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Essential Pseudospectra Involving Demicompact and Pseudodemicompact Operators and Some Perturbation Results

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Abstract. In this paper, we study the essential and the structured essential pseudospectra of closed densely defined linear operators acting on a Banach space X. We start by giving a refinement and investigating the stability of these essential pseudospectra by means of the class of demicompact linear operators. Moreover, we introduce the notion of pseudo demicompactness and we study its relationship with pseudo upper semi-Fredholm operators. Some stability results for the Gustafson essential pseudospectrum involving pseudo demicompact operators is given.

1. Introduction

Let X, Y and Z be Banach spaces. By an operator T from X into Y , we mean a linear operator with domain $\mathcal{D}(T) \subset X$ and range $\mathcal{R}(T) \subset Y$. By $C(X, Y)$ we denote the set of all closed, densely defined linear operators from X into Y , by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X into Y and by $\mathcal{K}(X, Y)$ the subset of compact operators of $\mathcal{L}(X, Y)$. If $T \in C(X, Y)$, then $\rho(T)$ denotes the resolvent set of T, $\sigma(T)$ the spectrum of T, $\alpha(T)$ the dimension of the kernel $\mathcal{N}(T)$ and $\beta(T)$ the codimension of $\mathcal{R}(T)$ in Y. The next sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from X into Y are, respectively, defined by

$$\Phi_+(X, Y) = \{T \in C(X, Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi_-(X, Y) = \{T \in C(X, Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi(X, Y) := \Phi_-(X, Y) \cap \Phi_+(X, Y),$$

and

$$\Phi_\pm(X, Y) := \Phi_-(X, Y) \cup \Phi_+(X, Y).$$

For $T \in \Phi_\pm(X, Y)$, we define the index by the following difference $i(T) := \alpha(T) - \beta(T)$. A complex number λ is in $\Phi_{+T}, \Phi_{-T}, \Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X, Y)$, $\Phi_-(X, Y)$, $\Phi_\pm(X, Y)$ or $\Phi(X, Y)$, respectively. If $X = Y$, then $\mathcal{L}(X, Y)$, $C(X, Y)$, $\mathcal{K}(X, Y)$, $\Phi(X, Y)$, $\Phi_+(X, Y)$, $\Phi_-(X, Y)$ and $\Phi_\pm(X, Y)$ are replaced by $\mathcal{L}(X)$, $C(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively.

Definition 1.1. *Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X, Y)$. The operator F is called:*
(i) Fredholm perturbation if $T + F \in \Phi(X, Y)$ whenever $T \in \Phi(X, Y)$.
(ii) Upper semi-Fredholm perturbation if $T + F \in \Phi_+(X, Y)$ whenever $T \in \Phi_+(X, Y)$.
(iii) Lower semi-Fredholm perturbation if $T + F \in \Phi_-(X, Y)$ whenever $T \in \Phi_-(X, Y)$.

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The set of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y), \mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively.
Let $T \in \mathcal{C}(X)$, for $x \in \mathcal{D}(T)$, the graph norm $\|.\|_T$ of x is defined by $\|x\|_T = \|x\| + \|Tx\|$. It follows from the closedness of T that $X_T := (\mathcal{D}(T), \|\cdot\|_T)$ is a Banach space. Clearly, for every $x \in \mathcal{D}(T)$ we have $\|Tx\| \leq \|x\|_T$, so that $T \in \mathcal{L}(X_T, X)$. We denote by $\widehat{T}$ the restriction of T to $\mathcal{D}(T)$, we observe that $\alpha(\widehat{T}) = \alpha(T)$ and $\beta(\widehat{T}) = \beta(T)$. A linear operator B is said to be T-defined if $\mathcal{D}(T) \subseteq \mathcal{D}(B)$. If $\widehat{B}$, the restriction of B to $\mathcal{D}(T)$ is bounded from X_T into X, we say that B is T-bounded.

Definition 1.2. *Let X and Y be two Banach spaces.*
(i) *An operator $T \in \mathcal{C}(X,Y)$ is said to have a left Fredholm inverse if there exists $T_l \in \mathcal{L}(Y, X_T)$ such that $I_{X_T} - T_l\widehat{T} \in \mathcal{K}(X_T)$.*
(ii) *An operator $T \in \mathcal{C}(X,Y)$ is said to have a right Fredholm inverse if there exists $T_r \in \mathcal{L}(Y, X_T)$ such that $I_Y - \widehat{T}T_r \in \mathcal{K}(Y)$.* ◊

The notion of pseudospectra can be introduced as a zone of spectral instability. This explains the importance of this concept for numerical calculus involving non-normal matrices. Historically, this concept was introduced since 1967 by J. M. Varah. Especially due to L. N. Trefethen [17], who developed this idea for matrices and operators. The pseudospectrum of a closed densely defined operator T on a Banach space X is defined as follows

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\}. \tag{1}$$

By convention, we write $\|(\lambda - T)^{-1}\| = \infty$ if $(\lambda - T)^{-1}$ is unbounded or nonexistent, i.e., if λ is in $\rho(T)$. We also refer the reader to E. B. Davies who defined the pseudospectrum otherwise, it was given in [6], equivalently to (1) as follows

$$\sigma_\varepsilon(T) := \bigcup_{\|D\|<\varepsilon} \sigma(T + D).$$

The notion of the pseudospectrum drew the attention of A. Ammar and A. Jeribi who defined in [1] an essential pseudospectrum of a densely defined, linear operator T acting on a Banach space X by

$$\sigma_{e5,\varepsilon}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(T + K).$$

Moreover, in the following theorem the authors gave a characterization of the essential spectrum by means of Fredholm perturbations.

