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$c-$Credibility Measure

Nebojša M. Ralević[a], Marija Paunović[a]

[a] *Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad*

Abstract. In this paper we introduce the concept of credibility measure and we show some of its basic properties. In this frame we present several results for credibility mappings. Our results generalize the notion of the credibility measure of Lee.

1. Introduction

Credibility theory (CT) in actuarial mathematics can be used to calculate the premium rate, as well as to determine the future premium rate based on experience and provision. Given that goal is to set the appropriate premium rates for the future, it is important to adjust the past premium rates to the expected value in the future period. Credibility factor Z is used to weight the observation, and its complement is attached to the other information in the given application. Alternative version of credibility theory in a fuzzy environment (CTF) and the credibility measure are formulated in Liu and Liu [6], and Liu [10].

Credibility measure, as a concept for the measure of a fuzzy event, is a set function satisfying normality, monotonicity, self-duality and maximality. Another difference in CTF refers to the weighted average based on the concepts of possibility measure and necessity measure.

In this paper, a generalized credibility theory is proposed. In that purpose in Section 2 the background knowledge are briefly introduced,terms of the operations defined on $[0, 1]$ interval (triangular norm, conorm and uninorm, fuzzy complement and aggregation function) as well as their properties. The third section contains an overview of the fuzzy measure. In Section 4, a new fuzzy measure is introduced, called $c-$ credibility measure. Some properties of the $c-$credibility measure are proved, such as, for example, subadditivity and semicontinuity.

Furthermore, an integral based on this measure is defined, in analogy to the existing integrals, and its properties. In the next section, the credibility in a fuzzy environment is introduced as the aggregation of the possibility and necessity measures. Examples of the application of such credibility and its comparison with the classical credibility are also shown.

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Email addresses: nralevic@uns.ac.rs (Nebojša M. Ralević), marija.paunovic@uns.ac.rs (Marija Paunović)

2. Aggregation functions

In this section, we recall some basic terms and properties of the operations defined on the interval $[0,1]$ which will be used in the paper (see Klement, Mesiar and Pap [4], Klir and Yuan [5], Yager and Rybalov [12]).

Definition 2.1. *Let the binary operation $F : [0,1]^2 \to [0,1]$, satisfying the following axioms:*

1. $(\forall a, b_1, b_2 \in [0,1])\ b_1 \leq b_2 \Rightarrow F(a,b_1) \leq F(a,b_2)$ *(monotonicity);*
2. $(\forall a, b \in [0,1])\ F(a,b) = F(b,a)$ *(commutativity);*
3. $(\forall a, b, c \in [0,1])\ F(a,F(b,c)) = F(F(a,b),c)$ *(associativity);*
4. $(\exists e \in [0,1])(\forall a \in [0,1])\ F(a,e) = a$ *(e is neutral element);.*

then we say that F is a norm. If $e = 0$, then F is a triangular conorm(shortly t−conorm) and instead of F we write S. If $e = 1$, then F is a triangular norm (shortly t−norm) and instead of F we write T. If $e \in (0,1)$, then F is a uninorm and instead of F we write U.

Remark 2.2. *From the conditions given in the previous definition follows the monotonicity by coordinates, i.e. for all $a_1, a_2, b_1, b_2 \in [0,1]$ it holds that*

$$a_1 \leq a_2 \wedge b_1 \leq b_2 \Rightarrow F(a_1,b_1) \leq F(a_2,b_2).$$

By replacing the given condition with the monotonic axiom in the definition of the t−norm, an equivalent definition is obtained.

If, in the definition, instead of the axiom of monotonicity, a strict monotonicity is valid, i.e.

$$a_1 < a_2 \wedge b_1 < b_2 \Rightarrow F(a_1,b_1) < F(a_2,b_2),$$

for all $a_1, a_2, b_1, b_2 \in [0,1]$, we say that F is *strict*.

Definition 2.3. *The power of the norm is given by formulas:*
$F^1(a_1,a_2) = F(a_1,a_2), \quad F^n(a_1,...,a_n,a_{n+1}) = F(F^{n-1}(a_1,...,a_n),a_{n+1}).$

The most commonly used triangular norms are:

1. $T(a,b) = \min(a,b) = a \wedge b$ (standard intersection);
2. $T(a,b) = ab$ (algebraic product);
3. $T(a,b) = \max(a+b-1,0)$ (bounded difference);
4. $T(a,b) = \begin{cases} a, & b = 1 \\ b, & a = 1 \\ 0, & \text{otherwise} \end{cases}$ (drastic intersection);

The most common triangular conorms are:

1. $S(a,b) = \max(a,b) = a \vee b$ (standard union);
2. $S(a,b) = a + b - ab$ (algebraic sum);
3. $S(a,b) = \min(1, a+b)$ (bounded sum);
4. $S(a,b) = \begin{cases} a, & b = 0 \\ b, & a = 0 \\ 1, & \text{otherwise} \end{cases}$ (drastic union).

Definition 2.4. *The function $c : [0,1] \to [0,1]$ is a fuzzy complement, if it satisfies the following conditions:*

c_1) $c(0) = 1$ *and* $c(1) = 0$, *(boundary conditions)*

c_2) $(\forall a, b \in [0,1])\ a \leq b \Rightarrow c(a) \geq c(b)$ (monotonicity).

If $c(c(a)) = a$ holds, for all $a \in [0,1]$, then the function c is involutive.
If c is a continuous function, then we say that c is a continuous fuzzy complement.

Lemma 2.5. *([5]). If $c : [0,1] \to [0,1]$ is an involutive, monotonic and non increasing function, than follows that c is a continuous bijective function for which boundary conditions are valid.*

The most commonly used continuous involutive fuzzy complements are:

1) $c(a) = 1 - a$, (standard fuzzy complement);

2) $c_\lambda(a) = \frac{1-a}{1+\lambda a}$, $\lambda \in (1, \infty)$ (Sugeno class fuzzy complement);

3) $c_\lambda(a) = (1 - a^\lambda)^{1/\lambda}$, $\lambda \in (0, \infty)$ (Yager class).

Definition 2.6. *The equilibrium of fuzzy complement is element $\epsilon \in [0,1]$, such that $c(\epsilon) = \epsilon$ is satisfied.*

Theorem 2.7. *Every fuzzy complement has at most one equilibrium.*
If fuzzy complement c has an equilibrium, then

$$a \geq \epsilon \Rightarrow \epsilon \geq c(a), \quad a \leq \epsilon \Rightarrow \epsilon \leq c(a).$$

If c is a continuous fuzzy complement, then c has a unique equilibrium.

