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On Characterization of Non-Newtonian Superposition Operators in Some Sequence Spaces

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Abstract. In this paper, we define a non-Newtonian superposition operator ${}_NP_f$ where $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ by ${}_NP_f(x) = (f(k, x_k))_{k=1}^{\infty}$ for every non-Newtonian real sequence $x = (x_k)$. Chew and Lee [4] have characterized $P_f : \ell_p \to \ell_1$ and $P_f : c_0 \to \ell_1$ for $1 \leq p < \infty$. The purpose of this paper is to generalize these works respect to the non-Newtonian calculus. We characterize ${}_NP_f : \ell_\infty(N) \to \ell_1(N)$, ${}_NP_f : c_0(N) \to \ell_1(N)$, ${}_NP_f : c(N) \to \ell_1(N)$ and ${}_NP_f : \ell_p(N) \to \ell_1(N)$, respectively. Then we show that such ${}_NP_f : \ell_\infty(N) \to \ell_1(N)$ is *-continuous if and only if $f(k, .)$ is *-continuous for every $k \in \mathbb{N}$.

1. Introduction and Preliminaries

Non-Newtonian calculus was firstly introduced and worked by Michael Grossman and Robert Katz between years 1967 and 1970. They published the book about fundamentals of non-Newtonian calculus and which includes some special calculus such as geometric, harmonic, bigeometric, anageometric. Türkmen and Başar [22] obtained some results on sequence spaces with respect to geometric calculus. After, Çakmak and Başar [5] obtained some properties of continuous functions in non-Newtonian calculus. Also, Duyar, Sağır and Oğur [8] studied on some properties of non-Newtonian real line.

The sequence spaces and operator theory have also wide application area. There exist many studies that are done until today on superposition operator which is one of the non-linear operators. Under the assumption that $f(k, .)$ is continuous on $\mathbb{R}$ for every $k \in \mathbb{N}$, Chew and Lee [4] have characterized $P_f : \ell_p \to \ell_1$ and $P_f : c_0 \to \ell_1$ for $1 \leq p < \infty$. Dedagich and Zabreiko [7] have given the necessary and sufficient conditions for the superposition operators on the sequence spaces ℓ_p, ℓ_∞ and c_0. After, some properties of superposition operator, such as boundedness, compactness, were studied by Sama-ae [19], Sağır and Güngör [11, 16], Kolk and Raidjoe [12], Oğur [14, 15] and many others. The purpose of this paper is to generalize these works respect to the non-Newtonian calculus. In this article, we define a non-Newtonian superposition operator ${}_NP_f$ where $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ by ${}_NP_f(x) = (f(k, x_k))_{k=1}^{\infty}$ for every non-Newtonian real sequence $x = (x_k)$ and we characterize non-Newtonian superposition operators on $\ell_\infty(N)$, $\ell_p(N)$, $c_0(N)$ and $c(N)$ into $\ell_1(N)$. Finally, we show that such ${}_NP_f$ which acting from $\ell_\infty(N)$ to $\ell_1(N)$, is *-continuous if and only if $f(k, .)$ is *-continuous for every $k \in \mathbb{N}$.

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A *generator* is defined as an injective function with domain $\mathbb{R}$ and the range of generator is a subset of $\mathbb{R}$. Let take any generator α with range $A = \mathbb{R}(N)_\alpha$. Let define $\alpha-$addition, $\alpha-$subtraction, $\alpha-$multiplication, $\alpha-$division and $\alpha-$order as follows;

$\alpha-$addition	$x\dot{+}y = \alpha\left(\alpha^{-1}(x) + \alpha^{-1}(y)\right)$
$\alpha-$subtraction	$x\dot{-}y = \alpha\left(\alpha^{-1}(x) - \alpha^{-1}(y)\right)$
$\alpha-$multiplication	$x\dot{\times}y = \alpha\left(\alpha^{-1}(x) \times \alpha^{-1}(y)\right)$
$\alpha-$division	$x\dot{/}y = \alpha\left(\alpha^{-1}(x) / \alpha^{-1}(y)\right), y \neq \dot{0}, \alpha^{-1}(y) \neq 0$
$\alpha-$order	$x\dot{<}y \ (x\dot{\leq}y) \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y) \ \left(\alpha^{-1}(x) \leq \alpha^{-1}(y)\right)$

for $x, y \in \mathbb{R}(N)_\alpha$ [10].

$(\mathbb{R}(N)_\alpha, \dot{+}, \dot{\times}, \dot{\leq})$ is totally ordered field [6].

The numbers $x\dot{>}\dot{0}$ are $\alpha-$positive numbers and the numbers $x\dot{<}\dot{0}$ are $\alpha-$negative numbers in $\mathbb{R}(N)_\alpha$. $\alpha-$integers are obtained by successive $\alpha-$addition of $\dot{1}$ to $\dot{0}$ and successive $\alpha-$subtraction of $\dot{1}$ from $\dot{0}$. For each integer n, we set $\dot{n} = \alpha(n)$.

$\alpha-$absolute value of a number $x \in \mathbb{R}(N)_\alpha$ is defined by

$$|x|_\alpha = \alpha\left(\left|\alpha^{-1}(x)\right|\right) = \begin{cases} x & if \quad x\dot{>}\dot{0} \\ \dot{0} & if \quad x = \dot{0} \\ \dot{0}\dot{-}x & if \quad x\dot{<}\dot{0} \end{cases}.$$

For $x \in \mathbb{R}(N)_\alpha$, $\sqrt[p]{x}^{\alpha} = \alpha\left(\sqrt[p]{\alpha^{-1}(x)}\right)$ and $x^{p_\alpha} = \alpha\left\{\left[\alpha^{-1}(x)\right]^p\right\}$.

Grossman and Katz described the *-calculus with the help of two arbitrary selected generators. In this study, we studied according to *-calculus. Let take any generators α and β and let * ("star") is shown the ordered pair of arithmetics ($\alpha-$arithmetic, $\beta-$arithmetic). The following notations will be used.

	$\alpha-$arithmetic	$\beta-$ arithmetic
Realm	$A\,(=\mathbb{R}(N)_\alpha)$	$B\left(=\mathbb{R}(N)_\beta\right)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Ordering	$\dot{<}$	$\ddot{<}$

In the $*-$calculus, $a-$arithmetic is used on arguments and $\beta-$arithmetic is used on values.

The isomorphism from $a-$arithmetic to $\beta-$arithmetic is the unique function ι(iota) that possesses the following three properties.

1. ι is one-to-one.
2. ι is on A and onto B.
3. For any numbers u and v in A,

$$\begin{aligned} \iota(u\dot{+}v) &= \iota(u)\ddot{+}\iota(v), \ \iota(u\dot{-}v) = \iota(u)\ddot{-}\iota(v), \\ \iota(u\dot{\times}v) &= \iota(u)\ddot{\times}\iota(v), \ \iota\left(u\dot{/}v\right) = \iota(u)\ddot{/}\iota(v), v \neq \dot{0} \\ u &\dot{<} v \Longleftrightarrow \iota(u)\ddot{<}\iota(v). \end{aligned}$$

It turns out that $\iota(x) = \beta\left\{\alpha^{-1}(x)\right\}$ for every number x in A and that for every integer n, we set $\iota(\dot{n}) = \ddot{n}$ [10].