Theorem 1.3. *[8] Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$.*

$$\sigma_{e5,\varepsilon}(T) = \bigcap_{K \in \mathcal{F}(X)} \sigma_\varepsilon(T + K). \qquad \diamond$$

Definition 1.4. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$.*
(i) *T is called a pseudo upper (resp. lower) semi-Fredholm operator if $T + D$ is an upper (resp. lower) semi-Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.*
(ii) *T is called a pseudo semi-Fredholm operator if $T + D$ is a semi-Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.*
(iii) *T is called a pseudo Fredholm operator if $T + D$ is a Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.* ◊

The sets of all pseudo Fredholm, pseudo upper Fredholm, pseudo lower Fredholm and pseudo semi-Fredholm operators are, respectively, denoted by $\Phi^\varepsilon(X)$, $\Phi^\varepsilon_+(X)$, $\Phi^\varepsilon_-(X)$ and $\Phi^\varepsilon_\pm(X)$. A complex number λ is in $\Phi^\varepsilon_{+T}, \Phi^\varepsilon_{-T}, \Phi^\varepsilon_{\pm T}$ or Φ^ε_T if, $\lambda - T$ is in $\Phi^\varepsilon_+(X)$, $\Phi^\varepsilon_-(X)$, $\Phi^\varepsilon_\pm(X)$ or $\Phi^\varepsilon(X)$, respectively.

In this work, we are interested in the following essential pseudospectra

$$\begin{aligned}
\sigma_{e1,\varepsilon}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi^{\varepsilon}_{+}(X)\} := \mathbb{C}\backslash\Phi^{\varepsilon}_{+T},\\
\sigma_{e2,\varepsilon}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi^{\varepsilon}_{-}(X)\} := \mathbb{C}\backslash\Phi^{\varepsilon}_{-T},\\
\sigma_{e3,\varepsilon}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi^{\varepsilon}_{\pm}(X)\} := \mathbb{C}\backslash\Phi^{\varepsilon}_{\pm T},\\
\sigma_{e4,\varepsilon}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi^{\varepsilon}(X)\} := \mathbb{C}\backslash\Phi^{\varepsilon}_{T},\\
\sigma_{e5,\varepsilon}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{\varepsilon}(T+K),\\
\sigma_{eap,\varepsilon}(T) &:= \sigma_{e1,\varepsilon}(T)\bigcap\{\lambda \in \mathbb{C} \text{ such that } i(\lambda - T - D) > 0, \text{ for all } \|D\| < \varepsilon\},
\end{aligned}$$

Note that if ε tends to 0, we recover the well-known definitions of essential spectra of T. In [18], M. P. H. Wolff has given a motivation to study the essential approximate pseudospectrum. In [3], the notion of the essential approximate pseudospectrum was extended by devoting the studies to the essential approximate spectrum. For $\varepsilon > 0$ and $T \in \mathcal{C}(X)$,

$$\sigma_{eap,\varepsilon}(T) := \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap,\varepsilon}(T+K),$$

where

$$\sigma_{ap,\varepsilon}(T) = \sigma_{ap}(T)\bigcup\left\{\lambda \in \mathbb{C} \text{ such that } \inf_{x\in\mathcal{D}(T);\|x\|=1} \|(\lambda - T)x\| < \varepsilon\right\},$$

and

$$\sigma_{ap}(T) = \left\{\lambda \in \mathbb{C} \text{ such that } \inf_{x\in\mathcal{D}(T);\|x\|=1} \|(\lambda - T)x\| = 0\right\}.$$

In the same work, the authors measured the sensitivity of the set $\sigma_{ap}(T)$ with respect to additive perturbations of T by an operator $D \in \mathcal{L}(X)$ with norm less than ε. So, they defined the approximate pseudospectrum of T by

$$\sigma_{ap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{ap}(T+D). \tag{2}$$

The authors also showed that there is an essential version of Eq. (2), that is

$$\sigma_{eap,\varepsilon}(T) = \bigcup_{\|D\|<\varepsilon} \sigma_{eap}(T+D),$$

which was refined in the following theorem as follows

Theorem 1.5. *[8] Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$.*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K\in\mathcal{F}_{+}(X)} \sigma_{ap,\varepsilon}(T+K). \qquad \diamond$$