Lemma 2.8. *De Morgan's laws hold, i.e.*

$$c(a \vee b) = c(a) \wedge c(b), \quad c(a \wedge b) = c(a) \vee c(b).$$

Definition 2.9. *An aggregation function is a function $\mathsf{A} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ such that*

A_1) $\mathsf{A}(0, ..., 0) = 0$ and $\mathsf{A}(1, ..., 1) = 1$ (boundary condition).

A_2) $\mathsf{A}(x_1, ..., x_n) \leq \mathsf{A}(y_1, ..., y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1, ..., n\}$ (A is monotonically nondecreasing function in all its arguments).

A_3) $\mathsf{A}(x) = x$ for all $x \in [0,1]$ (A is idempotent function).

The aggregation function A is

1. idempotent if $\mathsf{A}(x, ..., x) = x$ for all $x \in [0,1]$ (A is idempotent function).
2. continuous if A is continuous function.
3. commutative if A is symmetric function in all its arguments, i.e. $\mathsf{A}(x_1, ..., x_n) = \mathsf{A}(x_{p_1}, ..., x_{p_n})$ for any permutation $(p_1, ..., p_n)$ of set $\{1, ..., n\}$.

Remark 2.10. *The aggregation function is also defined as a function $\mathsf{A} : \bigcup_{n \in \mathbb{N}} \mathtt{I}^n \to \mathtt{I}$, where $\mathtt{I}$ is nonempty subinterval of the extended real, such that*

A_1) $\inf_{\mathtt{x} \in \mathtt{I}^n} \mathsf{A}(\mathtt{x}) = \inf \mathtt{I}$ and $\sup_{\mathtt{x} \in \mathtt{I}^n} \mathsf{A}(\mathtt{x}) = \sup \mathtt{I}$ (boundary condition).

A_2) $\mathsf{A}(\mathtt{x}) \leq \mathsf{A}(\mathtt{y})$ whenever $\mathtt{x} = (x_1, ..., x_n) \leq (y_1, ..., y_n) = \mathtt{y} \Leftrightarrow (\forall i \in \{1, ..., n\})\ x_i \leq y_i$ (A is monotonically nondecreasing function in all its arguments).

A_3) $\mathsf{A}(x) = x$ for all $x \in [0,1]$ (A is idempotent function).

Some examples of aggregation functions:

1) $WAM(x_1, x_2, \ldots, x_n) = \sum\limits_{i=1}^{n} w_{i_n} x_i$ weighted arithmetic mean associated with weight vector $w = (w_1, w_2, \ldots, w_n)$, $\sum\limits_{i=1}^{n} w_i = 1$, $w_i \in [0, 1]$.

2) $M_f(x_1, x_2, \ldots, x_n) = f^{-1}\Big(\frac{1}{n} \sum\limits_{i=1}^{n} f(x_i)\Big)$ ($f : [0, 1] \to [-\infty, +\infty]$ continuous and strictly monotonic function) quasi-arithmetic mean;

3) The power of the norm.

The root-power mean $M_p(x_1, x_2, \ldots, x_n) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$, $p \in (-\infty, 0) \cup (0, +\infty)$ is special case of 2). Marginal members of these classes are $M_0 = G = M_{\log x}$, which is the geometric mean, while $M_\infty = \max$ and $M_{-\infty} = \min$ which are not in the class of quasi-arithmetic means.

For details on aggregation functions see for example Grabish, Marichal, Mesiar and Pap [2] or Dubois and Prade [1].

Dual aggregation function of aggregation function A with respect to a fuzzy complement c is a function $\overline{\mathsf{A}}_c$ (briefly $\overline{\mathsf{A}}$) defined by

$$\overline{\mathsf{A}}_c(x_1, ..., x_n) = c(\mathsf{A}(c(x_1), ..., c(x_n))).$$

Lemma 2.11. *Dual aggregation function is aggregation function.*

Proof. A_1) From boundary conditions of function A, we have

$$\overline{\mathsf{A}}_c(0, ..., 0) = c(\mathsf{A}(c(0), ..., c(0))) = c(\mathsf{A}(1, ..., 1)) = c(1) = 0,$$

$$\overline{\mathsf{A}}_c(1, ..., 1) = c(\mathsf{A}(c(1), ..., c(1))) = c(\mathsf{A}(0, ..., 0)) = c(0) = 1.$$

A_2) Suppose $x_i \leq y_i$ for all $i \in I = \{1, ..., n\}$. The complement c is non-increasing function, then $c(x_i) \geq c(y_i)$, $i \in I$, and because A is monotonically non-decreasing function in all its arguments, it follows

$$\mathsf{A}(c(x_1), ..., c(x_n)) \geq \mathsf{A}(c(y_1), ..., c(y_n)),$$

$$c(\mathsf{A}(c(x_1), ..., c(x_n))) \leq c(\mathsf{A}(c(y_1), ..., c(y_n))),$$

$$\overline{\mathsf{A}}_c(x_1, ..., x_n) \leq \overline{\mathsf{A}}_c(y_1, ..., y_n).$$

A_3) The function A is idempotent

$$\overline{\mathsf{A}}_c(x) = c(\mathsf{A}(c(x))) = c(c(x)) = x,$$

if c is involutive.

1. If A is an idempotent aggregation function, we have

$$\overline{\mathsf{A}}_c(x, ..., x) = c(\mathsf{A}(c(x), ..., c(x))) = c(c(x)) = x,$$

if c is involutive fuzzy complement.

2. If A is a continuous function, supposing that complement c is continuous function, and from properties that composition of the continuous functions is a continuous function, it follow that $\overline{A}_c$ is a continuous function.

3. Assuming A is a symmetric function in all its arguments, for any permutation $(p_1, ..., p_n)$ of set $\{1, ..., n\}$, we have

$$\overline{\mathsf{A}}_c(x_1, ..., x_n) = c(\mathsf{A}(c(x_1), ..., c(x_n))) = c(\mathsf{A}(c(x_{p_1}), ..., c(x_{p_n})) = \overline{\mathsf{A}}_c(x_{p_1}, ..., x_{p_n}),$$

i.e. $\overline{\mathsf{A}}_c$ is a commutative function. □

3. Fuzzy measure

Definition 3.1. *Let X be a nonempty set and Σ be a nonempty class of subsets of X, such that $\emptyset \in \Sigma$. The map* $\mathsf{m} : \Sigma \to [0, \infty]$ *is called a fuzzy measure (fuzzy measure in the narrow sense)* m *on* Σ *if it holds that*

FM_1) $\mathsf{m}(\emptyset) = 0$,

FM_2) $(\forall A, B \in \Sigma)\ A \subset B \ \Rightarrow\ \mathsf{m}(A) \leq \mathsf{m}(B)$ *(monotonicity)*