In non-Newtonian metric space, the definitions of $\alpha-$accumulation point of a set, $\alpha-$convergence of a sequence and $\alpha-$bounded sequence are given in the studies which are numbered[6, 17]. The definitions of *-limit and *-continuity of the function $f : X \subset \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ are introduced by Sağır and Erdogan[17]. Duyar and Erdogan introduced $\alpha-$series and its $\alpha-$convergence[9]. Non-Newtonian interior point and non-Newtonian open set are defined by Binbaşıoğlu and others[3].

Let X be a vector space over the field $\mathbb{R}(N)_\alpha$ and $\|.\|_\alpha$ be a function from X to $\mathbb{R}^+(N)_\alpha \cup \left\{\dot{0}\right\}$ satisfying the following non-Newtonian norm axioms. For $x, y \in X$ and $\lambda \in \mathbb{R}(N)_\alpha$,

(NN1) $\|x\|_\alpha = \dot{0} \Leftrightarrow x = \dot{0}$,

(NN2) $\|\lambda \dot{\times} x\|_\alpha = |\lambda|_\alpha \dot{\times} \|x\|_\alpha$,

(NN3) $\left\|x \dot{+} y\right\|_\alpha \dot{\leq} \|x\|_\alpha \dot{+} \left\|y\right\|_\alpha$.

Then $(X, \|.\|_\alpha)$ is said to be a non-Newtonian normed space.

The non-Newtonian sequence spaces $S(N)$, $\ell_\infty(N)$, $c(N)$, $c_0(N)$ and $\ell_p(N)$ over the non-Newtonian real field $\mathbb{R}(N)_\alpha$ are defined as following:

$S(N) = \{x = (x_k) : \forall k \in \mathbb{N}, x_k \in \mathbb{R}(N)_\alpha\}$

$$\ell_\infty(N) = \left\{x = (x_k) \in S(N) : {}^{\alpha}\sup_{k\in\mathbb{N}} |x_k|_\alpha \dot{<} \dot{+}\infty\right\},$$

$$c(N) = \left\{x = (x_k) \in S(N) : \exists l \in \mathbb{R}(N)_\alpha \ni {}^{\alpha}\lim_{n\to\infty} |x_k \dot{-} l|_\alpha = \dot{0}\right\},$$

$$c_0(N) = \left\{x = (x_k) \in S(N) : {}^{\alpha}\lim_{n\to\infty} |x_k|_\alpha = \dot{0}\right\},$$

$$\ell_p(N) = \left\{x = (x_k) \in S(N) : {}_{\alpha}\sum_{k=1}^{\infty} |x_k|_\alpha^{p_\alpha} \dot{<} \dot{+}\infty\right\} \quad (1 \leq p < \infty).$$

The sequence spaces $\ell_\infty(N)$, $c(N)$, $c_0(N)$ are non-Newtonian normed spaces with the non-Newtonian norm $\|.\|_{\alpha,\infty}$ which is defined as $\|x\|_{\alpha,\infty} = {}^{\alpha}\sup_{k\in\mathbb{N}} |x_k|_\alpha$ and the sequence space $\ell_p(N)$ is a non-Newtonian normed space with the non-Newtonian norm $\|.\|_{\alpha,p}$ which is defined as $\|x\|_{\alpha,p} = \left({}_{\alpha}\sum_{k=1}^{\infty} |x_k|_\alpha^{p_\alpha}\right)^{\left(\frac{1}{p}\right)_\alpha}$ [6].

Let S be space of real number sequences, X and Y be two sequence spaces on $\mathbb{R}$. Let function $f : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ be given. Superposition operator P_f is mapping from X into S which is in the form of $P_f(x) = (f(k, x_k))_{k=1}^{\infty}$. If $P_f(x) \in Y$ for all $x = (x_k) \in X$, then it is said that P_f acts from X into Y and it is denoted by $P_f : X \to Y$.

The function f satisfies the following conditions:

(A_1) $f(k, 0) = 0$ for all $k \in \mathbb{N}$.

(A_2) $f(k, .)$ is continuous for all $k \in \mathbb{N}$.

(A_2') $f(k, .)$ is bounded on every bounded subset of $\mathbb{R}$ for all $k \in \mathbb{N}$.

If the function $f(k, .)$ satisfies the condition (A_2), then it satisfies the condition (A_2') [7, 20].

2. Results and Discussion

Definition 2.1. *Let $S(N)$ be space of non-Newtonian real number sequences, $X(N)$ be a sequence space on $\mathbb{R}(N)_\alpha$ and $Y(N)$ be a sequence space on $\mathbb{R}(N)_\beta$. Let function $f : \mathbb{N}\times\mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be given. Non-Newtonian superposition operator ${}_NP_f$ is mapping from $X(N)$ into $S(N)$ which is in the form of ${}_NP_f(x) = (f(k, x_k))_{k=1}^{\infty}$. If ${}_NP_f(x) \in Y(N)$ for all $x = (x_k) \in X(N)$, then it is said that ${}_NP_f$ acts from $X(N)$ into $Y(N)$ and it is denoted by ${}_NP_f : X(N) \to Y(N)$.*

The function f satisfies the following conditions:

(NA_1) $f(k, \dot{0}) = \ddot{0}$ for all $k \in \mathbb{N}$.

*(NA_2) $f(k, .)$ is *-continuous for all $k \in \mathbb{N}$.*

(NA_2') $f(k, .)$ is $\beta-$bounded on every $\alpha-$bounded subset of $\mathbb{R}(N)_\alpha$ for all $k \in \mathbb{N}$.

Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of $\alpha-$open sets at $\mathbb{R}(N)_\alpha$. If $E \subset \bigcup_{i\in I} A_i$ for subset $E \subset \mathbb{R}(N)_\alpha$, then the family $\mathcal{A}$ is called an $\alpha-$open cover of the set E. If $I_0 \subset I$ is a finite subset and $E \subset \bigcup_{i\in I_0} A_i$, then family $\mathcal{A}_0 = (A_i)_{i\in I_0}$ is called a finite $\alpha-$subcover of the set E.

If a finite $\alpha-$subcover can be selected from every $\alpha-$open cover of set $E \subset \mathbb{R}(N)_\alpha$, then it is said that E is $\alpha-$compact set.