Remark 1.6. *[7, 8] Let $T \in \mathcal{C}(X)$, $\varepsilon > 0$ and $i \in \{1, \cdots, 5, ap\}$.*
(i) $\sigma_{ei,\varepsilon}(T) \subset \sigma_{\varepsilon}(T)$.
(ii) $\bigcap_{\varepsilon>0}\sigma_{ei,\varepsilon}(T) = \sigma_{ei}(T)$.
(iii) If $\varepsilon_1 \leq \varepsilon_2$, then $\sigma_{ei,\varepsilon_1}(T) \subset \sigma_{ei,\varepsilon_2}(T)$.
(iv) $\sigma_{e5,\varepsilon}(T) = \sigma_{e5,\varepsilon}(T+K)$ for all $K \in \mathcal{K}(X)$.
(v) We have the following inclusions

$$\sigma_{e3,\varepsilon}(T) = \sigma_{e1,\varepsilon}(T) \cap \sigma_{e2,\varepsilon}(T) \subseteq \sigma_{e4,\varepsilon}(T) \subseteq \sigma_{e5,\varepsilon}(T) \qquad \diamond$$

The concept of the structured pseudospectrum, or spectral value sets of a closed densely defined linear operator A on X was defined by E. B. Davies in [6] by

$$\sigma(A, B, C, \varepsilon) := \bigcup_{\|D\|<\varepsilon} \sigma(A + CDB),$$

where $B \in \mathcal{L}(X, Y)$ and $C \in \mathcal{L}(Z, X)$. Based on this notion, A. Elleuch and A. Jeribi defined in [8] the structured essential pseudospectrum as follows

Definition 1.7. *Let $A \in \mathcal{C}(X)$, $B \in \mathcal{L}(X,Y)$, $C \in \mathcal{L}(Z,X)$ and $\varepsilon > 0$. We define the the structured essential pseudospectrum by*

$$\sigma_{e5}(A,B,C,\varepsilon) := \bigcup_{\|D\|<\varepsilon} \sigma_{e5}(A+CDB). \qquad \diamond$$

In the same work, the authors gave, in first time, the following description of the structured essential pseudospectrum.

Theorem 1.8. *[8] Let $A \in \mathcal{C}(X)$, $B \in \mathcal{L}(X,Y)$, $C \in \mathcal{L}(Z,X)$ and $\varepsilon > 0$.*

$$\sigma_{e5}(A,B,C,\varepsilon) = \bigcap_{K\in\mathcal{K}(X)} \sigma(A+K,B,C,\varepsilon). \qquad \diamond$$

It was given, in second time, a refinement of this description in the following theorem

Theorem 1.9. *[8] Let $A \in \mathcal{C}(X)$, $B \in \mathcal{L}(X,Y)$, $C \in \mathcal{L}(Z,X)$ and $\varepsilon > 0$.*

$$\sigma_{e5}(A,B,C,\varepsilon) = \bigcap_{F\in\mathcal{F}(X)} \sigma(A+F,B,C,\varepsilon). \qquad \diamond$$

An outline of this article is as follows. In Section 2, we give some preliminaries needed in the rest of the paper. In Section 3, we refine the description of the approximate and the structured essential pseudospectra of a closed densely defined linear operator. In Section 4, we provide abstract perturbation results for the essential pseudospectra of the sum of two linear operators. In Section 5, we introduce the notion of pseudo demicompactness and we give some sufficient conditions for closed densely defined linear operators to be pseudo upper-Fredholm and finally we give a perturbation result for the Gustafson essential pseudospectrum by the way of pseudo demicompactness.

2. Preliminaries

Definition 2.1. *[14] An operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be demicompact if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(I-T)x_n$ converges in X, there exists a convergent subsequence of $(x_n)_n$.* $\diamond$

The family of demicompact operators on X will be denoted by $\mathcal{DC}(X)$.

Theorem 2.2. *[4, 5] Let $T \in \mathcal{C}(X)$. Then, $T \in \mathcal{DC}(X)$ if, and only if, $I - T \in \Phi_+(X)$.* $\diamond$

Theorem 2.3. *[4] Let $T \in \mathcal{C}(X)$. If for each $\mu \in [0,1]$, $\mu T \in \mathcal{DC}(X)$, then*

$$I - T \in \Phi(X) \quad \text{and} \quad i(I-T) = 0. \qquad \diamond$$

For more results and applications of the concept of demicompactness, the reader may refer to [11, 12].

Theorem 2.4. *[13, 16] Let X, Y and Z be Banach spaces, $A \in \mathcal{L}(Y,Z)$ and $B \in \mathcal{L}(X,Y)$.*
(i) If $AB \in \Phi_+(X,Z)$, then $B \in \Phi_+(X,Y)$.
(ii) If $AB \in \Phi_-(X,Z)$, then $A \in \Phi_-(Y,Z)$.
(iii) If $X = Y = Z$, $AB \in \Phi(X)$ and $BA \in \Phi(X)$, then $A \in \Phi(X)$ and $B \in \Phi(X)$.
(iv) If $A \in \Phi_+(Y,Z)$ and $B \in \Phi_+(X,Y)$, then $AB \in \Phi_+(X,Z)$.
(v) If $A \in \Phi(Y,Z)$ and $B \in \Phi(X,Y)$, then $AB \in \Phi(X,Z)$ and $i(A+B) = i(A) + i(B)$. $\diamond$

Proposition 2.5. *[1] Let X be a Banach space, $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then, $\lambda \notin \sigma_{e5,\varepsilon}(T)$ if, and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have*

$$\lambda - T - D \in \Phi(X) \text{ and } i(\lambda - T - D) = 0. \qquad \diamond$$

Theorem 2.6. *[3] Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. The following properties are equivalent.*
(i) $\lambda \in \sigma_{ap,\varepsilon}(T)$.
(ii) There exists $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and $\lambda \in \sigma_{ap}(T+D)$. $\diamond$

Theorem 2.7. *[3] Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then, $\lambda \notin \sigma_{eap,\varepsilon}(T)$ if, and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, we have*

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T - D) \le 0. \qquad \diamond$$

Several measures of noncompactness were defined in the literature, the first one was defined and studied by K. Kuratowski [9] in 1930.