FM_3) $A_n \subset A_{n+1}\ ,\ A_n \in \Sigma\ ,\ n \in \mathbb{N}\ ,\ \bigcup\limits_{n=1}^{\infty} A_n \in \Sigma \ \Rightarrow\ \mathsf{m}(\bigcup\limits_{n=1}^{\infty} A_n) = \lim\limits_{n\to\infty} \mathsf{m}(A_n)$ *(continuity from below)*

FM_4) $A_n \supset A_{n+1}\ ,\ A_n \in \Sigma\ ,\ n \in \mathbb{N}\ ,\ \bigcap\limits_{n=1}^{\infty} A_n \in \Sigma$ *and there exist* $n_0 \in \mathbb{N}$ *such that* $\mathsf{m}(A_{n_0}) < \infty$

$\Rightarrow\ \mathsf{m}(\bigcap\limits_{n=1}^{\infty} A_n) = \lim\limits_{n\to\infty} \mathsf{m}(A_n)$, *(continuity from above)*

If also condition $\mathsf{m}(X) = 1$, where $X \in \Sigma$ holds, then it is a regular fuzzy measure. The triple (X, Σ, m) is a space with fuzzy measure or space with semi-continuous fuzzy measure.

We say m is lower or upper semi-continuous fuzzy measure if FM_1), FM_2), and FM_3) or FM_1), FM_2) and FM_4) are satisfied. If only one of those conditions holds, then m is a semi-continuous fuzzy measure. If for m also FM_1) and FM_2) hold, then it is a fuzzy measure in the broader sense.

Usually, Σ is a monotone class, semiring, $\sigma-$ring, $\sigma-$algebra, plump class or $\mathcal{P}(X)$ (power set of X). (see [13])

Definition 3.2. *Possibility function pos (see [14]) is a fuzzy measure on* (X, Σ) *if*

$$pos(\bigcup_{i\in I} A_i) = \sup_{i\in I} pos(A_i),$$

for any family $\{A_i | i \in I\}$ *in* Σ *such that* $\bigcup\limits_{i\in I} A_i \in \Sigma$, *where I is an arbitrary index set.*

Definition 3.3. *Necessity function nec (see [14]) is a fuzzy measure on* (X, Σ) *if*

$$nec(\bigcap_{i\in I} A_i) = \inf_{i\in I} nec(A_i),$$

for any family $\{A_i | i \in I\}$ *in* Σ *such that* $\bigcup\limits_{i\in I} A_i \in \Sigma$, *where I is an arbitrary index set.*

Let X be a nonempty sample set, $\mathcal{P}(X)$ the power set of X, and *pos* a possibility measure defined on X. Then the triplet $(X, \mathcal{P}(X), pos)$ is called a possibility space. The functions *pos* and *nec* are dual, i.e. $nec(A) = 1 - pos(\overline{A})$ and $pos(A) = 1 - nec(\overline{A})$.

Liu in his Uncertainty theory introduced the credibility measure (see [10], [7], [8], [9], [11]).

Definition 3.4. *Let X be a nonempty set and I an arbitrary index set. A set function* $Cr : \mathcal{P}(X) \to [0, 1]$ *such that for all* $A, B \subset X$:
C_1) $Cr(X) = 1$ *(normality);*
C_2) $A \subset B \Rightarrow Cr(A) \leq Cr(B)$ *(monotonicity);*
C_3) $Cr(A) + Cr(\overline{A}) = 1$ *(self-duality);*
C_4) $Cr(\bigcup\limits_{i\in I} A_i) = \sup\limits_{i\in I} Cr(A_i)$, *for any sets* $A_i \subset X, i \in I$ *for which it is* $\sup\limits_{i\in I} Cr(A_i) < 1/2$,
is called the credibility measure.

4. c−Credibility

Definition 4.1. *Let $c : [0,1] \to [0,1]$ be an involutive fuzzy complement, whose equilibrium is ϵ. The c−credibility measure on X is a set function $\mathsf{cr} : \mathcal{P}(X) \to [0,1]$ such that*

CR_1) $\mathsf{cr}(\emptyset) = 0$;

CR_2) $(\forall A, B \in \mathcal{P}(X))\ \ A \subset B \Rightarrow \mathsf{cr}(A) \le \mathsf{cr}(B)$;

CR_3) $(\forall A \in \mathcal{P}(X))\ \ \mathsf{cr}(\overline{A}) = c(\mathsf{cr}(A))$;

CR_4) $\mathsf{cr}(\bigcup\limits_{i\in I} A_i) = \sup\limits_{i\in I} \mathsf{cr}(A_i)$, *for any sets $A_i \in \mathcal{P}(X)$, $i \in I$, for which it is $\sup\limits_{i\in I} \mathsf{cr}(A_i) < \epsilon$, where I is an arbitrary index set.*

The triplet $(X, \mathcal{P}(X), \mathsf{cr})$ is called c−credibility space.

The fuzzy complement c is involutive function so

$$\mathsf{cr}(A) = c(\mathsf{cr}(\overline{A})).$$

It is clear that $\mathsf{cr}(X) = \mathsf{cr}(\overline{\emptyset}) = c(\mathsf{cr}(\emptyset)) = c(0) = 1$. Also $0 \le \mathsf{cr}(A) \le 1$. Indeed $\emptyset \subset A \subset X \Rightarrow 0 = \mathsf{cr}(\emptyset) \le \mathsf{cr}(A) \le \mathsf{cr}(X) = 1$.

Theorem 4.2. *Let cr be a c−credibility measure, then for all $A, B \in \mathcal{P}(X)$*

$$i)\ \ \mathsf{cr}(A \cup B) \le \epsilon \ \ \Rightarrow \ \ \mathsf{cr}(A \cup B) = \mathsf{cr}(A) \vee \mathsf{cr}(B).$$

$$ii)\ \ \mathsf{cr}(A \cap B) \ge \epsilon \ \ \Rightarrow \ \ \mathsf{cr}(A \cap B) = \mathsf{cr}(A) \wedge \mathsf{cr}(B).$$

Proof. i) If $\mathsf{cr}(A \cup B) < \epsilon$, then from monotonicity $\mathsf{cr}(A) \le \mathsf{cr}(A \cup B)$ and $\mathsf{cr}(B) \le \mathsf{cr}(A \cup B)$, it is $\mathsf{cr}(A) \vee \mathsf{cr}(B) \le \mathsf{cr}(A \cup B)$, i.e. $\mathsf{cr}(A) \vee \mathsf{cr}(B) < \epsilon$, due to CR_4) we have $\mathsf{cr}(A \cup B) = \mathsf{cr}(A) \vee \mathsf{cr}(B)$.

