of the nested intervals families

$$\{[st-\liminf u_n^-(\lambda), st-\liminf u_n^+(\lambda)] : \lambda \in [0,1]\} \text{ and } \{[st-\limsup u_n^-(\lambda), st-\limsup u_n^+(\lambda)] : \lambda \in [0,1]\} \quad (1)$$

,respectively. By using this construction, the statistical limit inferior and limit superior can be easily calculated. Furthermore, we obtain a necessary condition under which the nested interval families in (1) can determine a fuzzy number.

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2. Definitions and Notation

The concept of statistical convergence of sequences was first introduced by Fast [11] and further studied by Šalát [17], Fridy [12], Connor [9] and many others. First, we recall some definitions concerning this concept.

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{N}$ be the set of positive integers. The natural density of a subset A of $\mathbb{N}$ is given by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in A\}|$$

if this limit exists, where $|A|$ denotes the cardinality of the set A.

A sequence $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to some number l if for every $\varepsilon > 0$

$$\delta\left(\{k : |x_k - l| \ge \varepsilon\}\right) = 0.$$

In this case, we write st$-\lim x_k = l$. The sequence $x = (x_k)$ is said to be statistically bounded if there exists a real number M such that the set

$$\{k \in \mathbb{N} : |x_k| > M\}$$

has natural density zero. For a sequence $x = (x_k)$ of real numbers, the notions of statistical limit superior and limit inferior are defined as follows

$$st\text{-}\liminf x := \begin{cases} \inf A_x, & A_x \ne \emptyset, \\ \infty, & \text{otherwise}, \end{cases}$$

$$st\text{-}\limsup x := \begin{cases} \sup B_x, & B_x \ne \emptyset, \\ -\infty, & \text{otherwise}, \end{cases}$$

where $A_x := \{a \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k < a\}) \ne 0\}$ and $B_x := \{b \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k > b\}) \ne 0\}$.

Lemma 2.1. *[13] If $\beta = st - \limsup x$ is finite, then for every $\varepsilon > 0$,*

$$\delta\left(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}\right) \ne 0 \quad \textit{and} \quad \delta\left(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}\right) = 0. \tag{2}$$

Conversely, if (2) holds for every $\varepsilon > 0$ then $\beta = st - \limsup x$.

The dual statement for $st - \liminf x$ is as follows:

Lemma 2.2. *[13] If $\alpha = st - \liminf x$ is finite, then for every $\varepsilon > 0$,*

$$\delta\left(\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\}\right) \ne 0 \quad \textit{and} \quad \delta\left(\{k \in \mathbb{N} : x_k < \alpha - \varepsilon\}\right) = 0. \tag{3}$$

Conversely, if (3) holds for every $\varepsilon > 0$ then $\alpha = st - \liminf x$.

Theorem 2.3. *[13] The statistically bounded sequence $x = (x_k)$ of real numbers is statistically convergent if and only if $st - \liminf x = st - \limsup x$.*

In this section, we briefly recall some of the basic notions related with fuzzy numbers and we refer to [8, 10] for more details.

Fuzzy set $u \in E^1$ is called a fuzzy number if u is a normal, convex fuzzy set, upper semi-continuous and $supp\ u = \mathrm{cl}\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact. We use E^1 to denote the fuzzy number space. For $\lambda \in (0, 1]$ let $[u]_\lambda = \{x \in \mathbb{R} \mid u(x) \ge \lambda\}$ and $[u]_0 = supp\ u$.

For $r \in \mathbb{R}$, define a fuzzy number $\chi_{\{r\}}$ by

$$\chi_{\{r\}}(x) := \begin{cases} 1 \quad , & \text{if } x = r, \\ 0 \quad , & \text{if } x \ne r. \end{cases}$$

for any $x \in \mathbb{R}$.

Remark 2.4. *$u \in E^1$ if and only if $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$.*

From this characterization of fuzzy numbers, it can be seen that a fuzzy number is determined by the endpoints of the intervals.

Theorem 2.5. *[18, Theorem 1.1] Let $u \in E^1$ and $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for each $\lambda \in [0,1]$. Then the following statements hold:*

(i) *$u^-(\lambda)$ is a bounded and non-decreasing left continuous function on $(0,1]$;*

(ii) *$u^+(\lambda)$ is a bounded and non-increasing left continuous function on $(0,1]$;*

(iii) *The functions $u^-(\lambda)$ and $u^+(\lambda)$ are right continuous at the point $\lambda = 0$;*

(iv) *$u^-(1) \le u^+(1)$.*

Conversely, if the pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ satisfy the conditions (i)–(iv), then there exists a unique $u \in E^1$ such that $[u]_\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0,1]$.