Theorem 2.2. *(*-Heine Borel): Every $\alpha-$closed interval $\dot{[}a, b\dot{]}$ on $\mathbb{R}(N)_\alpha$ is $\alpha-$compact.*

Proof. Let $\mathcal{A} = \{A_i : i \in I\}$ be an $\alpha-$open cover of $\alpha-$closed interval $\dot{[}a,b\dot{]}$. Then it is written that $\dot{[}a,b\dot{]} \subset \underset{i\in I}{\cup} A_i$ and there exists an $\alpha-$number $r\dot{>}\dot{0}$ for every $a \in A_i$ such that $B_\alpha(a,r) \subset A_i$ since A_i is $\alpha-$open set. Since

$$\alpha(x) \in \dot{[}a,b\dot{]} \subset \underset{i\in I}{\cup} A_i \Rightarrow \exists i_0 \in I,\ \alpha(x) \in A_{i_0} \Rightarrow x \in \alpha^{-1}(A_{i_0}),$$

we have

$$\left[\alpha^{-1}(a), \alpha^{-1}(b)\right] \subset \underset{i\in I}{\cup} \alpha^{-1}(A_i)\ .$$

Hence the family $\{\alpha^{-1}(A_i)\}$ is an open cover of the closed interval $\left[\alpha^{-1}(a), \alpha^{-1}(b)\right]$. By Heine-Borel theorem, there exists finite subfamily $\{i_1, i_2, .., i_n\}$ such that

$$\left[\alpha^{-1}(a), \alpha^{-1}(b)\right] \subset \overset{n}{\underset{k=1}{\cup}} \alpha^{-1}(A_{i_k}).$$

Since

$$x \in \underset{i\in I}{\cup} A_i \Rightarrow \exists i_0 \in I,\ x \in A_{i_0} \Rightarrow \alpha^{-1}(x) \in \alpha^{-1}(A_{i_0}) \Rightarrow \alpha^{-1}(x) \in \underset{i\in I}{\cup} \alpha^{-1}(A_i)$$

$$\Rightarrow x \in \alpha\left(\underset{i\in I}{\cup} \alpha^{-1}(A_i)\right)$$

and

$$x \in \alpha\left(\underset{i\in I}{\cup} \alpha^{-1}(A_i)\right) \Rightarrow \alpha^{-1}(x) \in \underset{i\in I}{\cup} \alpha^{-1}(A_i) \Rightarrow \exists i_0 \in I,\ \alpha^{-1}(x) \in \alpha^{-1}(A_{i_0})$$

$$\Rightarrow x \in A_{i_0} \Rightarrow x \in \underset{i\in I}{\cup} A_i\ ,$$

we get $\dot{[}a,b\dot{]} = \alpha\left(\left[\alpha^{-1}(a), \alpha^{-1}(b)\right]\right) \subset \alpha\left(\overset{n}{\underset{k=1}{\cup}} \alpha^{-1}(A_{i_k})\right) = \overset{n}{\underset{k=1}{\cup}} A_{i_k}$. □

Proposition 2.3. *If the function $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ is *-continuous, it is $\beta-$bounded on every $\alpha-$bounded subset of $\mathbb{R}(N)_\alpha$.*

Proof. Let E be an $\alpha-$bounded subset of $\mathbb{R}(N)_\alpha$. Then there exist $a, b \in \mathbb{R}(N)_\alpha$ such that $E \subseteq \dot{[}a,b\dot{]}$. The function f is *-continuous for every $z \in \dot{[}a,b\dot{]}$. Then there exists an $\alpha-$real number $\delta_z\dot{>}\dot{0}$ such that $\left|f(x) \ddot{-} f(z)\right|_\beta \ddot{<} \lambda$ for $\beta-$real number $\lambda\ddot{>}\ddot{0}$ with $|x\dot{-}z|_\alpha \dot{<}\delta_z$. Then, we have

$$\left|f(x)\right|_\beta \ddot{<} \lambda \ddot{+} \left|f(z)\right|_\beta$$

for every $x \in B_\alpha(z, \delta_z)$. Additionally, there exist $z_1, z_2, ..., z_n \in \dot{[}a,b\dot{]}$ such that $\dot{[}a,b\dot{]} \subset \overset{n}{\underset{k=1}{\cup}} B_\alpha(z_k, \delta_{z_k})$ since the set $\dot{[}a,b\dot{]}$ is $\alpha-$compact and $\dot{[}a,b\dot{]} \subset \underset{z\in\dot{[}a,b\dot{]}}{\cup} B_\alpha(z, \delta_z)$. Thus, we have $\left|f(x)\right|_\beta \ddot{<} \lambda_k \ddot{+} \left|f(z_k)\right|_\beta$ for every $x \in B_\alpha(z_k, \delta_{z_k})$, where $k = 1, 2, ..., n$. If $M = {}^{\beta}\max\left\{\lambda_k \ddot{+} \left|f(z_k)\right|_\beta : 1 \le k \le n\right\}$, then $\left|f(x)\right|_\beta \ddot{\le} M$ for all $x \in B_\alpha(z_k, \delta_{z_k})$. We get $\left|f(x)\right|_\beta \ddot{\le} M$ for all $x \in E$ since $E \subseteq \dot{[}a,b\dot{]} \subset \overset{n}{\underset{k=1}{\cup}} B_\alpha(z_k, \delta_{z_k})$. □

Corollary 2.4. *If the function $f(k,.)$ satisfies the condition (NA_2), then it satisfies the condition (NA_2').*

Theorem 2.5. *Let us suppose that $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ satisfies the condition (NA_2'). Then ${}_NP_f : \ell_\infty(N) \to \ell_1(N)$ if and only if there exists a $\beta-$sequence $(c_k) \in \ell_1(N)$ such that $\left|f(k,t)\right|_\beta \ddot{\le} c_k$ for each $\alpha-$number $\mu\dot{>}\dot{0}$ and $k \in \mathbb{N}$ whenever $|t|_\alpha \dot{\le}\mu$.*

Proof. Let $x = (x_k) \in \ell_\infty(N)$. Then, there exists an $\alpha-$real number $M \dot{>} \dot{0}$ such that $|x_k|_\alpha \dot{\leq} M$ for all $k \in \mathbb{N}$. By assumption, there exists a $\beta-$sequence $c_k \in \ell_1(N)$ such that $\left|f(k,x_k)\right|_\beta \ddot{\leq} c_k$ for all $k \in \mathbb{N}$. Then, we have ${}_\beta\sum_{k=1}^{\infty}\left|f(k,x_k)\right|_\beta \ddot{\leq} {}_\beta\sum_{k=1}^{\infty} c_k = \|c_k\|_{\beta,1}$. Since ${}_NP_f(x) = (f(k,x_k))_{k=1}^{\infty}$, we obtain that ${}_NP_f(x) \in \ell_1(N)$.

Conversely, let ${}_NP_f : \ell_\infty(N) \to \ell_1(N)$. For each $\alpha-$number $\mu \dot{>} \dot{0}$ and for all $k \in \mathbb{N}$, we define

$$A(\mu) = \{t \in \mathbb{R}(N)_\alpha : |t|_\alpha \dot{\leq} \mu\}$$

and

$$B(k,\mu) = {}^{\beta}\sup\left\{\left|f(k,t)\right|_\beta : t \in A(\mu)\right\}.$$

We see that $\left|f(k,t)\right|_\beta \ddot{\leq} B(k,\mu)$ whenever $|t|_\alpha \dot{\leq} \mu$. We shall show that $(B(k,\mu))_{k=1}^{\infty} \in \ell_1(N)$ for all $\alpha-$real number $\mu \dot{>} \dot{0}$. Suppose that there exists an $\alpha-$number $\mu_1 \dot{>} \dot{0}$ such that $(B(k,\mu_1))_{k=1}^{\infty} \notin \ell_1(N)$. Then, we have ${}_\beta\sum_{k=1}^{\infty} B(k,\mu_1) = \ddot{+}\infty$. So, there exists a sequence of positive integers $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$ such that ${}_\beta\sum_{k=n_{i-1}+1}^{n_i} B(k,\mu_1) \ddot{>} \ddot{1}$ for all $i \in \mathbb{N}$ and there exists a $\beta-$number ε_i such that

$${}_\beta\sum_{k=n_{i-1}+1}^{n_i} B(k,\mu_1) \ddot{-} (\ddot{n}_i \ddot{-} \ddot{n}_{i-1}) \ddot{\times} \varepsilon_i \ddot{>} \ddot{1}. \tag{1}$$