If $\mathsf{cr}(A \cup B) = \epsilon$ and we suppose $\mathsf{cr}(A \cup B) > \mathsf{cr}(A) \vee \mathsf{cr}(B)$, must be that $\mathsf{cr}(A) \vee \mathsf{cr}(B) < \epsilon$, so we can apply CR_4):

$$\mathsf{cr}(A \cup B) = \mathsf{cr}(A) \vee \mathsf{cr}(B) < \epsilon,$$

which gives contradiction with the assumption.

ii) From $\mathsf{cr}(A \cap B) \ge \epsilon$, monotonicity of the fuzzy complement and the fact that ϵ is equilibrium, it follows $\epsilon = c(\epsilon) \ge c(\mathsf{cr}(A \cap B))$. Now from CR_2) we have $c(\mathsf{cr}(A \cap B)) = c(\mathsf{cr}(\overline{\overline{A} \cup \overline{B}})) = \mathsf{cr}(\overline{A} \cup \overline{B})$ and from the property i), we have

$\mathsf{cr}(A \cap B) = c(\mathsf{cr}(\overline{A \cap B})) = c(\mathsf{cr}(\overline{A} \cup \overline{B})) = c(\mathsf{cr}(\overline{A}) \vee \mathsf{cr}(\overline{B})) = c(\mathsf{cr}(\overline{A})) \wedge c(\mathsf{cr}(\overline{B})) = \mathsf{cr}(A) \wedge \mathsf{cr}(B)$ □

Theorem 4.3. *(Subadditivity Law) Let c be an involutive fuzzy complement, such that $c(x) \ge 1 - x$ for all $x \in [0,1]$. Then subadditivity holds, i.e.*

$$(\forall A, B \in \mathcal{P}(X))\ \ \ \mathsf{cr}(A \cup B) \le \mathsf{cr}(A) + \mathsf{cr}(B).$$

Proof. If $\mathsf{cr}(A), \mathsf{cr}(B) \in [0, \epsilon)$, then from CR_4) follows

$$\mathsf{cr}(A \cup B) = \mathsf{cr}(A) \vee \mathsf{cr}(B) \le \mathsf{cr}(A) + \mathsf{cr}(B).$$

Let $\mathsf{cr}(A) \in [\epsilon, 1]$ or $\mathsf{cr}(B) \in [\epsilon, 1]$. Let $\mathsf{cr}(A) \ge \epsilon$. Then

$$\mathsf{cr}(\overline{A}) = c(\mathsf{cr}(A)) \le c(\epsilon) = \epsilon.$$

From $\overline{A} = \overline{A} \cap (B \cup \overline{B}) = (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})$, and from $\overline{A} \cap B \subset \overline{A} \Rightarrow \mathrm{cr}(\overline{A} \cap B) \leq \mathrm{cr}(\overline{A}) \leq \epsilon$, $\overline{A} \cap \overline{B} \subset \overline{A} \Rightarrow \mathrm{cr}(\overline{A} \cap \overline{B}) \leq \mathrm{cr}(\overline{A}) \leq \epsilon$, we can use property *i*), so

$$\mathrm{cr}(\overline{A}) = \mathrm{cr}((\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})) = \mathrm{cr}(\overline{A} \cap B) \vee \mathrm{cr}(\overline{A} \cap \overline{B}).$$

From Lema 2.8 we have $c(x \vee y) = c(x) \wedge c(y)$ and it follows that

$$\mathrm{cr}(A) = c(\mathrm{cr}(\overline{A})) = c(\mathrm{cr}(\overline{A} \cap B)) \wedge c(\mathrm{cr}(\overline{A} \cap \overline{B})).$$

Now, from property $(x \wedge y) + z = (x + z) \wedge (y + z)$ we get

$$\mathrm{cr}(A) + \mathrm{cr}(B) = [c(\mathrm{cr}(\overline{A} \cap B)) + \mathrm{cr}(B)] \wedge [c(\mathrm{cr}(\overline{A} \cap \overline{B})) + \mathrm{cr}(B)]$$

$$= [c(\mathrm{cr}(\overline{A \cup \overline{B}})) + \mathrm{cr}(B)] \wedge [c(\mathrm{cr}(\overline{A \cup B})) + \mathrm{cr}(B)]$$

$$= [\mathrm{cr}(A \cup \overline{B}) + \mathrm{cr}(B)] \wedge [\mathrm{cr}(A \cup B) + \mathrm{cr}(B)].$$

In order that subadditivity is fulfilled, i.e. $\mathrm{cr}(A) + \mathrm{cr}(B) \geq \mathrm{cr}(A \cup B)$, it must be

$$\mathrm{cr}(A \cup \overline{B}) + \mathrm{cr}(B) \geq \mathrm{cr}(A \cup B) \quad \text{i} \quad \mathrm{cr}(A \cup B) + \mathrm{cr}(B) \geq \mathrm{cr}(A \cup B).$$

The second inequality is obvious. To show the first we assume the opposite i.e. $\mathrm{cr}(A \cup B) > \mathrm{cr}(A \cup \overline{B}) + \mathrm{cr}(B)$. If we use axioms CR_2) and CR_3) and assumption $c(x) \geq 1 - x$, we have

$$\mathrm{cr}(A \cup B) > \mathrm{cr}(A \cup \overline{B}) + \mathrm{cr}(B) \geq \mathrm{cr}(\overline{B}) + \mathrm{cr}(B) = c(\mathrm{cr}(B)) + \mathrm{cr}(B) \geq 1,$$

and that is impossible. Thus, the second inequality is true and subadditivity is fulfilled. □

Theorem 4.4. *Let $A_n \downarrow$, i.e. $A_n \supset A_{n+1}$, $A_n \subset X$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathrm{cr}(A_n) = 0$. Then for all $A \in \mathcal{P}(X)$:*

$$\lim_{n\to\infty} \mathrm{cr}(A \cup A_n) = \lim_{n\to\infty} \mathrm{cr}(A \setminus A_n) = \mathrm{cr}(A).$$

Proof. From monotonicity and subadditivity of cr, we obtain

$$A \subset A \cup A_n \Rightarrow \mathrm{cr}(A) \leq \mathrm{cr}(A \cup A_n) \leq \mathrm{cr}(A) + \mathrm{cr}(A_n).$$

From the squeeze theorem, because of $\lim_{n\to\infty} \mathrm{cr}(A_n) = 0$, it follows that $\lim_{n\to\infty} \mathrm{cr}(A \cup A_n) = \mathrm{cr}(A)$.