For $u, v, w \in E^1$ and $k \in \mathbb{R}$ the addition and the scalar multiplication are defined respectively by

$$\begin{aligned} u + v = w \quad &\Longleftrightarrow \quad [w]_\lambda = [u]_\lambda + [v]_\lambda \text{ for all } \lambda \in [0,1] \\ &\Longleftrightarrow \quad w^-(\lambda) = u^-(\lambda) + v^-(\lambda) \text{ and } w^+(\lambda) = u^+(\lambda) + v^+(\lambda) \text{ for all } \lambda \in [0,1], \end{aligned}$$

$$[ku]_\lambda = k[u]_\lambda \text{ for all } \lambda \in [0,1].$$

The partial ordering relation on E^1 is defined as follows:

$$u \preceq v \Longleftrightarrow u^-(\lambda) \le v^-(\lambda) \text{ and } u^+(\lambda) \le v^+(\lambda) \text{ for all } \lambda \in [0,1].$$

$u \prec v$ means $u \preceq v$ and at least one of $u^-(\lambda) < v^-(\lambda)$ or $u^+(\lambda) < v^+(\lambda)$ holds for some $\lambda \in [0,1]$. If $u \preceq v$ or $v \preceq u$, we say u, v are comparable.

Let us denote by W the set of all nonempty compact intervals of the real line $\mathbb{R}$. Hausdorff metric d_H on W is defined by

$$d_H(A,B) := \max\{|A^- - B^-|, |A^+ - B^+|\}$$

where $A = [A^-, A^+], B = [B^-, B^+] \in W$. Now, we may define the metric D on E^1 by means of the Hausdorff metric d_H as follows

$$D(u,v) := \sup_{\lambda \in [0,1]} d_H([u]_\lambda, [v]_\lambda) := \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$

Several types of convergence of sequences of fuzzy numbers have been introduced (see [10, 19, 21, 22]). Let (u_n) be a sequence of fuzzy numbers and $\mu \in E^1$.

(u_n) is said to be convergent to μ with respect to metric D if $\lim_{n\to\infty} D(u_n, \mu) = 0$. In this case we write $u_n \xrightarrow{D} \mu$.

(u_n) is said to be levelwise convergent to $\mu \in E^1$, written as $u_n \xrightarrow{l} \mu$, if $\lim_{n\to\infty} d_H([u_n]_\lambda, [\mu]_\lambda) = 0$ for all $\lambda \in [0,1]$ or equivalently,

$$\lim_{n\to\infty} u_n^-(\lambda) = \mu^-(\lambda) \text{ and } \lim_{n\to\infty} u_n^+(\lambda) = \mu^+(\lambda)$$

for all $\lambda \in [0,1]$.

(u_n) is said to be almost everywhere converges to μ if $\lim_{n\to\infty} d_H([u_n]_\lambda, [\mu]_\lambda) = 0$ holds for λ almost everywhere on $[0,1]$. In this case we write $u_n \xrightarrow{a.e.} \mu$.

Lemma 2.6. *[21, Lemma 2.1] Let $u, v \in E^1$. If $[u]_\lambda = [v]_\lambda$ for almost everywhere on $[0,1]$, then $u = v$.*

The statistical boundedness of a sequence of fuzzy numbers was introduced and studied by Aytar and Pehlivan [5]. The sequence $u = (u_k)$ is said to be statistically bounded if there exists a real number M such that the set

$$\{k \in \mathbb{N} : D(u_k, \overline{0}) > M\}$$

has natural density zero.

Wu and Wu [20] proved the existence of supremum and infimum for a bounded set of fuzzy numbers according to relation $\preceq$. Fang and Huang [10] improved the expressions of the supremum and infimum. By means of the concepts of "sup" and "inf" of sets of fuzzy numbers, Aytar et al.[2] defined the concept of statistical limit superior and limit inferior of statistically bounded sequences of fuzzy numbers. Given $u = (u_k)$, define the following sets:

$$\begin{aligned} A_u &= \left\{\mu \in E^1 : \delta\left(\{k \in \mathbb{N} : u_k \prec \mu\}\right) \neq 0\right\}, \\ \overline{A}_u &= \left\{\mu \in E^1 : \delta\left(\{k \in \mathbb{N} : u_k \succ \mu\}\right) = 1\right\}, \\ B_u &= \left\{\mu \in E^1 : \delta(\{k \in \mathbb{N} : u_k \succ \mu\}) \neq 0\right\}, \\ \overline{B}_u &= \left\{\mu \in E^1 : \delta(\{k \in \mathbb{N} : u_k \prec \mu\}) = 1\right\}. \end{aligned}$$

Theorem 2.7. *[1, Theorem 1] If the sequence $u = (u_k) \subseteq E^1$ is statistically bounded, then $\inf A_u = \sup \overline{A}_u$ and $\sup B_u = \inf \overline{B}_u$.*

For $u = (u_k)$, statistical limit inferior and limit superior defined as follows:

$$\begin{aligned} st - \operatorname{Lim\,inf} u_k &= \inf A_u \\ st - \operatorname{Lim\,sup} u_k &= \sup B_u. \end{aligned}$$

3. Main results

In this section, we give more useful expressions for endpoints of level sets of statistical limit inferior and limit superior.