Let $i \in \mathbb{N}$ be fixed. Since f satisfies (NA_2'), $\ddot{0} \ddot{\leq} B(k,\mu_1) \ddot{<} \ddot{+}\infty$ for all $k \in \mathbb{N}$ such that $n_{i-1}+1 \leq k \leq n_i$. From the definition of $B(k,\mu_1)$, there exists a $(x_k) \in A(\mu_1)$ such that

$$\left|f(k,x_k)\right|_\beta \ddot{>} B(k,\mu_1) \ddot{-} \varepsilon_i\,. \tag{2}$$

From 1 and 2, we have

$${}_\beta\sum_{k=n_{i-1}+1}^{n_i}\left|f(k,x_k)\right|_\beta \ddot{>} {}_\beta\sum_{k=n_{i-1}+1}^{n_i} B(k,\mu_1) \ddot{-} {}_\beta\sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i \ddot{>} \ddot{1}$$

for all $i \in \mathbb{N}$. Thus, we have $(f(k,x_k))_{k=1}^{\infty} \notin \ell_1(N)$. Since $|x_k|_\alpha \dot{\leq} \mu_1$, we get $(x_k) \in \ell_\infty(N)$ for all $k \in \mathbb{N}$. This contradicts the assumption. The proof is completed by putting $B(k,\mu) = c_k$ for all $k \in \mathbb{N}$. □

Example 2.6. *Let $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be defined by $f(k,t) = \dfrac{\left|\iota(t) \ddot{-} \ddot{1}\right|_\beta}{\ddot{8}^{k_\beta}}\beta$ for all $k \in \mathbb{N}$ and for all $t \in \mathbb{R}(N)_\alpha$. Since f is $*-$continuous, it is clear that f satisfies (NA_2'). Let $\mu, t \in \mathbb{R}(N)_\alpha$ such that $\mu \dot{>} \dot{0}$ and $|t|_\alpha \dot{\leq} \mu$. Then*

$$\begin{aligned}\left|f(k,t)\right|_\beta &= \frac{\left|\iota(t) \ddot{-} \ddot{1}\right|_\beta}{\ddot{8}^{k_\beta}}\beta \\ &\ddot{\leq} \frac{|\iota(t)|_\beta \ddot{+} \ddot{1}}{\ddot{8}^{k_\beta}}\beta \\ &\ddot{\leq} \frac{\iota(\mu) \ddot{+} \ddot{1}}{\ddot{8}^{k_\beta}}\beta.\end{aligned}$$

If we choose $(c_k) = \left(\dfrac{\iota(\mu) \ddot{+} \ddot{1}}{\ddot{8}^{k_\beta}}\beta\right)$, we get ${}_NP_f : \ell_\infty(N) \to \ell_1(N)$ by Theorem 2.5.

The next theorem characterizes non-Newtonian superposition operator acting from $c_0(N)$ into $\ell_1(N)$.

Theorem 2.7. *Let us suppose that $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ satisfies the condition (NA_2'). Then ${}_NP_f : c_0(N) \to \ell_1(N)$ if and only if there exist an $\alpha-$number $\mu \dot{>} \dot{0}$ and a $\beta-$sequence $(c_k) \in \ell_1(N)$ such that $\left|f(k,t)\right|_\beta \ddot{\leq} c_k$ when $|t|_\alpha \dot{\leq} \mu$ for all $k \in \mathbb{N}$.*

Proof. Let $x = (x_k) \in c_0(N)$. Since ${}^{\alpha}\lim_{k\to\infty} x_k = \dot{0}$, there exists an integer $i \in \mathbb{N}$ such that $|x_k|_\alpha \dot{\leq} \mu$ especially for $\alpha-$number $\mu \dot{>} \dot{0}$ when $k \geq i$. By assumption, there exists a $c_k \in \ell_1(N)$ such that $\left|f(k,x_k)\right|_\beta \ddot{\leq} c_k$ for all $k \geq i$. Then

$$\begin{aligned} {}_\beta\sum_{k=i}^{\infty} \left|f(k,x_k)\right|_\beta &\ddot{\leq} {}_\beta\sum_{k=i}^{\infty} c_k \\ &\ddot{\leq} \|c_k\|_{\beta,1}. \end{aligned}$$

Hence we get ${}_NP_f(x) \in \ell_1(N)$.

Conversely, let ${}_NP_f : c_0(N) \to \ell_1(N)$. For each $\mu \dot{>} \dot{0}$ and for all $k \in \mathbb{N}$, we define

$$A(\mu) = \{t \in \mathbb{R}(N)_\alpha : |t|_\alpha \dot{\leq} \mu\}$$

and

$$B(k,\mu) = {}^{\beta}\sup\left\{\left|f(k,t)\right|_\beta : t \in A(\mu)\right\}.$$

Then, we see that $\left|f(k,t)\right|_\beta \ddot{\leq} B(k,\mu)$ whenever $|t|_\alpha \dot{\leq} \mu$. We shall show that there exists an $\alpha-$number $\mu_1 \dot{>} \dot{0}$ such that $(B(k,\mu_1))_{k=1}^{\infty} \in \ell_1(N)$. Suppose that $(B(k,\mu))_{k=1}^{\infty} \notin \ell_1(N)$ for all $\mu \dot{>} \dot{0}$. Then, we can write ${}_\beta\sum_{k=1}^{\infty} B\left(k, \dot{2}^{(-i)_\alpha}\right) = \ddot{+}\infty$ for all $i \in \mathbb{N}$. Thus, there exists a sequence of positive integers $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$ such that ${}_\beta\sum_{k=n_{i-1}+1}^{n_i} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{>} \ddot{1}$ for all $i \in \mathbb{N}$ and there exists a $\beta-$number ε_i such that

$${}_\beta\sum_{k=n_{i-1}+1}^{n_i} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{-} (\ddot{n}_i \ddot{-} \ddot{n}_{i-1}) \ddot{\times} \varepsilon_i \ddot{>} \ddot{1}. \tag{3}$$

Let $i \in \mathbb{N}$ be fixed. Then, we have $\ddot{0} \ddot{\leq} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{<} \ddot{+}\infty$ for all $k \in \mathbb{N}$ with $n_{i-1}+1 \leq k \leq n_i$. From the definition of $B\left(k, \dot{2}^{(-i)_\alpha}\right)$, there exists a $(x_k) \in A\left(\dot{2}^{(-i)_\alpha}\right)$ such that

$$\left|f(k,x_k)\right|_\beta \ddot{>} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{-} \varepsilon_i. \tag{4}$$