Analogously

$$A \setminus A_n \subset A \subset (A \setminus A_n) \cup A_n \Rightarrow$$

$$\mathrm{cr}(A \setminus A_n) \leq \mathrm{cr}(A) \leq \mathrm{cr}((A \setminus A_n) \cup A_n) \leq \mathrm{cr}(A \setminus A_n) + \mathrm{cr}(A_n).$$

And follows $\lim_{n\to\infty} \mathrm{cr}(A \setminus A_n) \leq \mathrm{cr}(A) \leq \lim_{n\to\infty} \mathrm{cr}(A \setminus A_n) = \mathrm{cr}(A)$, then we have

$$\lim_{n\to\infty} \mathrm{cr}(A \setminus A_n) = \mathrm{cr}(A). \quad \square$$

Theorem 4.5. *(Semicontinuity Law) For any series $\{A_n\}$, $\lim_{n\to\infty} \mathrm{cr}(A_n) = \mathrm{cr}(\lim_{n\to\infty} A_n)$ is true if one of the following conditions is satisfied*
i) $A_n \uparrow A$ and ($\mathrm{cr}(A) \leq \epsilon$ or $\lim_{n\to\infty} \mathrm{cr}(A_n) < \epsilon$);
ii) $A_n \downarrow A$ and ($\mathrm{cr}(A) \geq \epsilon$ or $\lim_{n\to\infty} \mathrm{cr}(A_n) > \epsilon$).

Proof. *i*) First, we can notice that for all monotone series of sets $\{A_n\}$ there exits $\lim\limits_{n\to\infty} A_n$ and it is equal $\bigcup\limits_{n\in\mathbb{N}} A_n$, i.e. $\bigcap\limits_{n\in\mathbb{N}} A_n$ when $A_n \uparrow A$, i.e. $A_n \downarrow A$, respectively. Also exits $\lim\limits_{n\to\infty} \overline{A_n}$ and we have $\lim\limits_{n\to\infty} \overline{A_n} = \overline{\lim\limits_{n\to\infty} A_n}$.

Since $\mathsf{cr}(A) \leq \epsilon$, from the credibility monotonicity it follows $A_n \subset \bigcup\limits_{n\in\mathbb{N}} A_n = A \Rightarrow \mathsf{cr}(A_n) \leq \mathsf{cr}(A) \leq \epsilon$ for all $n \in \mathbb{N}$.

From $A_n \subset A_{n+1}$ we have $\mathsf{cr}(A_n) \leq \mathsf{cr}(A_{n+1})$, then series of real numbers $\{\mathsf{cr}(A_n)\}$ which is limited (from the upper side) converges to its supremum.

Hence from CR_4) we have $\mathsf{cr}(A) = \mathsf{cr}(\bigcup\limits_{n\in\mathbb{N}} A_n) = \sup\limits_{n\in\mathbb{N}} \mathsf{cr}(A_n) = \lim\limits_{n\to\infty} \mathsf{cr}(A_n)$.

In the case that $\lim\limits_{n\to\infty} \mathsf{cr}(A_n) < \epsilon$ from the previous consideration we have $\lim\limits_{i\to\infty} \mathsf{cr}(A_n) = \sup\limits_{n} \mathsf{cr}(A_n) < \epsilon$, by using axiom CR_4), it follows $\mathsf{cr}(A) = \mathsf{cr}(\bigcup\limits_{n\in\mathbb{N}} A_n) = \sup\limits_{n\in\mathbb{N}} \mathsf{cr}(A_n) = \lim\limits_{n\to\infty} \mathsf{cr}(A_n)$.

ii) Assuming $\mathsf{cr}(A) \geq \epsilon$ from CR_3) we have $\mathsf{cr}(\overline{A}) = c(\mathsf{cr}(A)) \leq c(\epsilon) = \epsilon$, and from $A_n \downarrow A$, i.e. $A_n \supset A_{n+1}$, $n \in \mathbb{N}$ and from the fuzzy complement monotonicity follows $\overline{A_n} \subset \overline{A_{n+1}}$, $n \in \mathbb{N}$, i.e. $\overline{A_n} \uparrow \overline{A}$. Now, from i), $\lim\limits_{n\to\infty} \mathsf{cr}(\overline{A_n}) = \mathsf{cr}(\overline{A})$ and from the continuity of c (follows from Lemma 2.5), we have

$$\lim_{n\to\infty} \mathsf{cr}(A_n) = \lim_{n\to\infty} c(\mathsf{cr}(\overline{A_n})) = c(\lim_{n\to\infty} \mathsf{cr}(\overline{A_n})) = c(\mathsf{cr}(\overline{A})) = \mathsf{cr}(A).$$

Suppose that $\lim\limits_{n\to\infty} \mathsf{cr}(A_n) > \epsilon$. Based on the previous one, from the assumption $A_n \downarrow A$, we get $\overline{A_n} \uparrow \overline{A}$.

From Lemma 2.5 it follows that fuzzy complement c is a bijective function, and then it is a monotonic decreasing function. Therefore $\lim\limits_{n\to\infty} \mathsf{cr}(A_n) > \epsilon \Rightarrow c(\lim\limits_{n\to\infty} \mathsf{cr}(A_n)) < c(\epsilon)$. Now, from continuity of c we have

$$\lim_{n\to\infty} \mathsf{cr}(\overline{A_n}) = \lim_{n\to\infty} c(\mathsf{cr}(A_n)) = c(\lim_{n\to\infty} \mathsf{cr}(A_n)) < c(\epsilon) = \epsilon.$$

By using i), we have $\lim\limits_{n\to\infty} \mathsf{cr}(\overline{A_n}) = \mathsf{cr}(\lim\limits_{n\to\infty} \overline{A_n})$, i.e. $\lim\limits_{n\to\infty} c(\mathsf{cr}(A_n)) = \mathsf{cr}(\overline{\lim\limits_{n\to\infty} A_n})$, and from $c(\lim\limits_{n\to\infty} \mathsf{cr}(A_n)) = c(\mathsf{cr}(\lim\limits_{n\to\infty} A_n))$, finally $\lim\limits_{n\to\infty} \mathsf{cr}(A_n) = \mathsf{cr}(\lim\limits_{n\to\infty} A_n)$. □

Theorem 4.6. *A credibility measure on X is additive if and only if there are at most two singletons in $\mathcal{P}(X)$ taking nonzero credibility values.*

Proof. Let the credibility measure cr be additive. Suppose there are more than two singletons taking nonzero credibility values, for example $\{x_1\}, \{x_2\}$ i $\{x_3\}$ such that $\mathsf{cr}(\{x_1\}) \geq \mathsf{cr}(\{x_2\}) \geq \mathsf{cr}(\{x_3\}) > 0$.