Theorem 3.1. *Let (u_n) be a statistically bounded sequence of fuzzy numbers. Then $st - \operatorname{Lim\,sup} u_n = \mu$ has the following representation:*

$$\mu = st - \operatorname{Lim\,sup} u_n = \bigcup_{\lambda \in (0,1]} \lambda \left[st\text{-}\limsup u_n^-(\lambda), st - \limsup u_n^+(\lambda)\right],$$

$$\mu^-(\lambda) = \sup_{r<\lambda} st\text{-}\limsup u_n^-(r), \quad \mu^+(\lambda) = \inf_{r<\lambda} st\text{-}\limsup u_n^+(r), \tag{4}$$

$$\mu^-(0) = \inf_{\lambda>0} st\text{-}\limsup u_n^-(\lambda), \quad \mu^+(0) = \sup_{\lambda>0} st\text{-}\limsup u_n^+(\lambda) \tag{5}$$

for each $\lambda \in (0,1]$. Dually, $\nu = st - \operatorname{Lim\,inf} \mathrm{u_n}$ has the following representation:

$$\begin{aligned} \nu = st\text{-}\operatorname{Lim\,inf} u_n &= \bigcup_{\lambda \in (0,1]} \lambda \left[st\text{-}\liminf u_n^-(\lambda), st\text{-}\liminf u_n^+(\lambda)\right], \\ \nu^-(\lambda) &= \sup_{r<\lambda} st\text{-}\liminf u_n^-(r), \quad \nu^+(\lambda) = \inf_{r<\lambda} st\text{-}\liminf u_n^+(r), \\ \nu^-(0) &= \inf_{\lambda>0} st\text{-}\liminf u_n^-(\lambda), \quad \nu^+(0) = \sup_{\lambda>0} st\text{-}\liminf u_n^+(\lambda) \end{aligned}$$

for each $\lambda \in (0,1]$.

Proof. We prove the result only for st−Lim sup. Since (u_n) is a statistically bounded sequence, for each $\lambda \in [0,1]$, $(u_n^-(\lambda))$ and $(u_n^+(\lambda))$ are statistically bounded sequences. Therefore the real numbers $st\text{-}\limsup u_n^-(\lambda)$ and $st\text{-}\limsup u_n^+(\lambda)$ exist. So the interval

$$H(\lambda) = [st\text{-}\limsup u_n^-(\lambda), st\text{-}\limsup u_n^+(\lambda)]$$

can be defined. By Theorem 2.5, $u_n^-(\lambda)$ and $u_n^+(\lambda)$ are nondecreasing and nonincreasing functions with respect to λ for fixed n, respectively. So we obtain $st\text{-}\limsup u_n^-(\lambda)$ and $st\text{-}\limsup u_n^+(\lambda)$ are nondecreasing and nonincreasing functions on $[0,1]$, respectively. That is, for $0 < r < \lambda \leq 1$,

$$st\text{-}\limsup u_n^-(r) \leq st\text{-}\limsup u_n^-(\lambda) \quad \text{and} \quad st\text{-}\limsup u_n^+(r) \geq st\text{-}\limsup u_n^+(\lambda). \tag{6}$$

Thus, we have $H(\lambda) \subseteq H(r)$. So, there exists a fuzzy set μ on $\mathbb{R}$ such that

$$\mu = \bigcup_{\lambda \in (0,1]} \lambda \, [st\text{-}\limsup u_n^-(\lambda), st\text{-}\limsup u_n^+(\lambda)]$$

and

$$\begin{aligned}
[\mu]_\lambda &= \bigcap_{r<\lambda} H(r) \\
&= \bigcap_{r<\lambda} [st\text{-}\limsup u_n^-(r), st\text{-}\limsup u_n^+(r)] \\
&= \left[\sup_{r<\lambda} st\text{-}\limsup u_n^-(r), \inf_{r<\lambda} st\text{-}\limsup u_n^-(r)\right]
\end{aligned} \tag{7}$$

for each $\lambda \in (0,1]$. Furthermore, for each $\lambda \in (0,1]$ and $r \in (0,\lambda)$, we have

$$\begin{aligned}
[\mu]_\lambda &\subseteq [st\text{-}\limsup u_n^-(r), st\text{-}\limsup u_n^+(r)] \\
&\subseteq \left[\inf_{\lambda>0} st\text{-}\limsup u_n^-(\lambda), \sup_{\lambda>0} st\text{-}\limsup u_n^+(\lambda)\right].
\end{aligned}$$

This implies that

$$[\mu]_0 = cl\left(\bigcup_{\lambda \in (0,1]} [\mu]_\lambda\right) \subseteq \left[\inf_{\lambda>0} st\text{-}\limsup u_n^-(\lambda), \sup_{\lambda>0} st\text{-}\limsup u_n^+(\lambda)\right].$$