By 3 and 4, we have

$${}_\beta\sum_{k=n_{i-1}+1}^{n_i} \left|f(k,x_k)\right|_\beta \ddot{>} {}_\beta\sum_{k=n_{i-1}+1}^{n_i} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{-} {}_\beta\sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i \ddot{>} \ddot{1}$$

for all $i \in \mathbb{N}$. Thus, we obtain that $(f(k,x_k))_{k=1}^{\infty} \notin \ell_1(N)$. For all $k \in \mathbb{N}$, since $(x_k) \in A\left(\dot{2}^{(-i)_\alpha}\right)$ with $n_{i-1}+1 \leq k \leq n_i$, we have $|x_k|_\alpha \dot{\leq} \dot{2}^{(-i)_\alpha}$. Hence ${}^{\alpha}\lim_{k\to\infty} x_k = \dot{0}$ and $(x_k) \in c_0(N)$. This contradicts the assumption. Then, there exists an $\alpha-$number $\mu_1 \dot{>} \dot{0}$ such that $(B(k,\mu_1))_{k=1}^{\infty} \in \ell_1(N)$. The proof is completed by putting $c_k = B(k,\mu_1)$ for all $k \in \mathbb{N}$. □

Example 2.8. *Let $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be defined by $f(k,t) = \dfrac{\left|\iota(t) \ddot{\times} \left(\iota(t) \ddot{-} \ddot{1}\right)\right|_\beta}{\ddot{3}^{k_\beta}} \beta$ for all $k \in \mathbb{N}$ and for all $t \in \mathbb{R}(N)_\alpha$. Since f is $*-$continuous, it is clear that f satisfies (NA_2'). Let $\mu, t \in \mathbb{R}(N)_\alpha$ such that $\mu = \dot{1}$ and $|t|_\alpha \dot{\leq} \dot{1}$.*

Then

$$\begin{aligned}\left|f(k,t)\right|_\beta &= \frac{|\iota(t)|_\beta \ddot{\times} \left|\iota(t) \ddot{-} \ddot{1}\right|_\beta}{\ddot{3}^{k_\beta}}\beta \\ &\ddot{\leq} \frac{|\iota(t)|_\beta \ddot{\times} \left(|\iota(t)|_\beta \ddot{+} \ddot{1}\right)}{\ddot{3}^{k_\beta}}\beta \\ &\ddot{\leq} \frac{\ddot{2}}{\ddot{3}^{k_\beta}}\beta.\end{aligned}$$

If we choose $(c_k) = \left(\frac{\ddot{2}}{\ddot{3}^{k_\beta}}\beta\right)$ *for all* $k \in \mathbb{N}$*, then, we get* ${}_NP_f : c_0(N) \to \ell_1(N)$ *by Theorem 2.7.*

Theorem 2.9. *Let us suppose that* $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ *satisfies the condition* (NA_2')*. Then* ${}_NP_f : c(N) \to \ell_1(N)$ *if and only if there exist an* $\alpha-$*number* $\mu \dot{>} \dot{0}$ *and a* $\beta-$*sequence* $(c_k) \in \ell_1(N)$ *such that* $\left|f(k,t)\right|_\beta \ddot{\leq} c_k$ *when* $|t \dot{-} z|_\alpha \dot{\leq} \mu$ *for all* $z \in \mathbb{R}(N)_\alpha$ *and for all* $k \in \mathbb{N}$*.*

Proof. Let $x = (x_k) \in c(N)$. Then, there exists a $z \in \mathbb{R}(N)_\alpha$ such that ${}^\alpha \lim_{k\to\infty} x_k = z$ and ${}^\alpha \lim_{k\to\infty} (x_k \dot{-} z) = \dot{0}$. Thus, there exists an integer $i \in \mathbb{N}$ such that $|x_k \dot{-} z|_\alpha \dot{\leq} \mu$ especially for $\alpha-$number $\mu \dot{>} \dot{0}$ when $k \geq i$. By assumption, there exists a $c_k \in \ell_1(N)$ such that $\left|f(k,x_k)\right|_\beta \ddot{\leq} c_k$ for all $k \geq i$. Then

$$\begin{aligned}{}_\beta\sum_{k=i}^{\infty} \left|f(k,x_k)\right|_\beta &\ddot{\leq} {}_\beta\sum_{k=i}^{\infty} c_k \\ &\ddot{\leq} \|c_k\|_{\beta,1}.\end{aligned}$$

Hence, we obtain that ${}_NP_f(x) \in \ell_1(N)$.

Conversely, let ${}_NP_f : c(N) \to \ell_1(N)$. For each $z \in \mathbb{R}(N)_\alpha$, $\mu \dot{>} \dot{0}$ and for all $k \in \mathbb{N}$, we define

$$A(z,\mu) = \{t \in \mathbb{R}(N)_\alpha : |t \dot{-} z|_\alpha \dot{\leq} \mu\}$$

and

$$B(k,\mu) = {}^\beta\sup\left\{\left|f(k,t)\right|_\beta : t \in A(z,\mu)\right\}.$$

Then, we can see that $\left|f(k,t)\right|_\beta \ddot{\leq} B(k,\mu)$ whenever $|t \dot{-} z|_\alpha \dot{\leq} \mu$. Thus, there exists an $\alpha-$number $\mu_1 \dot{>} \dot{0}$ such that $(B(k,\mu_1))_{k=1}^\infty \in \ell_1(N)$. Suppose that $(B(k,\mu))_{k=1}^\infty \notin \ell_1(N)$ for all $\mu \dot{>} \dot{0}$. Then, we can write ${}_\beta\sum_{k=1}^{\infty} B\left(k, \dot{2}^{(-i)_\alpha}\right) = \dot{+}\infty$ for all $i \in \mathbb{N}$. Hence, there exists a sequence of positive integers $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$ such that ${}_\beta\sum_{k=n_{i-1}+1}^{n_i} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{>} \ddot{1}$ for all $i \in \mathbb{N}$ and there exists a $\beta-$number ε_i such that

$${}_\beta\sum_{k=n_{i-1}+1}^{n_i} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{-} (\ddot{n}_i \ddot{-} \ddot{n}_{i-1}) \ddot{\times} \varepsilon_i \ddot{>} \ddot{1}. \tag{5}$$

Let $i \in \mathbb{N}$ be fixed. Then, we have that $\ddot{0} \ddot{\leq} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{<} \dot{+}\infty$ for all $k \in \mathbb{N}$ with $n_{i-1} + 1 \leq k \leq n_i$. From the definition of $B\left(k, \dot{2}^{(-i)_\alpha}\right)$, there exists a $(x_k) \in A\left(z, \dot{2}^{(-i)_\alpha}\right)$ such that

$$\left|f(k,x_k)\right|_\beta \ddot{>} B\left(k, \dot{2}^{(-i)_\alpha}\right) \ddot{-} \varepsilon_i. \tag{6}$$