If $\mathsf{cr}(\{x_1\}) \geq \epsilon$, then from CR_3) follows $\mathsf{cr}(\overline{\{x_1\}}) = c(\mathsf{cr}(\{x_1\})) \leq c(\epsilon) = \epsilon$, and we have $\{x_2, x_3\} \subset \overline{\{x_1\}}$, from CR_2) we obtain $\mathsf{cr}(\{x_2, x_3\}) \leq \mathsf{cr}(\overline{\{x_1\}}) \leq \epsilon$.

By using CR_4) we get

$$\mathsf{cr}(\{x_2, x_3\}) = \mathsf{cr}(\{x_2\}) \vee \mathsf{cr}(\{x_3\}) < \mathsf{cr}(\{x_2\}) + \mathsf{cr}(\{x_3\}),$$

and that is a contradiction with the additivity assumption.

If $\mathsf{cr}(\{x_1\}) < \epsilon$, then $\mathsf{cr}(\{x_3\}) \leq \mathsf{cr}(\{x_2\}) < \epsilon$, and by using CR_4):

$$\mathsf{cr}(\{x_2, x_3\}) = \mathsf{cr}(\{x_2\}) \vee \mathsf{cr}(\{x_3\}) < \epsilon.$$

It follows

$$\mathsf{cr}(\{x_2, x_3\}) = \mathsf{cr}(\{x_2\}) \vee \mathsf{cr}(\{x_3\}) < \mathsf{cr}(\{x_2\}) + \mathsf{cr}(\{x_3\}),$$

an it is contradiction with the additivity assumption, hence there are at most two singletons taking nonzero credibility values.

Conversely, suppose that there are at most two singletons, for example $\{x_1\}$ i $\{x_2\}$ such that $\mathsf{cr}(\{x_1\}), \mathsf{cr}(\{x_2\}) > 0$.

Let us consider two arbitrary disjunctive sets A and B. If $\mathrm{cr}(A) = 0$ or $\mathrm{cr}(B) = 0$, then from subadditivity theorem we have

$$\mathrm{cr}(A \cup B) \leq \mathrm{cr}(A) + \mathrm{cr}(B) = \mathrm{cr}(A) \vee \mathrm{cr}(B),$$

and by using credibility monotonicity $A, B \subset A \cup B \Rightarrow \mathrm{cr}(A), \mathrm{cr}(B) \leq \mathrm{cr}(A \cup B)$, we have $\mathrm{cr}(A) \vee \mathrm{cr}(B) \leq \mathrm{cr}(A \cup B)$, and then $\mathrm{cr}(A \cup B) = \mathrm{cr}(A) \vee \mathrm{cr}(B)$, holds and in this case subadditivity too.

In case that $\mathrm{cr}(A) > 0$ and $\mathrm{cr}(B) > 0$, follows that each set A and B must contain one of the elements x_1 and x_2, for example $x_1 \in A$ and $x_2 \in B$. Otherwise, for example if $x_1, x_2 \notin A$, we would have $\mathrm{cr}(A) = \mathrm{cr}(\bigcup\limits_{x \in A} \{x\}) = \sup\limits_{x \in A}(\{x\}) = 0$. For the same reason, any set $(\overline{A \cup B})$ that does not contain x_1 and x_2 is of measure 0, i.e. $\mathrm{cr}(\overline{A \cup B}) = 0$.

From monotonicity and subadditivity of the credibility measure, we have

$$\mathrm{cr}(A \cup B) \leq \mathrm{cr}(A \cup B \cup \overline{A \cup B}) \leq \mathrm{cr}(A \cup B) + \mathrm{cr}(\overline{A \cup B}) = \mathrm{cr}(A \cup B),$$

and from $\mathrm{cr}(A \cup B \cup \overline{A \cup B}) = \mathrm{cr}(X) = 1$, then $\mathrm{cr}(A \cup B) = 1$.

Similarly

$$\mathrm{cr}(A) \leq \mathrm{cr}(A \cup \overline{A \cup B}) \leq \mathrm{cr}(A) + \mathrm{cr}(\overline{A \cup B}) = \mathrm{cr}(A),$$

$\mathrm{cr}(A \cup \overline{A \cup B}) = \mathrm{cr}(A)$, then

$$\mathrm{cr}(A) + \mathrm{cr}(B) = \mathrm{cr}(A \cup \overline{A \cup B}) + \mathrm{cr}(B) \geq \mathrm{cr}(A \cup \overline{A \cup B} \cup B) = \mathrm{cr}(X) = 1,$$

and it must be that $\mathrm{cr}(A) + \mathrm{cr}(B) = 1$.

Therefore, $\mathrm{cr}(A \cup B) = \mathrm{cr}(A) + \mathrm{cr}(B)$, and the additivity is proved. □

5. Integral based on c−credibility measure

Integrals based on different fuzzy measures can be defined in various ways (see for example [2], [3], [5], [10], [13]).

We will introduce integral based on c−credibility measure. Suppose that $(X, \mathcal{P}(X), \mathrm{cr})$ is c−credibility space and S continuous t−conorm on $[0, 1]$ and $\mu \in \mathcal{M} = \left\{\mu \,\middle|\, \mu : X \to [0, 1]\right\}$ fuzzy sets on X, i.e. their membership functions.

Definition 5.1. *An integral based on c−credibility measure, of $\mu \in \mathcal{M}$ is defined as*

$$\int\limits_A \mu(x) d\mathrm{cr} = \inf_{\alpha \in [0,1]} S\left(\alpha, \mathrm{cr}\left(A \cap {}^{\alpha}\mu\right)\right),$$

where ${}^{\alpha}\mu = \{x \in X | \mu(x) \geq \alpha\}$.

Theorem 5.2. *Let $\mathrm{cr}, \mathrm{cr}_1$ and cr_2 be the c−credibility measures. For arbitrary sets $A, B \subset X$ and $\mu, \mu_1, \mu_2 \in \mathcal{M}$, the following statements hold:*

i) $\int\limits_A \mu(x) d\mathrm{cr} \in [0, 1]$;

ii) $\mu_1 \leq \mu_2 \Rightarrow \int\limits_A \mu_1(x) d\mathrm{cr} \leq \int\limits_A \mu_2(x) d\mathrm{cr}$;

iii) $A \subset B \Rightarrow \int\limits_A \mu(x) d\mathrm{cr} \leq \int\limits_B \mu(x) d\mathrm{cr}$;

iv) $\mathrm{cr}_1 \leq \mathrm{cr}_2 \Rightarrow \int\limits_A \mu(x) d\mathrm{cr}_1 \leq \int\limits_A \mu(x) d\mathrm{cr}_2$;

v) $\mathrm{cr}(A) = 0 \Rightarrow \int\limits_A \mu(x) d\mathrm{cr} = 0$;

vi) $k \in [0, 1] \Rightarrow \int\limits_A k d\mathrm{cr} = \mathrm{cr}(A) \wedge k$.