Definition 2.8. *[9] Let D be a bounded subset of X. We define $\gamma(D)$, the Kuratowski measure of noncompactness of D, to be* $\inf\{d > 0$ *such that D can be covered by a finite number of sets of diameter less than or equal to $d\}$.* ◇

The following proposition gives some properties of the Kuratowski measure of noncompactness which are frequently used.

Proposition 2.9. *Let D and D' be two bounded subsets of X, then we have the following properties.*
(i) *$\gamma(D) = 0$ if, and only if, D is relatively compact.*
(ii) *If $D \subseteq D'$, then $\gamma(D) \leq \gamma(D')$.*
(iii) *$\gamma(D + D') \leq \gamma(D) + \gamma(D')$.*
(vi) *For every $\alpha \in \mathbb{C}$, $\gamma(\alpha D) = |\alpha|\gamma(D)$.* ◇

Definition 2.10. *[10] Let $T : \mathcal{D}(T) \subset X \longrightarrow X$ be a continuous operator, and $\gamma(.)$ be the Kuratowski measure of noncompactness in X. Let $k \geq 0$, then T is said to be k-set-contraction if for any bounded subset B of $\mathcal{D}(T)$, $T(B)$ is a bounded subset of X and $\gamma(T(B)) \leq k\gamma(B)$.* ◇

Lemma 2.11. *[4] Let $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ be a closed linear operator. If T is a 1-set-contraction, then αT is demicompact for each $\alpha \in [0, 1)$.* ◇

3. Characterization of the approximate and the structured essential pseudospectra

In this section, we will give a description of essential pseudospectra of closed densely defined operators by means of demicompactness. In order to state our results, the following notations will be convenient

$$\Lambda_X := \{J \in \mathcal{L}(X) \text{ such that } \mu J \in \mathcal{DC}(X),\ \forall \mu \in [0, 1]\},$$

$$\Psi_T(X) := \bigcap_{\varepsilon>0} \bigcap_{\|D\|<\varepsilon} \Big\{K \in \mathcal{L}(X) \ \text{ such that} - (\lambda - T - K - D)^{-1}K \in \Lambda_X,\ \forall \lambda \in \rho(T + K + D)\Big\}$$

and

$$\Upsilon_T(X) := \bigcap_{\varepsilon>0} \bigcap_{\|D\|<\varepsilon} \mathcal{H}_{\varepsilon,D}(T, X),$$

where

$$\mathcal{H}_{\varepsilon,D}(T, X) = \Big\{K \text{ is } T\text{-bounded such that } - K(\lambda - T - K - D)^{-1} \in \Lambda_X,\ \forall \lambda \in \rho(T + K + D)\Big\}.$$

Theorem 3.1. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e5,\varepsilon}(T) = \bigcap_{K\in\Psi_T(X)} \sigma_\varepsilon(T + K) = \bigcap_{K\in\Upsilon_T(X)} \sigma_\varepsilon(T + K).$$ ◇

Proof. Let $T \in \mathcal{C}(X)$ and $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ for each $\varepsilon > 0$. K be a T-bounded operator and $\lambda \in \rho(T + K + D)$, then according to Lemma 2.1 in [15], $K(\lambda - T - K - D)^{-1}$ is a closed linear operator defined on X and therefore bounded. Clearly, $\mathcal{K}(X) \subset \Psi_T(X)$ (resp. $\mathcal{K}(X) \subset \Upsilon_T(X)$). Then,

$$\bigcap_{K\in\Psi_T(X)} \sigma_\varepsilon(T + K) \subset \sigma_{e5,\varepsilon}(T),$$

$$(\text{resp. } \bigcap_{K\in\Upsilon_T(X)} \sigma_\varepsilon(T + K) \subset \sigma_{e5,\varepsilon}(T)),$$

Conversely, let $\lambda \notin \bigcap_{K\in\Psi_T(X)} \sigma_\varepsilon(T+K)$, then there exists $K \in \Psi_T(X)$ (resp. $K \in \Upsilon_T(X)$) such that $\lambda \notin \sigma_\varepsilon(T+K)$. Thus, by Theorem 9.2.13 (ii) in [6] we deduce that $\lambda \in \rho(T+K+D)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. So,

$$\lambda - T - K - D \in \Phi(X) \text{ and } i(\lambda - T - K - D) = 0.$$

Moreover, $-(\lambda - T - K - D)^{-1}K \in \Lambda_X$ (resp. $-K(\lambda - T - K - D)^{-1} \in \Lambda_X$). It follows from Theorem 2.3 that

$$I + (\lambda - T - K - D)^{-1}K \in \Phi(X) \text{ and } i[I + (\lambda - T - K - D)^{-1}K] = 0.$$

$$(\text{resp. } I + K(\lambda - T - K - D)^{-1} \in \Phi(X) \text{ and } i[I + K(\lambda - T - K - D)^{-1}] = 0).$$