From 5 and 6,

$$ {}_{\beta}\sum_{k=n_{i-1}+1}^{n_i}\left|f\left(k,x_k\right)\right|_{\beta}\ddot{>}{}_{\beta}\sum_{k=n_{i-1}+1}^{n_i}B\left(k,\dot{2}^{(-i)_{\alpha}}\right)\ddot{-}{}_{\beta}\sum_{k=n_{i-1}+1}^{n_i}\varepsilon_i\ddot{>}\ddot{1} $$

for all $i\in\mathbb{N}$. Thus, we obtain that $\left(f\left(k,x_k\right)\right)_{k=1}^{\infty}\notin\ell_1\left(N\right)$. Since $\left(x_k\right)\in A\left(z,\dot{2}^{(-i)_{\alpha}}\right)$ with $n_{i-1}+1\leq k\leq n_i$ for all $k\in\mathbb{N}$, we have that $\left|x_k\dot{-}z\right|_{\alpha}\dot{\leq}\dot{2}^{(-i)_{\alpha}}$. Hence ${}^{\alpha}\lim_{k\to\infty}x_k=z$ and $\left(x_k\right)\in c\left(N\right)$. This contradicts the assumption. Then there exists an $\alpha-$number $\mu_1\dot{>}\dot{0}$ such that $\left(B\left(k,\mu_1\right)\right)_{k=1}^{\infty}\in\ell_1\left(N\right)$. The proof is completed by putting $c_k=B\left(k,\mu_1\right)$ for all $k\in\mathbb{N}$. □

Example 2.10. *In geometric calculus, $\alpha=I$ and $\beta=\exp$, let $f:\mathbb{N}\times\mathbb{R}(N)_{\alpha}\to\mathbb{R}(N)_{\beta}$ be defined by $f\left(k,t\right)=\dfrac{\left|\iota\left(t\right)\right|_{\beta}}{\ddot{3}^{k_{\beta}}}\beta$ for all $k\in\mathbb{N}$ and for all $t\in\mathbb{R}(N)_{\alpha}$. Then the function $f:\mathbb{N}\times\mathbb{R}\to\mathbb{R}^{+}$ is in the form $f\left(k,t\right)=e^{\left(\frac{t}{3^k}\right)}$. It is clear that f satisfies (NA_2'). Let take any $z>0$. Let $t\in\mathbb{R}$ be such that $\mu=1$ and $|t-z|\leq 1$. Then $|t|\leq 1+|z|$ and since $\dfrac{|t|}{3^k}\leq\dfrac{1+|z|}{3^k}$, it is obtained that*

$$ \left|f\left(k,x_k\right)\right|_{\beta}=\exp\left\{\ln\left(e^{\left(\frac{|t|}{3^k}\right)}\right)\right\}=e^{\left(\frac{|t|}{3^k}\right)}\ddot{\leq}e^{\left(\frac{1+|z|}{3^k}\right)}. $$

If we choose $\left(c_k\right)=\left(e^{\left(\frac{1+|z|}{3^k}\right)}\right)$ for all $k\in\mathbb{N}$, then, we get ${}_{N}P_f:c\left(N\right)\to\ell_1\left(N\right)$ by Theorem 2.9.

Theorem 2.11. *Let us suppose that $f:\mathbb{N}\times\mathbb{R}(N)_{\alpha}\to\mathbb{R}(N)_{\beta}$ satisfies the condition (NA_2'). Then ${}_{N}P_f:\ell_p\left(N\right)\to\ell_1\left(N\right)$ if and only if there exist $\alpha-$numbers $\mu,\eta\dot{>}\dot{0}$ and a $\beta-$sequence $\left(c_k\right)\in\ell_1\left(N\right)$ such that $\left|f\left(k,t\right)\right|_{\beta}\ddot{\leq}c_k\ddot{+}\iota\left(\eta\right)\ddot{\times}\left|\iota\left(t\right)\right|_{\beta}^{p_{\beta}}$ when $|t|_{\alpha}\dot{\leq}\mu$ for all $k\in\mathbb{N}$.*

Proof. Let $x=\left(x_k\right)\in\ell_p\left(N\right)$. Since ${}_{\alpha}\sum_{k=1}^{\infty}|x_k|_{\alpha}^{p_{\alpha}}\dot{<}\dot{+}\infty$, then ${}^{\alpha}\lim_{k\to\infty}|x_k|_{\alpha}^{p_{\alpha}}=\dot{0}$. There exists an integer $i\in\mathbb{N}$ such that $|x_k|_{\alpha}^{p_{\alpha}}\dot{\leq}\mu^{p_{\alpha}}$ especially for $\alpha-$number $\mu\dot{>}\dot{0}$ when $k\geq i$. By assumption, there exists a $c_k\in\ell_1\left(N\right)$ such that $\left|f\left(k,x_k\right)\right|_{\beta}\ddot{\leq}c_k\ddot{+}\iota\left(\eta\right)\ddot{\times}\left|\iota\left(t\right)\right|_{\beta}^{p_{\beta}}$ when for all $k\geq i$. Then

$$ \begin{aligned} {}_{\beta}\sum_{k=i}^{\infty}\left|f\left(k,x_k\right)\right|_{\beta} &\ddot{\leq} {}_{\beta}\sum_{k=i}^{\infty}c_k\ddot{+}\iota\left(\eta\right)\ddot{\times}{}_{\beta}\sum_{k=i}^{\infty}\left|\iota\left(x_k\right)\right|_{\beta}^{p_{\beta}} \\ &\ddot{\leq} \left\|c_k\right\|_{\beta,1}\ddot{+}\iota\left(\eta\right)\ddot{\times}\left\|\iota\left(x\right)\right\|_{\beta,p}. \end{aligned} $$

Hence, we obtain ${}_{N}P_f\left(x\right)\in\ell_1\left(N\right)$.

Conversely, let ${}_{N}P_f:\ell_p\left(N\right)\to\ell_1\left(N\right)$. For each $\alpha-$numbers $\mu,\eta\dot{>}\dot{0}$ and for all $k\in\mathbb{N}$, we define

$$ A\left(k,\eta,\mu\right)=\left\{t\in\mathbb{R}(N)_{\alpha}:|t|_{\alpha}^{p_{\alpha}}\dot{\leq}{}^{\alpha}\min\left\{\mu,\eta^{(-1)_{\alpha}}\dot{\times}\iota^{-1}\left(\left|f\left(k,x_k\right)\right|_{\beta}\right)\right\}\right\} $$

and

$$ B\left(k,\eta,\mu\right)={}^{\beta}\sup\left\{\left|f\left(k,t\right)\right|_{\beta}:t\in A\left(k,\eta,\mu\right)\right\}. $$

Then, we can see that $\left|f\left(k,t\right)\right|_{\beta}\ddot{\leq}B\left(k,\eta,\mu\right)$ whenever $t\in A\left(k,\eta,\mu\right)$ and $|t|_{\alpha}\dot{\leq}\eta$. Additionally, if $t\notin A\left(k,\eta,\mu\right)$ and $|t|_{\alpha}\dot{\leq}\mu$, then, we have $\left|f\left(k,t\right)\right|_{\beta}\ddot{\leq}\iota\left(\mu\right)\ddot{\times}\left|\iota\left(t\right)\right|_{\beta}^{p_{\beta}}$. Hence, we get $\left|f\left(k,t\right)\right|_{\beta}\ddot{\leq}B\left(k,\eta,\mu\right)\ddot{+}\iota\left(\mu\right)\ddot{\times}\left|\iota\left(t\right)\right|_{\beta}^{p_{\beta}}$ when