Hence $[\mu]_0$ is a closed interval. Therefore, we know that $\mu \in E^1$ by Remark 2.4. By (7) we have

$$\mu^-(\lambda) = \sup_{r<\lambda} st\text{-}\limsup u_n^-(r), \quad \mu^+(\lambda) = \inf_{r<\lambda} st\text{-}\limsup u_n^+(r) \quad \text{for } \lambda \in (0,1]$$

$$\mu^-(0) = \inf_{\lambda>0} \sup_{r<\lambda} st\text{-}\limsup u_n^-(r), \quad \mu^+(0) = \sup_{\lambda>0} \inf_{r<\lambda} st\text{-}\limsup u_n^+(r).$$

(4) is proved. We prove the first equation in (5). Using the similar way the second equation in (5) can be proved. For $r \in (0,\lambda)$ since

$$st\text{-}\limsup u_n^-(r) \geq \inf_{\lambda>0} st\text{-}\limsup u_n^-(\lambda),$$

we have

$$\inf_{\lambda>0} \sup_{r<\lambda} st\text{-}\limsup u_n^-(r) \geq \inf_{\lambda>0} st\text{-}\limsup u_n^-(\lambda). \tag{8}$$

By (6), for $r \in (0,\lambda)$ we obtain

$$st\text{-}\limsup u_n^-(r) \leq st\text{-}\limsup u_n^-(\lambda).$$

Therefore for each $\lambda \in (0,1]$

$$\sup_{r<\lambda} st\text{-}\limsup u_n^-(r) \le st\text{-}\limsup u_n^-(\lambda).$$

So we have

$$\inf_{\lambda>0}\sup_{r<\lambda} st\text{-}\limsup u_n^-(r) \le \inf_{\lambda>0} st\text{-}\limsup u_n^-(\lambda). \tag{9}$$

From (8) and (9) we obtain the first equation in (5).

Now, we prove that $\mu = st\text{-}\mathrm{Lim}\sup u_n$. Let $b \in B_u$. Then $\delta(\{k \in \mathbb{N} : u_k \succ b\}) \neq 0$. So, for each $\lambda \in (0,1]$

$$\delta\left(\{k \in \mathbb{N} : u_k^-(\lambda) \ge b^-(\lambda)\}\right) \neq 0 \text{ and } \delta\left(\{k \in \mathbb{N} : u_k^+(\lambda) \ge b^+(\lambda)\}\right) \neq 0.$$

This implies that

$$st\text{-}\limsup u_n^-(\lambda) \ge b^-(\lambda) \quad \text{and} \quad st\text{-}\limsup u_n^+(\lambda) \ge b^+(\lambda).$$

Therefore, we have $\mu \succeq b$. Since b is an arbitrary element of B_u, we get

$$\mu \succeq \sup B_u. \tag{10}$$

Conversely, let $b \in \overline{B}_u$ be given. Then $\delta(\{k \in \mathbb{N} : u_k \prec \mu\}) = 1$. For each $\lambda \in (0,1]$ we get

$$\delta\left(\{k \in \mathbb{N} : u_k^-(\lambda) \le b^-(\lambda)\}\right) = 1, \quad \delta\left(\{k \in \mathbb{N} : u_k^+(\lambda) \le b^+(\lambda)\}\right) = 1.$$

This implies

$$st\text{-}\limsup u_n^-(\lambda) \le b^-(\lambda) \quad \text{and} \quad st\text{-}\limsup u_n^+(\lambda) \le b^+(\lambda).$$

Therefore $\mu \preceq b$. Since b is an arbitrary element of $\overline{B}_u$ we have

$$\mu \preceq \inf \overline{B}_u. \tag{11}$$

Combining (10) with (11) we get $\sup B_u \preceq \mu \preceq \inf \overline{B}_u$. By Theorem 2.7 we have $\mu = st\text{-}\mathrm{Lim}\sup u_n$. □

Example 3.2. *Define the sequence $u = (u_n)$ of fuzzy numbers as follows:*

$$u_n = \begin{cases} w_n, & \text{if } n \text{ is an odd nonsquare,} \\ \chi_{\{-n\}}, & \text{if } n \text{ is an odd square,} \\ \chi_{\{n\}}, & \text{if } n \text{ is an even square,} \\ v_n, & \text{if } n \text{ is an even nonsquare,} \end{cases}$$

,where

$$w_n(x) = \begin{cases} 1\ , & \text{if } x \in [-\frac{1}{2}, 0], \\ \frac{n-1}{2n}\ , & \text{if } x \in [-1, -\frac{1}{2}), \\ 0\ , & \text{otherwise,} \end{cases}$$

and

$$v_n(x) = \begin{cases} 1 - \sqrt[n]{x-1}\ , & \text{if } x \in [1,2], \\ 0\ , & \text{otherwise.} \end{cases}$$