Now, using the equality

$$\lambda - T - D = (\lambda - T - K - D)[I + (\lambda - T - K - D)^{-1}K],$$

$$(\text{resp. } \lambda - T - D = [I + K(\lambda - T - K - D)^{-1}](\lambda - T - K - D)),$$

together with Theorem 2.4 (v) one gets

$$\lambda - T - D \in \Phi(X) \text{ and } i(\lambda - T - D) = 0,$$

which shows that $\lambda \notin \sigma_{e5,\varepsilon}(T)$. Thereby,

$$\sigma_{e5,\varepsilon}(T) \subset \bigcap_{K \in \Psi_T(X)} \sigma_\varepsilon(T + K),$$

$$(\text{resp. } \sigma_{e5,\varepsilon}(T) \subset \bigcap_{K \in \Upsilon_T(X)} \sigma_\varepsilon(T + K)). \qquad \text{Q.E.D.}$$

Corollary 3.2. *Let $T \in \mathcal{C}(X)$, $\Sigma(X)$ be a subset of X containing $\mathcal{K}(X)$ and $\varepsilon > 0$. If $\Sigma(X) \subset \Psi_T(X)$ (resp. $\Sigma(X) \subset \Upsilon_T(X)$), then*

$$\sigma_{e5,\varepsilon}(T) = \bigcap_{K \in \Sigma(X)} \sigma_\varepsilon(T + K). \qquad \diamond$$

Proof. From the following inclusions $\mathcal{K}(X) \subset \Sigma(X) \subset \Psi_T(X)$, we infer that

$$\bigcap_{K \in \Psi_T(X)} \sigma_\varepsilon(T + K) \subset \bigcap_{K \in \Sigma(X)} \sigma_\varepsilon(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(T + K).$$

By virtue of Theorem 3.1, we obtain

$$\sigma_{e5,\varepsilon}(T) = \bigcap_{K \in \Sigma(X)} \sigma_\varepsilon(T + K). \qquad \text{Q.E.D.}$$

Next, we give a refinement of the essential approximate pseudospectrum.

Theorem 3.3. *Let $T \in \mathcal{C}(X)$ and $\varepsilon > 0$, then*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K \in \Psi_T(X)} \sigma_{ap,\varepsilon}(T + K). \qquad \diamond$$

Proof. We first should remark that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have

$$\lambda - T - D = (\lambda - T - K - D)[I + (\lambda - T - K - D)^{-1}K], \tag{3}$$

We start by showing that $\sigma_{eap,\varepsilon}(T) \subset \bigcap_{K \in \Psi_T(X)} \sigma_{ap,\varepsilon}(T + K)$. In fact, for $\lambda \notin \bigcap_{K \in \Psi_T(X)} \sigma_{ap,\varepsilon}(T + K)$ there exists $K \in \Psi_T(X)$ such that $\lambda \notin \sigma_{ap,\varepsilon}(T + K)$. Based on Theorem 2.6, we infer that for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, $\lambda \notin \sigma_{ap}(T + K + D)$ and so, $\lambda - T - K - D$ is injective. This yields to

$$\lambda - T - K - D \in \Phi_+(X) \text{ and } i(\lambda - T - K - D) \leq 0.$$

It follows immediately from Theorem 2.7 that $\lambda \notin \sigma_{eap,\varepsilon}(T + K)$. Now, since $K \in \Psi_T(X)$, we have $-(\lambda - T - K - D)^{-1}K \in \Lambda_X$, whenever $\lambda \in \rho(T + K + D)$. By virtue of Theorem 2.3 we show that

$$I + (\lambda - T - K - D)^{-1}K \in \Phi(X) \text{ and } i(I + (\lambda - T - K - D)^{-1}K) = 0.$$

Which implies that

$$I + (\lambda - T - K - D)^{-1}K \in \Phi_+(X) \text{ and } i(I + (\lambda - T - K - D)^{-1}K) \leq 0.$$

Hence, applying Theorem 2.4 (v) on Eq. (3), we obtain

$$\lambda - T - D \in \Phi_+(X) \text{ and } i(\lambda - T) \leq 0.$$

The use of Theorem 2.7 gives us $\lambda \notin \sigma_{eap,\varepsilon}(T)$. The reverse inclusion follows by remarking that $\mathcal{K}(X) \subset \Psi(X)$. Q.E.D.

We can drive from Theorem 3.3 the following result.

Corollary 3.4. *Let $T \in \mathcal{C}(X)$, $\Gamma(X)$ be a subset of X containing $\mathcal{K}(X)$ and $\varepsilon > 0$. If $\Gamma(X) \subset \Psi_T(X)$, then*

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K \in \Gamma(X)} \sigma_{ap,\varepsilon}(T + K) \qquad \diamond$$

Proof. Since $\mathcal{K}(X) \subset \Gamma(X) \subset \Psi_T(X)$, we obtain

$$\bigcap_{K \in \Psi_T(X)} \sigma_{ap,\varepsilon}(T + K) \subset \bigcap_{K \in \Gamma(X)} \sigma_{ap,\varepsilon}(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap,\varepsilon}(T + K) := \sigma_{eap,\varepsilon}(T).$$

The use of Theorem 3.3 allows us to conclude that

$$\sigma_{eap,\varepsilon}(T) = \bigcap_{K \in \Gamma(X)} \sigma_{ap,\varepsilon}(T + K).$$

Hence, we get the desired result. Q.E.D.