$|t|_\alpha \dot{\leq} \mu$. We shall show there exist $\alpha-$numbers $\mu_1, \eta_1 \dot{>} \dot{0}$ such that $(B(k, \eta_1, \mu_1))_{k=1}^{\infty} \in \ell_1(N)$. Suppose that $(B(k, \eta, \mu))_{k=1}^{\infty} \notin \ell_1(N)$ for all $\mu, \eta \dot{>} \dot{0}$. Then it is written that ${}_\beta \sum_{k=1}^{\infty} B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right) = \ddot{+}\infty$ for all $i \in \mathbb{N}$. Thus there exists a sequence of positive integers $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$ such that ${}_\beta \sum_{k=n_{i-1}+1}^{n_i} B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right) \ddot{>} \ddot{1}$ for all $i \in \mathbb{N}$ and n_i is the smallest integer that satisfies this condition. Otherwise, there exists a $\beta-$number ε_i such that

$$ {}_\beta \sum_{k=n_{i-1}+1}^{n_i} B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right) \ddot{-} (\ddot{n}_i \ddot{-} \ddot{n}_{i-1}) \ddot{\times} \varepsilon_i \ddot{>} \ddot{1}. \tag{7} $$

Let $i \in \mathbb{N}$ be fixed. Then, we have that $\ddot{0} \ddot{\leq} B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right) \ddot{<} \ddot{+}\infty$ for all $k \in \mathbb{N}$ with $n_{i-1} + 1 \leq k \leq n_i$. From the definition of $B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right)$, there exists a $(x_k) \in A\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right)$ such that

$$ \left|f(k, x_k)\right|_\beta \ddot{>} B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right) \ddot{-} \varepsilon_i. \tag{8} $$

From 7 and 8 we have

$$ {}_\beta \sum_{k=n_{i-1}+1}^{n_i} \left|f(k, x_k)\right|_\beta \ddot{>} {}_\beta \sum_{k=n_{i-1}+1}^{n_i} B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right) \ddot{-} {}_\beta \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i \ddot{>} \ddot{1} $$

for all $i \in \mathbb{N}$. Thus, we get $(f(k, x_k))_{k=1}^{\infty} \notin \ell_1(N)$. Since $(x_k) \in A\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right)$ with $n_{i-1} + 1 \leq k \leq n_i$ for all $k \in \mathbb{N}$, we have

$$ |x_k|_\alpha \dot{\leq} \dot{2}^{(-i)_\alpha} \text{ and } |x_k|_\alpha^{p_\alpha} \dot{\leq} \dot{2}^{(-i)_\alpha} \dot{\times} \iota^{-1}\left(\left|f(k, x_k)\right|_\beta\right). \tag{9} $$

Additionally, ${}_\beta \sum_{k=n_{i-1}+1}^{n_i - 1} B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right) \ddot{\leq} \ddot{1}$. From 9, it is obtained that

$$ \begin{aligned} {}_\alpha \sum_{k=n_{i-1}+1}^{n_i} |x_k|_\alpha^{p_\alpha} &= {}_\alpha \sum_{k=n_{i-1}+1}^{n_i - 1} |x_k|_\alpha^{p_\alpha} \dot{+} \left|x_{n_i}\right|_\alpha \\ &\dot{\leq} {}_\alpha \sum_{k=n_{i-1}+1}^{n_i - 1} \dot{2}^{(-i)_\alpha} \dot{\times} \iota^{-1}\left(\left|f(k, x_k)\right|_\beta\right) \dot{+} \dot{2}^{(-i)_\alpha} \\ &\dot{\leq} \dot{2}^{(-i)_\alpha} \dot{\times} {}_\alpha \sum_{k=n_{i-1}+1}^{n_i - 1} B\left(k, \dot{2}^{i_\alpha}, \dot{2}^{(-i)_\alpha}\right) \dot{+} \dot{2}^{(-i)_\alpha} \\ &\dot{\leq} \frac{\dot{2}}{\dot{2}^{i_\alpha}} \alpha. \end{aligned} $$

Hence we have that $(x_k) \in \ell_p(N)$. This contradicts the assumption. Then there exist $\alpha-$numbers $\mu_1, \eta_1 \dot{>} \dot{0}$ such that $(B(k, \eta_1, \mu_1))_{k=1}^{\infty} \in \ell_1(N)$. The proof is completed by putting $c_k = B(k, \eta_1, \mu_1)$ for all $k \in \mathbb{N}$. □

Example 2.12. *Let $f : \mathbb{N} \times \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be defined by $f(k, t) = \left(\frac{\ddot{1}}{\ddot{9}^{k_\beta}} \beta \ddot{+} |\iota(t)|_\beta^{p_\beta}\right) \ddot{\times} |\iota(t)|_\beta$ for all $k \in \mathbb{N}$ and $t \in \mathbb{R}(N)_\alpha$. It is clear that f satisfies the condition (NA_2'). Let $\mu, t \in \mathbb{R}(N)_\alpha$ such that $\mu = \dot{2}$ and $|t|_\alpha \dot{\leq} \dot{2}$. Then*

$$ \begin{aligned} \left|f(k, t)\right|_\beta &= \left(\frac{\ddot{1}}{\ddot{9}^{k_\beta}} \beta \ddot{+} |\iota(t)|_\beta^{p_\beta}\right) \ddot{\times} |\iota(t)|_\beta \\ &\ddot{\leq} \frac{\ddot{2}}{\ddot{9}^{k_\beta}} \beta \ddot{+} \ddot{2} \ddot{\times} |\iota(t)|_\beta^{p_\beta}. \end{aligned} $$

If we choose $(c_k) = \left(\frac{\ddot{2}}{\ddot{9}^{k_\beta}}\beta\right)$ *for all* $k \in \mathbb{N}$ *and* $\eta = \dot{2}$*, then, we obtain* ${}_NP_f : \ell_p(N) \to \ell_1(N)$ *by Theorem 2.11.*

Theorem 2.13. *Non-Newtonian superposition operator* ${}_NP_f : \ell_\infty(N) \to \ell_1(N)$ *is *-continuous on* $\ell_\infty(N)$ *if and only if the function* $f(k,.)$ *is *-continuous* $\mathbb{R}(N)_\alpha$ *for all* $k \in \mathbb{N}$.

Proof. Suppose that ${}_NP_f$ is non-Newtonian continuous on $\ell_\infty(N)$. Let $k \in \mathbb{N}$, $t_0 \in \mathbb{R}(N)_\alpha$ and $\varepsilon \ddot{>} \ddot{0}$ be given. Since ${}_NP_f$ is non-Newtonian continuous at $t_0 \dot{\times} e^{(k)} \in \ell_\infty(N)$ with $e_n^{(k)} = \begin{cases} \dot{1} & n = k \\ \dot{0}, & n \neq k \end{cases}$, we have

$$\left\| {}_NP_f(z) \ddot{-} {}_NP_f\left(t_0 \dot{\times} e^{(k)}\right)\right\|_{\beta,1} \ddot{<} \varepsilon \tag{10}$$

when $\left\|t \dot{-} \left(t_0 \dot{\times} e^{(k)}\right)\right\|_{\alpha,\infty} \dot{<} \delta$ for all $z = (z_k) \in \ell_\infty(N)$. Let $t \in \mathbb{R}(N)_\alpha$ be such that $|t \dot{-} t_0|_\alpha \dot{<} \delta$. If y_n is defined in the form of $y_n = \begin{cases} t & n = k \\ \dot{0}, & n \neq k \end{cases}$, $(y_n) \in \ell_\infty(N)$ and

$$|t \dot{-} t_0|_\alpha = \left\|y \dot{-} \left(t_0 \dot{\times} e^{(k)}\right)\right\|_{\alpha,\infty} \dot{<} \delta.$$

By 10, we get

$$\left|f(k,t) \ddot{-} f(k,t_0)\right|_\beta = \left\| {}_NP_f(y) \ddot{-} {}_NP_f\left(t_0 \dot{\times} e^{(k)}\right)\right\|_{\beta,1} \ddot{<} \varepsilon.$$

Hence, the function $f(k,.)$ is *-continuous on $\mathbb{R}(N)_\alpha$ for all $k \in \mathbb{N}$.