*The sequence is statistically bounded since the set of squares has density zero. So st-*Lim sup u_n *and st-*Lim inf u_n *exist. Now we calculate these. Firstly, we find endpoints of $\lambda-$level sets $u = (u_n)$ as follows:*

$$u_n^+(\lambda) = \begin{cases} 0 & \text{if } n \text{ is an odd nonsquare,} \\ -n, & \text{if } n \text{ is an odd square,} \\ n, & \text{if } n \text{ is an even square,} \\ 1 + (1-\lambda)^n, & \text{if } n \text{ is an even nonsquare} \end{cases}$$

and

$$u_n^-(\lambda)=\begin{cases}1, & \text{if } n \text{ is an even nonsquare},\\ -n, & \text{if } n \text{ is an odd square},\\ n, & \text{if } n \text{ is an even square},\\ \left.\begin{array}{ll}-1\ , & \text{if } \lambda\in\left[0,\frac{1}{2}-\frac{1}{2n}\right],\\ -\frac{1}{2}\ , & \text{if } \lambda\in\left(\frac{1}{2}-\frac{1}{2n},1\right].\end{array}\right\} & \text{if } n \text{ is an odd nonsquare}.\end{cases}$$

Then

$$st\text{-}\limsup u_n^-(\lambda)=1,\quad st\text{-}\limsup u_n^+(\lambda)=\begin{cases}2\ , & \text{if } \lambda=0,\\ 1\ , & \text{if } \lambda\in(0,1],\end{cases}$$

$$st\text{-}\liminf u_n^+(\lambda)=0,\quad st\text{-}\liminf u_n^-(\lambda)=\begin{cases}-1\ , & \text{if } \lambda\in[0,\frac{1}{2}),\\ -\frac{1}{2}\ , & \text{if } \lambda\in[\frac{1}{2},1].\end{cases}$$

We can see that $st\text{-}\limsup u_n^+(\lambda)$ is not right continuous at $\lambda=0$ and $st\text{-}\liminf u_n^-(\lambda)$ is not left continuous at $\lambda=\frac{1}{2}$. By Theorem 3.1, we have

$$(st-\operatorname{Lim\,inf} u_n)^+(\lambda)=0,\qquad (st-\operatorname{Lim\,inf} u_n)^-(\lambda)=\begin{cases}-1, & \text{if } \lambda\in[0,\frac{1}{2}],\\ -\frac{1}{2}, & \text{if } \lambda\in(\frac{1}{2},1],\end{cases}$$

$$(st-\operatorname{Lim\,sup} u_n)^+(\lambda)=1,\qquad (st-\operatorname{Lim\,sup} u_n)^-(\lambda)=1.$$

So,

$$(st-\operatorname{Lim\,inf} u_n)(x)=\begin{cases}1, & \text{if } x\in\left[-\frac{1}{2},0\right],\\ \frac{1}{2}, & \text{if } x\in\left[-1,-\frac{1}{2}\right),\\ 0, & \text{otherwise},\end{cases}\qquad st-\operatorname{Lim\,sup} u_n=\chi_{\{1\}}.$$

It is evident that if there exist a fuzzy number $\mu\in E^1$ such that $[\mu]_\lambda=[st-\limsup u_n^-(\lambda), st-\liminf u_n^+(\lambda)]$ for all $\lambda\in[0,1]$, then $\mu=\operatorname{Lim\,sup} u_n$. However by Example 3.2, we see that $st\text{-}\limsup u_n^-(\lambda)$ and $st\text{-}\limsup u_n^+(\lambda)$ may not determine a fuzzy number. The following theorem gives necessary conditions under which the family $\{[st-\limsup u_n^-(\lambda), st-\limsup u_n^+(\lambda)] : \lambda\in[0,1]\}$ can define a fuzzy number. First, we need to define the notions of statistical equi-left and right-continuity of a sequence of functions.

Definition 3.3. *[14] Let $\{f_k\}$ be a sequence of functions defined on $[a,b]$ and $\lambda_0\in(a,b]$. Then, $\{f_k\}$ is said to be statistically equi-left-continuous (SELC) at λ_0 if for any $\varepsilon>0$ there exists $\varepsilon'>0$ such that*

$$\delta\left(\{k\in\mathbb{N} : |f_k(\lambda)-f_k(\lambda_0)|\geq\varepsilon\}\right)=0,$$

whenever $\lambda\in(\lambda_0-\varepsilon',\lambda_0]$.

Statistical equi-right continuity (SERC) at $\lambda_0\in[a,b)$ can be defined similarly.

Theorem 3.4. *Let (u_k) be a statistically bounded sequence of fuzzy numbers such that*

$$st-\limsup u_k^-(\lambda)=\mu^-(\lambda)\quad \text{and}\quad st-\limsup u_k^+(\lambda)=\mu^+(\lambda) \tag{12}$$

for each $\lambda\in[0,1]$. If the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are SELC at each $\lambda\in(0,1]$ and SERC at $\lambda=0$, then the pair of functions $\mu^-(\lambda)$ and $\mu^+(\lambda)$ define a fuzzy number.