Theorem 3.5. *Let $A \in \mathcal{C}(X)$, $B \in \mathcal{L}(X, Y)$, $C \in \mathcal{L}(Z, X)$ and $\varepsilon > 0$. Then,*

$$\sigma_{e5}(A, B, C, \varepsilon) = \bigcap_{K \in \Psi_T(X)} \sigma(A + K, B, C, \varepsilon) = \bigcap_{K \in \Upsilon_T(X)} \sigma(A + K, B, C, \varepsilon). \qquad \diamond$$

Proof. As $\Psi_T(X)$ contains $\mathcal{K}(X)$, it follows that

$$\bigcap_{K \in \Psi_T(X)} \sigma(A + K, B, C) \subset \sigma_{e5}(A, B, C, \varepsilon).$$

Conversely, we argue by contradiction, we suppose that there exists $K \in \Psi_T(X)$ such that $\lambda \notin \sigma(A+K, B, C)$. Thus, $\lambda \notin \sigma(A + K + CDB)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. So,

$$\lambda - A - K - CDB \in \Phi(X) \text{ and } i(\lambda - A - K - CDB) = 0.$$

The fact that $K \in \Psi_T(X)$ implies that

$$I + (\lambda - A - K - CDB)^{-1}K \in \Phi(X) \text{ and } i(I + (\lambda - A - K - CDB)^{-1}K) = 0.$$

Applying Theorem 2.4 (v) to the following equality

$$\lambda - A - CDB = (\lambda - A - K - CDB)[I + (\lambda - A - K - CDB)^{-1}K],$$

we conclude that

$$\lambda - A - CDB \in \Phi(X) \text{ and } i(\lambda - A - CDB) = 0.$$

Which shows that $\lambda \notin \sigma_{e5}(A + CDB)$. Thereby, $\lambda \notin \sigma_{e5}(A, B, C, \varepsilon)$. The second equality can be checked in the same way. Q.E.D.

As a consequence of the previous theorem, we may state:

Corollary 3.6. *Let $A \in \mathcal{C}(X)$, $B \in \mathcal{L}(X, Y)$, $C \in \mathcal{L}(Z, X)$, $\Gamma(X)$ of X containing $\mathcal{K}(X)$ and $\varepsilon > 0$. If $\Gamma(X) \subset \Psi_T(X)$ (resp. $\Gamma(X) \subset \Upsilon_T(X)$), then*

$$\sigma_{e5}(A, B, C, \varepsilon) = \bigcap_{K \in \Gamma(X)} \sigma(A + K, B, C, \varepsilon) \qquad \diamond$$

4. Some perturbation results

In this Section, we give some results of stability for some essential pseudospectra.

Theorem 4.1. *Let $T, S \in \mathcal{L}(X)$, $\varepsilon > 0$ and let $\lambda \notin \sigma_{e1,\varepsilon}(T)$. If for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, the operator $\lambda - T - D$ has a left (resp. right) Fredholm inverse $T_{\lambda l}$ (resp. $T_{\lambda r}$) such that $ST_{\lambda l} \in \mathcal{DC}(X)$ (resp. $T_{\lambda r}S \in \mathcal{DC}(X)$), then*

$$\sigma_{e1,\varepsilon}(T + S) \subset \sigma_{e1,\varepsilon}(T). \qquad \diamond$$

Proof. Let $\lambda \in \mathbb{C}$ and $T_{\lambda l}$ be a left Fredholm inverse of $\lambda - T - D$, then there exists $K \in \mathcal{K}(X)$ such that $T_{\lambda l}(\lambda - T - D) = I - K$. Thus, we may easily observe that

$$\lambda - T - S - D = (I - ST_{\lambda l})(\lambda - T - D) - SK. \tag{4}$$

Now, let $\lambda \notin \sigma_{e1,\varepsilon}(T)$, then $\lambda - T - D \in \Phi_+(X)$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. As $ST_{\lambda l} \in \mathcal{DC}(X)$, likewise Theorem 2.2 gives $I - ST_{\lambda l} \in \Phi_+(X)$. Applying Akinson's theorem on Eq. (4) and using the fact that $SK \in \mathcal{K}(X)$, we conclude that $\lambda - T - S - D \in \Phi_+(X)$. Which allows us to reach the desired result. Q.E.D.