Proof. *i*) How infimum preserves the order, i.e. $f(\alpha) \leq g(\alpha) \Rightarrow \inf\limits_{\alpha} f(\alpha) \leq \inf\limits_{\alpha} g(\alpha)$, from $0 \leq S\left(\alpha, \mathrm{cr}\left(A \cap {}^{\alpha}\mu\right)\right) \leq 1$ follows the claim.

ii) From the property of $\alpha-$cut $\mu_1 \leq \mu_2 \;\Rightarrow\; {}^{\alpha}\mu_1 \subset {}^{\alpha}\mu_2$, monotonicity of measures and conorms, and properties of infimum, we have $A \cap {}^{\alpha}\mu_1 \subset A \cap {}^{\alpha}\mu_2 \;\Rightarrow\; \mathsf{cr}\,(A \cap {}^{\alpha}\mu_1) \leq \mathsf{cr}\,(A \cap {}^{\alpha}\mu_2) \;\Rightarrow\; S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}\mu_1)) \leq S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}\mu_2)) \;\Rightarrow\; \inf\limits_{\alpha\in[0,1]} S\,(\mathsf{cr}\,(A \cap {}^{\alpha}\mu_1)) \leq \inf\limits_{\alpha\in[0,1]} S\,(\mathsf{cr}\,(A \cap {}^{\alpha}\mu_2))$.

iii) $A \subset B \;\Rightarrow\; A \cap {}^{\alpha}\mu \subset B \cap {}^{\alpha}\mu \;\Rightarrow\; \mathsf{cr}\,(A \cap {}^{\alpha}\mu) \leq \mathsf{cr}\,(B \cap {}^{\alpha}\mu) \;\Rightarrow\; S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}\mu)) \leq S\,(\alpha, \mathsf{cr}\,(B \cap {}^{\alpha}\mu)) \;\Rightarrow\; \inf\limits_{\alpha\in[0,1]} S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}\mu)) \leq \inf\limits_{\alpha\in[0,1]} S\,(\alpha, \mathsf{cr}\,(B \cap {}^{\alpha}\mu))$.

iv) $\mathsf{cr}_1 \leq \mathsf{cr}_2 \;\Rightarrow\; \mathsf{cr}_1\,(A \cap {}^{\alpha}\mu) \leq \mathsf{cr}_2\,(A \cap {}^{\alpha}\mu) \;\Rightarrow\; S\,(\alpha, \mathsf{cr}_1\,(A \cap {}^{\alpha}\mu)) \leq S\,(\alpha, \mathsf{cr}_2\,(A \cap {}^{\alpha}\mu)) \;\Rightarrow\; \inf\limits_{\alpha\in[0,1]} S\,(\alpha, \mathsf{cr}_1\,(A \cap {}^{\alpha}\mu)) \leq \inf\limits_{\alpha\in[0,1]} S\,(\alpha, \mathsf{cr}_2\,(A \cap {}^{\alpha}\mu))$.

v) $A \cap {}^{\alpha}\mu \subset A \;\Rightarrow\; \mathsf{cr}\,(A \cap {}^{\alpha}\mu) \leq \mathsf{cr}\,(A) = 0 \;\Rightarrow\; \mathsf{cr}\,(A \cap {}^{\alpha}\mu) = 0 \;\Rightarrow\; S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}\mu)) = S\,(\alpha, 0) = \alpha \;\Rightarrow\; \inf\limits_{\alpha\in[0,1]} S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}\mu)) = 0$.

vi) Based on the facts $\alpha \in [0,k] \;\Rightarrow\; {}^{\alpha}k = \{x \in X | k \geq \alpha\} = X,\; \alpha \in (k,1] \;\Rightarrow\; {}^{\alpha}k = \{x \in X | k \geq \alpha\} = \emptyset$,we obtain

$$\begin{aligned}
\int_A k d\mathsf{cr} &= \inf_{\alpha\in[0,1]} S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}k)) \\
&= \inf_{\alpha\in[0,k]} S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}k)) \wedge \inf_{\alpha\in(k,1]} S\,(\alpha, \mathsf{cr}\,(A \cap {}^{\alpha}k)) \\
&= \inf_{\alpha\in[0,k]} S\,(\alpha, \mathsf{cr}\,(A)) \wedge \inf_{\alpha\in(k,1]} S\,(\alpha, \mathsf{cr}\,(\emptyset)) \\
&= \inf_{\alpha\in[0,k]} S\,(\alpha, \mathsf{cr}\,(A)) \wedge \inf_{\alpha\in(k,1]} S\,(\alpha, 0) \\
&= S\,(0, \mathsf{cr}\,(A)) \wedge S\,(k, 0) \\
&= \mathsf{cr}\,(A) \wedge k. \quad \square
\end{aligned}$$

6. Possibility, Necessity and Credibility of a Fuzzy Events

Liu and Liu in [6] introduced the credibility in a fuzzy environment as the average of the possibility and necessity measures:

$$Cr(A) = \frac{1}{2}(pos(A) + nec(A)),$$

where A is a set on the possibility space $(X, \mathcal{P}(X), pos)$. In the classical credibility theory the main task is to find the weight of measures, but as we can see here a choice of 0.5 is preliminary made. Further we will generalize credibility measure.

Let X be a triangular fuzzy number on $(X, \mathcal{P}(X), pos)$, with the membership function

$$\mu(x) = \begin{cases} \frac{x-\ell}{m-\ell}, & \ell < x < m \\ \frac{r-x}{r-m}, & m \leq x < r \\ 0, & x \leq \ell \vee r \leq x \end{cases}.$$

Possibility of a fuzzy event $\{X \leq x\}$ is defined with

$$pos(\{X \leq x\}) = \sup_{z \leq x} \mu(z)$$

and for triangular fuzzy number is given by

$$pos_{\alpha}(\{X \leq x\}) = \begin{cases} 0, & x \leq \ell \\ \frac{x-\ell}{r-\ell}, & \ell < x < r \\ 1, & r \leq x \end{cases}.$$

Necessity of a fuzzy event $\{X \leq x\}$ is given with

$$nec(\{X \leq x\}) = 1 - pos(\{X > x\}) = 1 - \sup_{z > x} \mu(z) = \begin{cases} 0, & x \leq m \\ \frac{x-m}{r-m}, & m < x < r \\ 1, & r \leq x \end{cases}.$$