Proof. Since the sequences $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are SELC at each $\lambda\in(0,1]$, Then for any $\varepsilon>0$, there exist $\varepsilon'>0$ such that

$$\delta\left(\left\{k\in\mathbb{N} : u_k^-(\lambda)-u_k^-(r)\geq\frac{\varepsilon}{3}\right\}\right)=0\ \text{ and }\ \delta\left(\left\{k\in\mathbb{N} : u_k^+(r)-u_k^+(\lambda)\geq\frac{\varepsilon}{3}\right\}\right)=0 \tag{13}$$

whenever $r \in (\lambda - \varepsilon', \lambda]$. Let us define

$$K_1 = \left\{k \in \mathbb{N} : u_k^-(\lambda) - u_k^-(r) < \frac{\varepsilon}{3}\right\},$$
$$K_2 = \left\{k \in \mathbb{N} : u_k^+(r) - u_k^+(\lambda)| < \frac{\varepsilon}{3}\right\}.$$

We have $\delta(K_1) = 1$ and $\delta(K_2) = 1$. We define

$$K_3 = \left\{k \in \mathbb{N} : u_k^-(r) \leq \mu^-(r) + \frac{\varepsilon}{3}\right\},$$
$$K_4 = \left\{k \in \mathbb{N} : u_k^+(\lambda) \leq \mu^+(\lambda) + \frac{\varepsilon}{3}\right\},$$
$$K_5 = \left\{k \in \mathbb{N} : u_k^-(\lambda) > \mu^-(\lambda) - \frac{\varepsilon}{3}\right\},$$
$$K_6 = \left\{k \in \mathbb{N} : u_k^+(r) > \mu^+(r) - \frac{\varepsilon}{3}\right\}.$$

By (12), (13) and Lemma 2.1 we have $\delta(K_3) = 1$, $\delta(K_4) = 1$, $\delta(K_5) \neq 0$ and $\delta(K_6) \neq 0$. So there exist $k \in K_1 \cap K_3 \cap K_5$ and $m \in K_2 \cap K_4 \cap K_6$ such that

$$0 \leq \mu^-(\lambda) - \mu^-(r) \leq u_k^-(\lambda) + \frac{\varepsilon}{3} - \left(u_k^-(r) - \frac{\varepsilon}{3}\right) < \varepsilon,$$
$$0 \leq \mu^+(r) - \mu^+(\lambda) \leq u_m^+(r) + \frac{\varepsilon}{3} - \left(u_m^+(\lambda) - \frac{\varepsilon}{3}\right) < \varepsilon.$$

This means that $\mu^-(\lambda)$ and $\mu^+(\lambda)$ are left continuous at $\lambda \in (0, 1]$.

Since the sequences $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are SERC at 0, it can be easily prove that $\mu^-(\lambda)$ and $\mu^+(\lambda)$ are right continuous at $\lambda = 0$. From the proof of Theorem 3.1 it can be seen that $\mu^-(\lambda)$ is nondecreasing, $\mu^+(\lambda)$ is nonincreasing. Furtermore, we have $u_k^-(1) \leq u_k^+(1)$ for all k. So

$$st\text{-}\limsup u_k^-(1) \leq st\text{-}\limsup u_k^+(1).$$

That is $\mu^-(1) \leq \mu^+(1)$. Consequently by Theorem 2.5 we obtain $\mu \in E^1$. This completes the proof. □

The dual statement of Theorem 3.4 for $st - \operatorname{Lim\,inf} u_n$ may be given as follows.

Theorem 3.5. *Let (u_k) be a statistically bounded sequence of fuzzy numbers such that*

$$st - \liminf u_k^-(\lambda) = \nu^-(\lambda) \quad \text{and} \quad st - \liminf u_k^+(\lambda) = \nu^+(\lambda)$$

for each $\lambda \in [0, 1]$. If the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are SELC at each $\lambda \in (0, 1]$ and SERC at $\lambda = 0$, then the pair of functions $\nu^-(\lambda)$ and $\nu^+(\lambda)$ define a fuzzy number.

4. Almost everywhere statistical convergence of sequences of fuzzy numbers

Now we give some definitions for statistical convergence of sequences of fuzzy numbers and we refer to [2–4, 6, 7, 15, 16] for more details.

Let $(u_k)_{k=0}^{\infty}$ be a sequence of fuzzy numbers and $\mu \in E^1$.

If st$-\lim_{k\to\infty} D(u_k, \mu) = 0$, we say that (u_k) statistically converges to μ with respect to the metric D. In this case we write $u_k \xrightarrow{D} \mu(st)$.