Theorem 4.2. *Let $T, S \in \mathcal{L}(X)$, $\varepsilon > 0$ and let $\lambda \notin \sigma_{ei,\varepsilon}(T)$, where $i \in \{2, 3, 4, 5, ap\}$. If for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, the operator $\lambda - T - D$ has a left (resp. right) Fredholm inverse $T_{\lambda l}$ (resp. $T_{\lambda r}$) such that $ST_{\lambda l} \in \Lambda_X$ (resp. $T_{\lambda r}S \in \Lambda_X$), then*

$$\sigma_{ei,\varepsilon}(T + S) \subset \sigma_{ei,\varepsilon}(T). \qquad \diamond$$

Proof. We give the proof for $i = 5$. Note that the other cases can be checked in the same manner. We will proceed by contradiction, we suppose that $\lambda \notin \sigma_{e5,\varepsilon}(T)$. In view of Proposition 2.5, we get $\lambda - T - D \in \Phi(X)$ and $i(\lambda - T - D) = 0$ for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Let $T_{\lambda l}$ (resp. $T_{\lambda r}$) be a left (resp. right) Fredholm inverse of $\lambda - T - D$, we have $T_{\lambda l}(\lambda - T - D) = I - K$, (resp. $(\lambda - T - D)T_{\lambda r} = I - K'$), where $K \in \mathcal{K}(X)$ (resp. $K' \in \mathcal{K}(X)$). By making some simple calculations, we get

$$\lambda - T - S - D = (I - ST_{\lambda l})(\lambda - T - D) - SK. \tag{5}$$

$$(\text{resp. } \lambda - T - S - D = (\lambda - T - D)(I - T_{\lambda r}S) - K'S)\,. \tag{6}$$

Since $ST_{\lambda l} \in \Lambda_X$ (resp. $T_{\lambda r}S \in \Lambda_X$), then combining this result together with Theorem 2.3, we get $I - ST_{\lambda l} \in \Phi(X)$ (resp. $I - T_{\lambda r}S \in \Phi(X)$) and $i(I - ST_{\lambda l}) = 0$, (resp. $i(I - T_{\lambda r}S) = 0$). Thus, the use of Theorem 2.4 (v) on Eq. (5) (resp. Eq. (6)) leads to

$$\lambda - T - S - D \in \Phi(X) \text{ and } i(\lambda - T - S - D) = 0.$$

Consequently, from Theorem 2.3, we deduce that $\lambda \notin \sigma_{e5,\varepsilon}(T + S)$. This allows us to conclude that

$$\sigma_{e5,\varepsilon}(T + S) \subset \sigma_{e5,\varepsilon}(T). \qquad \text{Q.E.D.}$$

Similarly to the proof of Theorem 4.2, we show the following theorem

Theorem 4.3. *Let $A \in \mathcal{C}(X)$, $B, C, S \in \mathcal{L}(X)$ and $\varepsilon > 0$ and let $\lambda \notin \sigma_{e5}(A, B, C, \varepsilon)$. If for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, the operator $\lambda - A - CDB$ has a left (resp. right) Fredholm inverse $A_{\lambda l}$ (resp. $A_{\lambda r}$) such that $SA_{\lambda l} \in \Lambda_X$ (resp. $A_{\lambda r}S \in \Lambda_X$), then*

$$\sigma_{e5}(A + S, B, C, \varepsilon) \subset \sigma_{e5}(A, B, C, \varepsilon). \qquad \diamond$$

5. Pseudo-demicompactness and some related results

Definition 5.1. *Let $\varepsilon > 0$. An operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be pseudo demicompact if for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ and for all bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $(I - T - D)x_n$ converges in X, there exists a convergent subsequence of $(x_n)_n$.* $\diamond$

The set of all pseudo demicompact operators on X will be denoted by $\mathcal{DC}_\varepsilon(X)$.

Remark 5.2. *$T \in \mathcal{DC}_\varepsilon(X)$ if, and only if, for all $D \in \mathcal{L}(X)$ satisfying $\|D\| < \varepsilon$, the operator $T + D \in \mathcal{DC}(X)$.* $\diamond$

Lemma 5.3. *Let* $0 < \varepsilon \leq 1$ *and* $K \in \mathcal{K}(X)$*. Then,* $K \in \mathcal{DC}_\varepsilon(X)$*.* ◊

Proof. Suppose that $0 < \varepsilon \leq 1$ and let $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Then, using the fact that K is a compact operator, we get $\gamma(K + D) = \gamma(D)$. It follows that $\gamma(K + D) < 1$. Therefore, according to Lemma 2.11 in [4], $K + D$ is k-set-contractive, where $k < 1$. It follows from Remark 2.5 in [4] that $K + D \in \mathcal{DC}(X)$ and consequently, $K \in \mathcal{DC}_\varepsilon(X)$. Q.E.D.

Proposition 5.4. *Let* $T \in \mathcal{C}(X)$ *and* $\varepsilon > 0$*. Then,* $T \in \mathcal{DC}_\varepsilon(X)$ *if, and only if,* $I - T \in \Phi^\varepsilon_+(X)$*.* ◊

Proof. Let $\varepsilon > 0$ and $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Assume that $T \in \mathcal{DC}_\varepsilon(X)$, then by Remark 5.2, we get $T + D \in \mathcal{DC}(X)$. From Theorem 2.2, we deduce that $I - T - D \in \Phi_+(X)$ and so $I - T \in \Phi^\varepsilon_+(X)$. The reverse implication is similar. Q.E.D.