Credibility of a fuzzy event $\{X \leq x\}$ is

$$Cr(\{X \leq x\}) = \frac{1}{2}(pos(\{X \leq x\}) + nec(\{X \leq x\})) = \begin{cases} 0, & x \leq \ell \\ \frac{x-\ell}{2(m-\ell)}, & \ell < x < m \\ \frac{x+r-2m}{2(r-m)}, & m \leq x < r \\ 1, & r \leq x \end{cases}.$$

Now, we can define the $c-$credibility in a fuzzy environment in the relation on aggregation function h with

$$\mathrm{cr}_h(A) = h(pos(A), nec(A)).$$

Theorem 6.1. *The $c-$credibility in a fuzzy environment is a regular fuzzy measure in the broader sense.*

Proof. $\mathrm{cr}_h(\emptyset) = h(pos(\emptyset), nec(\emptyset)) = h(0,0) = 0$.
$\mathrm{cr}_h(X) = h(pos(X), nec(X) = h(1,1) = 1$.
$A \subset B \Rightarrow pos(A) \leq pos(B) \wedge nec(A) \leq nec(B)$
$\Rightarrow \mathrm{cr}_h(A) = h(pos(A), nec(A)) \leq h(pos(B), nec(B)) = \mathrm{cr}_h(A)$. □

If c is a standard fuzzy complement, then we also have additional property

$$\mathrm{cr}_h(\overline{A}) = 1 - \mathrm{cr}_{\overline{h}}(A).$$

Indeed,
$\mathrm{cr}_h(\overline{A}) = h(pos(\overline{A}), nec(\overline{A})) = h(1 - nec(A), 1 - pos(A)) = h(1 - pos(A), 1 - nec(A)) = 1 - \overline{h}(pos(A), nec(A)) = 1 - \mathrm{cr}_{\overline{h}}(A)$.

If aggregation function is weighted arithmetic mean: $h(x,y) = \lambda \cdot x + (1-\lambda) \cdot y$, $\lambda \in [0,1]$ (which is not symmetric in the general case), then $c-$credibility in a fuzzy environment is

$$\mathrm{cr}_\lambda(A) = \lambda \cdot pos(A) + (1-\lambda) \cdot nec(A),$$

and $c-$credibility (in a fuzzy environment) of a fuzzy events $\{X \leq x\}$ is given with (see figure)

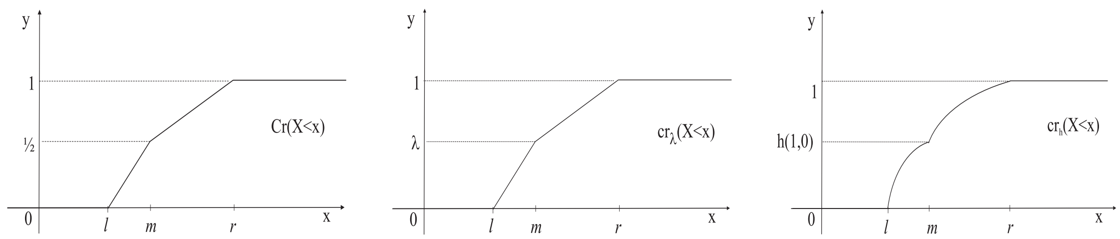
$$\mathrm{cr}_\lambda(\{X \leq x\}) = \lambda \cdot pos(\{X \leq x\}) + (1-\lambda) \cdot nec(\{X \leq x\}) = \begin{cases} 0, & x \leq \ell \\ \lambda \cdot \frac{x-\ell}{m-\ell}, & \ell < x \leq m \\ \lambda + (1-\lambda) \cdot \frac{x-m}{r-m}, & m < x < r \\ 1, & r \leq x \end{cases}.$$

The $c-$credibility (in a fuzzy environment) (in the relation with aggregation function h) of a fuzzy events $\{X \leq x\}$ is given with

$$\mathrm{cr}_h(\{X \leq x\}) = h(pos(\{X \leq x\}), nec(\{X \leq x\})) = \begin{cases} 0, & x \leq \ell \\ h(\frac{x-\ell}{m-\ell}, 0), & \ell < x \leq m \\ h(1, \frac{x-m}{r-m}), & m < x < r \\ 1, & r \leq x \end{cases}$$

The expected value (which we will use in applications) we define by

$$E(X) = \int_0^{+\infty} \mathrm{cr}_\lambda(\{X \geq x\})dx - \int_{-\infty}^{0} \mathrm{cr}_\lambda(\{X \leq x\})dx.$$



Credibility is the estimate of the prediction value in the given application that the actuary assigns to a particular set of data. As we mentioned before, in the classical credibility theory the main task is to find the weight of measures i.e. Z in equation

$$\text{Estimated} = Z \cdot [\text{Observation}] + (1 - Z) \cdot [\text{Other information}]\,, \quad 0 \leq Z \leq 1.$$

In the next example we will show comparative results in determining indicated premium rate changes using classical credibility and $c-$credibility.

Example 6.2. *The main task is to determine the new premium rates for each premium class in the function of the potential loss measure, which together gives the total average rate change. For each risk classification variables there is a vector of differentials. Suppose that there are 3 classes of risk variables x, y and z, with the differentials respectively i, j and k, and that the differentials are multiplicative. The previous formula can be transformed to*

$$\textit{Adopted differential} = Z \cdot D_I + (1 - Z) \cdot D_E.$$

Formulated in the case of the $c-$credibility (the aggregation function is root-power mean, see Remark 2.10)

$$\textit{New differential} = \sqrt[p]{Z \cdot D_I^p + (1 - Z) D_E^p}, \quad \textit{where} \;\; D_I = D_E^i \frac{LR_i}{LR_b}, \quad D_E = \frac{R_c^i}{R_b}.$$

Territory	*Current Base rates*	*Earned premium at current rates*	*Incurred Losses*
x	*150*	*850000*	*330000*
y	*64*	*970000*	*525000*
z	*100*	*600000*	*290000*

Territory	*Current diff.*	*Loss Ratio*	*Indicated*	*Z*	*Adopted*	*New (p=1)*	*New (p=2)*	*New (p=3)*	*New (p=4)*
x	*2,344*	*0,388*	*1,681*	*0,850*	*1,781*	*1,781*	*1,796*	*1,814*	*1,834*
y	*1,000*	*0,541*	*1,000*	*1,000*	*1,000*	*1,000*	*1,000*	*1,000*	*1,000*
z	*1,563*	*0,483*	*1,395*	*0,550*	*1,471*	*1,471*	*1,473*	*1,475*	*1,478*

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