If st$-\lim_{k\to\infty} d_H([u_k]_\lambda, [\mu]_\lambda) = 0$ for all $\lambda \in [0, 1]$ or equivalently,

$$\text{st}-\lim_{k\to\infty} u_k^-(\lambda) = \mu^-(\lambda) \text{ and } \text{st}-\lim_{k\to\infty} u_k^+(\lambda) = \mu^+(\lambda)$$

for all $\lambda \in [0, 1]$, then (u_k) is said to be levelwise statistically convergent to μ, denoted by $u_k \xrightarrow{l} \mu(st)$.

If st$-\lim_{k\to\infty} d_H([u_k]_\lambda, [\mu]_\lambda) = 0$ holds for λ almost everywhere on $[0,1]$ then we say that (u_k) almost everywhere statistically converges to μ. In this case we write $u_k \xrightarrow{a.e.} \mu(st)$.

Clearly $u_k \xrightarrow{D} \mu(st)$ if and only if $[u_k]_\lambda$ is uniformly statistically convergent to $[\mu]_\lambda$ with respect to λ. So we have the following implication

$$u_k \xrightarrow{D} \mu(st) \Rightarrow u_k \xrightarrow{l} \mu(st) \Rightarrow u_k \xrightarrow{a.e.} \mu(st).$$

In fuzzy number space Theorem 2.3 is not valid for levelwise statistical convergence and statistical convergence with respect to the metric D. It can be seen the following example.

Example 4.1. *Let us define*

$$u_n(x) = \begin{cases} \left.\begin{array}{cc} x-n, & \text{for } n \le x \le n+1, \\ -x+n+2, & \text{for } n+1 \le x \le n+2, \\ 0, & \text{otherwise} \end{array}\right\} & \text{if } n \text{ is a square,} \\ \left.\begin{array}{cc} 1-\frac{1}{n}\ , & \text{if } x \in [0,1), \\ 1\ , & \text{if } x = 1, \\ 0\ , & \text{otherwise.} \end{array}\right\} & \text{if } n \text{ is a nonsquare} \end{cases}$$

Then, if n is a nonsquare, we have

$$u_n^+(\lambda) = 1, \text{ and } u_n^-(\lambda) = \begin{cases} 1\ , & \text{if } \lambda \in (1-\frac{1}{n}, 1], \\ 0\ , & \text{if } \lambda \in [0, 1-\frac{1}{n}]. \end{cases}$$

Therefore

$$st\text{-}\liminf_{n\to\infty} u_n^-(\lambda) = st\text{-}\limsup_{n\to\infty} u_n^-(\lambda) = \begin{cases} 1\ , & \text{if } \lambda = 1, \\ 0\ , & \text{if } \lambda \in [0,1), \end{cases}$$
$$st\text{-}\liminf_{n\to\infty} u_n^+(\lambda) = st\text{-}\limsup_{n\to\infty} u_n^+(\lambda) = 1.$$

By Theorem 3.1 we obtain $st - \operatorname{Lim\,inf} u_n = st - \operatorname{Lim\,sup} u_n = \chi_{[0,1]}$. However, if n is a nonsquare, then $d_H([u_n]_1, [\chi_{[0,1]}]_1) = 1$. $u = (u_n)$ is neither statistically convergent to $\chi_{[0,1]}$ with respect to the metric D nor levelwise.

We obtain Theorem 2.3 for almost everywhere statistical convergence. This can be seen following theorem.

Theorem 4.2. *Let (u_n) be a statistically bounded sequence of fuzzy numbers and $\mu \in E^1$. Then* st$-\operatorname{Lim\,sup} u_n =$ st$-\operatorname{Lim\,inf} u_n = \mu$ *if and only if* $u_n \xrightarrow{a.e.} \mu(st)$.

Proof. Necessity: Assume that st$-\operatorname{Lim\,sup} u_n =$ st$-\operatorname{Lim\,inf} u_n = \mu$. Since $st\text{-}\limsup u_n^-(\lambda)$ and $st\text{-}\liminf u_n^-(\lambda)$ are nondecreasing and bounded functions in λ, they have at most countably many discontinuities. We denote these discontinuities by D^-.

Similarly, $st\text{-}\limsup u_n^+(\lambda)$ and $st\text{-}\liminf u_n^+(\lambda)$ are nonincreasing and bounded functions in λ and they have at most countably many discontinuities. We denote these discontinuities by D^+. We define $D = D^- \cup D^+$. D is countable set. For all $\lambda \in (0,1]\backslash D$ we have

$$(st - \operatorname{Lim\,sup} u_n)^-(\lambda) = \sup_{r<\lambda} st\text{-}\limsup u_n^-(r) = \lim_{r\to\lambda^-} st\text{-}\limsup u_n^-(r) = st\text{-}\limsup u_n^-(\lambda),$$
$$(st - \operatorname{Lim\,sup} u_n)^+(\lambda) = \inf_{r<\lambda} st\text{-}\limsup u_n^+(r) = \lim_{r\to\lambda^-} st\text{-}\limsup u_n^+(r) = st\text{-}\limsup u_n^+(\lambda).$$