Lemma 5.5. *Let* $A, B \in \mathcal{L}(X)$ *and* $\varepsilon > 0$*.*
(*i*) *If* $\|A\| < 1$ *and* $AB \in \Phi^\varepsilon_+(X)$*, then* $B \in \Phi^\varepsilon_+(X)$*.*
(*ii*) *If* $\|B\| < 1$ *and* $AB \in \Phi^\varepsilon_-(X)$*, then* $A \in \Phi^\varepsilon_-(X)$*.*
(*iii*) *If* $\|B\| < 1$*,* $AB \in \Phi^\varepsilon(X)$ *and* $BA \in \Phi^\varepsilon(X)$*, then* $A \in \Phi^\varepsilon(X)$ *and* $B \in \Phi(X)$*.*

Proof. (*i*) Let $D \in \mathcal{L}(X)$ be such that $\|D\| < \varepsilon$. We have

$$A(B + D) = AB + AD. \tag{7}$$

Clearly, $\|AD\| < \varepsilon$. Using the fact that $AB \in \Phi^\varepsilon_+(X)$, it follows that $AB + AD \in \Phi_+(X)$. Now, combining Eq. (7) together with Theorem 2.4 (*i*), we obtain $B + D \in \Phi_+(X)$ and so $B \in \Phi^\varepsilon_+(X)$.
(*ii*) Let $D \in \mathcal{L}(X)$ be such that $\|D\| < \varepsilon$. We have

$$(A + D)B = AB + DB. \tag{8}$$

Since $\|DB\| < \varepsilon$ and $AB \in \Phi^\varepsilon_-(X)$, it follow from Eq. (8) together with Theorem 2.4 (*ii*) that $A + D \in \Phi_-(X)$, that is $A \in \Phi^\varepsilon_-(X)$.
(*iii*) Since $AB \in \Phi^\varepsilon(X)$ and taking into account Eq. (8), we infer that $(A + D)B \in \Phi(X)$. In the same way, by using the following equality $B(A + D) = BA + BD$, we prove that $B(A + D) \in \Phi(X)$. Hence, we conclude from Theorem 2.4 (*iii*) that $A \in \Phi^\varepsilon(X)$ and $B \in \Phi(X)$. Q.E.D.

Lemma 5.6. *Let* $A,\ B \in \mathcal{L}(X)$ *and* $\varepsilon > 0$*.*
(*i*) *If* $A \in \Phi(X)$*,* $B \in \Phi^\varepsilon(X)$ *and* $(I - A)D \in \mathcal{F}(X)$*, then* $AB \in \Phi^\varepsilon(X)$ *and* $i(AB + D) = i(A) + i(B + D)$ *for all* $D \in \mathcal{L}(X)$ *satisfying* $\|D\| < \varepsilon$*.*
(*ii*) *If* $A \in \Phi_+(X)$*,* $B \in \Phi^\varepsilon_+(X)$ *and* $(I - A)D \in \mathcal{F}_+(X)$*, then* $AB \in \Phi^\varepsilon_+(X)$*.* ◊

Proof. (*i*) For each $D \in \mathcal{L}(X)$ satisfying $\|D\| < \varepsilon$, we have

$$AB + D = A(B + D) + (I - A)D. \tag{9}$$

Since $A \in \Phi(X)$ and $B + D \in \Phi(X)$, then applying Theorem 2.4 (*v*) on Eq. (9) and using the fact that $(I - A)D \in \mathcal{F}(X)$, we get $AB \in \Phi^\varepsilon(X)$ and $i(AB + D) = i(A) + i(B + D)$.
(*ii*) We reason in the same way as the proof of (*i*). Q.E.D.

Theorem 5.7. *Let* $T, S \in \mathcal{L}(X)$ *and* $\varepsilon > 0$*. If for every* $\lambda \notin \sigma_{e1,\varepsilon}(T)$ *and for every* $D \in \mathcal{L}(X)$ *such that* $\|D\| < \varepsilon$*, there exists a left Fredholm inverse* T_0 *of* $\lambda - T - D$ *satisfying* $ST_0 \in \mathcal{DC}_\varepsilon(X)$ *and* $D(\lambda - T - D) \in \mathcal{F}_+(X)$*, then*

$$\sigma_{e1,\varepsilon}(T + S) \subset \sigma_{e1,\varepsilon}(T).$$ ◊

Proof. Let $\lambda \in \mathbb{C}$ and T_0 be a left Fredholm inverse of $\lambda - T - D$, then there exists a compact operator K such that

$$\lambda - T - S - D = (I - ST_0 - D)(\lambda - T - D) + SK - D(\lambda - T - D). \tag{10}$$

As $ST_0 \in \mathcal{DC}_\varepsilon(X)$, it follows from Proposition 5.4 that $I - ST_0 - D \in \Phi_+(X)$. Since $\lambda \notin \sigma_{e1,\varepsilon}(T)$, then $\lambda - T - D \in \Phi_+(X)$. Using the fact that $SK \in \mathcal{K}(X) \subset \mathcal{F}_+(X)$ and $D(\lambda - T - D) \in \mathcal{F}_+(X)$ and applying Theorem 2.4 (*iv*) on Eq. (10), we get $\lambda - T - S - D \in \Phi_+(X)$. This yields to $\lambda \notin \sigma_{e1,\varepsilon}(T + S)$, which is equivalent to the state estimate. Q.E.D.

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