Conversely, suppose that the function $f(k,.)$ is *-continuous on $\mathbb{R}(N)_\alpha$ for all $k \in \mathbb{N}$. We shall show that ${}_NP_f$ is non-Newtonian continuous on $\ell_\infty(N)$. Let $x = (x_k) \in \ell_\infty(N)$ and $\varepsilon \ddot{>} \ddot{0}$ be given. Since f is *-continuous, it is clear that f satisfies (NA_2'). Since ${}_NP_f : \ell_\infty(N) \to \ell_1(N)$, there exists a $\beta-$sequence $(c_k) \in \ell_1(N)$ such that

$$\left|f(k,t)\right|_\beta \ddot{\leq} c_k \text{ with } |t|_\alpha \dot{\leq} \mu \tag{11}$$

for all $\mu \dot{>} \dot{0}$ and $k \in \mathbb{N}$ by Theorem 2.5. Since $x \in \ell_\infty(N)$, there exists an $\alpha-$number $\gamma \dot{>} \dot{0}$ such that $|x_k|_\alpha \dot{\leq} \frac{\gamma}{\dot{2}}\alpha$ for all $k \in \mathbb{N}$. Hence, by 11, there exists a $\beta-$sequence $(c_k) \in \ell_1(N)$ such that

$$\left|f(k,t)\right|_\beta \ddot{\leq} c_k \tag{12}$$

for all $k \in \mathbb{N}$. Additionally, by 11, there exists a $\beta-$sequence $\left(c_k^*\right) \in \ell_1(N)$ such that

$$\left|f(k,t)\right|_\beta \ddot{\leq} c_k^* \text{ with } |t|_\alpha \dot{\leq} \gamma \tag{13}$$

for all $k \in \mathbb{N}$. Since $(c_k), \left(c_k^*\right) \in \ell_1(N)$, there exists a $N \in \mathbb{N}$ such that

$$\beta \sum_{k=N}^{\infty} c_k \ddot{<} \frac{\varepsilon}{\ddot{3}}\beta \text{ and } \beta \sum_{k=N}^{\infty} c_k^* \ddot{<} \frac{\varepsilon}{\ddot{3}}\beta. \tag{14}$$

Since $f(k,.)$ is *-continuous at x_k, there exists an $\alpha-$number $\delta \dot{>} \dot{0}$ with $\delta = {}^{\alpha}\min\left\{\dot{1}, \frac{\gamma}{\dot{2}}\alpha\right\}$ such that

$$\left|f(k,t) \ddot{-} f(k,x_k)\right|_\beta \ddot{<} \frac{\varepsilon}{\ddot{3} \ddot{\times} \left(\ddot{N} \ddot{-} \ddot{1}\right)}\beta \text{ whenever } |t \dot{-} x_k|_\alpha \dot{<} \delta \tag{15}$$

for all $k \in \{1, 2, ..., N-1\}$ and $t \in \mathbb{R}(N)_\alpha$. Let $z = (z_k) \in \ell_\infty(N)$ be given with $\|z \dot{-} x\|_{\alpha,\infty} \dot{<} \delta$. Then $|z_k \dot{-} x_k|_\alpha \dot{<} \delta$ for all $k \in \mathbb{N}$ and $\left|f(k, z_k) \ddot{-} f(k, x_k)\right|_\beta \ddot{<} \dfrac{\varepsilon}{\ddot{3} \ddot{\times} \left(\ddot{N} \ddot{-} \ddot{1}\right)} \beta$ for all $k \in \{1, 2, ..., N-1\}$ from 15. Then, we obtain

$$\beta \sum_{k=1}^{N-1} \left|f(k, z_k) \ddot{-} f(k, x_k)\right|_\beta \ddot{<} \frac{\varepsilon}{\ddot{3}} \beta. \tag{16}$$

Since $|z_k|_\alpha \dot{\leq} |z_k \dot{+} x_k|_\alpha \dot{+} |x_k|_\alpha \dot{<} \delta \dot{+} \dfrac{\gamma}{\dot{2}} \alpha \dot{<} \gamma$, by 13, $\left|f(k, z_k)\right|_\beta \ddot{\leq} c_k^*$ for all $k \in \mathbb{N}$. Thus,we have that

$$\begin{aligned}
\beta \sum_{k=N}^{\infty} \left|f(k, x_k)\right|_\beta &\ddot{\leq} \beta \sum_{k=N}^{\infty} c_k && (17)\\
&\ddot{<} \frac{\varepsilon}{\ddot{3}} \beta && (18)\\
\beta \sum_{k=N}^{\infty} \left|f(k, z_k)\right|_\beta &\ddot{\leq} \beta \sum_{k=N}^{\infty} c_k^* && (19)\\
&\ddot{<} \frac{\varepsilon}{\ddot{3}} \beta && (20)
\end{aligned}$$

from 12 and 14. Then, by 16, 17 and 19,

$$\begin{aligned}
\left\| {}_N P_f(z) \ddot{-} {}_N P_f(x) \right\|_\beta &= \beta \sum_{k=1}^{\infty} \left|f(k, z_k) \ddot{-} f(k, x_k)\right|_\beta \\
&\ddot{\leq} \beta \sum_{k=1}^{N-1} \left|f(k, z_k) \ddot{-} f(k, x_k)\right|_\beta \ddot{+} \beta \sum_{k=N}^{\infty} \left|f(k, z_k)\right|_\beta \\
&\quad \ddot{+} \beta \sum_{k=N}^{\infty} \left|f(k, x_k)\right|_\beta \\
&\ddot{<} \frac{\varepsilon}{\ddot{3}} \beta \ddot{+} \frac{\varepsilon}{\ddot{3}} \beta \ddot{+} \frac{\varepsilon}{\ddot{3}} \beta \\
&= \varepsilon.
\end{aligned}$$

This completes the proof. □

3. Concluding Remarks

The necessary and sufficient conditions for the characterization of non-Newtonian superposition operators have been formulated, as stated in Theorem 2.5, Theorem 2.7, Theorem 2.9 and Theorem 2.11. For the future, we will formulate the necessary and sufficient conditions for $\beta-$boundedness and characterization from $\ell_p(N)$ into $\ell_q(N)$ of non-Newtonian superposition operators.

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