Consequently

$$[st - \operatorname{Lim\,sup} u_n]_\lambda = [st\text{-}\limsup u_n^-(\lambda), st\text{-}\limsup u_n^+(\lambda)]$$

holds for every $\lambda \in (0,1]\backslash D$. Similarly, it can be seen that

$$[st-\operatorname{Lim\,inf} u_n]_\lambda = [st\text{-}\liminf u_n^-(\lambda), st\text{-}\liminf u_n^+(\lambda)]$$

holds for every $\lambda \in (0,1]\backslash D$. By the assumption we have

$$st\text{-}\limsup u_n^-(\lambda) = st\text{-}\liminf u_n^-(\lambda) = \mu^-(\lambda)$$

and

$$st\text{-}\limsup u_n^+(\lambda) = st\text{-}\liminf u_n^+(\lambda) = \mu^+(\lambda)$$

for every $\lambda \in (0,1]\backslash D$. This implies that

$$st\text{-}\lim u_n^+(\lambda) = \mu^+(\lambda) \quad \text{and} \quad st\text{-}\lim u_n^-(\lambda) = \mu^-(\lambda)$$

for every $\lambda \in (0,1]\backslash D$. Therefore, $u_n \xrightarrow{a.e.} \mu(st)$.

Sufficiency: Suppose that $u_n \xrightarrow{a.e.} \mu(st)$. So there exist a set D with zero measure such that

$$st\text{-}\lim u_n^+(\lambda) = \mu^+(\lambda) \quad \text{and} \quad st\text{-}\lim u_n^-(\lambda) = \mu^-(\lambda)$$

holds for every $\lambda \in [0,1]\backslash D$. For $\lambda_0 \in [0,1]\backslash D$ and $\lambda_0 \neq 0$, taking $r_n \in [0,1]\backslash D$ is increasing and $r_n \to \lambda_0$, then we have

$$\begin{aligned}
(st-\operatorname{Lim\,sup} u_n)^-(\lambda_0) &= \lim_{n\to\infty} st\text{-}\limsup u_n^-(r_n) = \lim_{n\to\infty} \mu^-(r_n) = \mu^-(\lambda_0),\\
(st-\operatorname{Lim\,sup} u_n)^+(\lambda_0) &= \lim_{n\to\infty} st\text{-}\limsup u_n^+(r_n) = \lim_{n\to\infty} \mu^+(r_n) = \mu^+(\lambda_0),\\
(st-\operatorname{Lim\,inf} u_n)^-(\lambda_0) &= \lim_{n\to\infty} st\text{-}\liminf u_n^-(r_n) = \lim_{n\to\infty} \mu^-(r_n) = \mu^-(\lambda_0),\\
(st-\operatorname{Lim\,inf} u_n)^+(\lambda_0) &= \lim_{n\to\infty} st\text{-}\liminf u_n^+(r_n) = \lim_{n\to\infty} \mu^+(r_n) = \mu^+(\lambda_0).
\end{aligned}$$

As a consequence, $[st-\operatorname{Lim\,sup} \mathrm{u_n}]_\lambda = [\mathrm{st}-\operatorname{Lim\,inf} \mathrm{u_n}]_\lambda = [\mu]_\lambda$ for every $\lambda \in (0,1]\backslash D$. By Lemma 2.6 we have st$-\operatorname{Lim\,sup} u_n =$ st$-\operatorname{Lim\,inf} u_n = \mu$ and the proof is completed. □

Remark 4.3. *The limit inferior and superior of a bounded sequence of fuzzy numbers have been defined by Aytar et.al.[1]. By using the similar way in Theorem 4.2, for bounded sequence (u_k) we can prove that* $\operatorname{Lim\,sup} u_n = \operatorname{Lim\,inf} u_n = \mu$ *if and only if* $u_n \xrightarrow{a.e.} \mu$. *Besides Zhao and Wu [22] proved that for a bounded sequence of fuzzy numbers almost everywhere convergence, convergence with respect to the endograph metric and d_p metric are equivalent. So we can obtain the following theorem.*

Theorem 4.4. *Let (u_k) be a bounded sequence of fuzzy numbers and $\mu \in E^1$, then the following properties are equivalent:*

(i) $\operatorname{Lim\,sup} u_n = \operatorname{Lim\,inf} u_n = \mu$,

(ii) $u_k \xrightarrow{a.e.} \mu$,

(iii) $u_k \xrightarrow{d_p} \mu$,

(iv) $u_k \xrightarrow{D_{end}} \mu$,

where $d_p(u,v) = \left(\int_0^1 (d_H([u]_\lambda,[v]_\lambda))^p d\lambda\right)^{\frac{1}{p}}$, $1 \leq p < \infty$ *and* $D_{end}(u,v) = d_H(end(u), end(v))$, $end(u) = \{(x,y) : x \in \mathbb{R}, 0 \leq y \leq u(x)\}$, *for* $u,v \in E^1$.

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