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# On Nikodým and Rainwater sets for $ba(\mathcal{R})$ and a Problem of M. Valdivia

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**Abstract.** If $\mathcal{R}$ is a ring of subsets of a set $\Omega$ and $ba(\mathcal{R})$ is the Banach space of bounded finitely additive measures defined on $\mathcal{R}$ equipped with the supremum norm, a subfamily $\Delta$ of $\mathcal{R}$ is called a *Nikodým set* for $ba(\mathcal{R})$ if each set $\{\mu_\alpha : \alpha \in \Lambda\}$ in $ba(\mathcal{R})$ which is pointwise bounded on $\Delta$ is norm-bounded in $ba(\mathcal{R})$. If the whole ring $\mathcal{R}$ is a Nikodým set, $\mathcal{R}$ is said to have property $(N)$, which means that $\mathcal{R}$ satisfies the Nikodým-Grothendieck boundedness theorem. In this paper we find a class of rings with property $(N)$ that fail Grothendieck's property $(G)$ and prove that a ring $\mathcal{R}$ has property $(G)$ if and only if the set of the evaluations on the sets of $\mathcal{R}$ is a so-called *Rainwater set* for $ba(\mathcal{R})$. Recalling that $\mathcal{R}$ is called a $(wN)$-ring if each increasing web in $\mathcal{R}$ contains a strand consisting of Nikodým sets, we also give a partial solution to a question raised by Valdivia by providing a class of rings without property $(G)$ for which the relation $(N) \Leftrightarrow (wN)$ holds.

## 1. Preliminaries

Let $\mathcal{R}$ be a ring of subsets of a nonempty set $\Omega$, $\chi_A$ the characteristic function of a set $A \in \mathcal{R}$ and $\ell_0^\infty(\mathcal{R}) := \operatorname{span}\{\chi_A : A \in \mathcal{R}\}$ the linear space of all $\mathbb{K}$-valued $\mathcal{R}$-simple functions, $\mathbb{K}$ being the scalar field of the real or complex numbers. Since $A \cap B \in \mathcal{R}$ and $A \,\Delta\, B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$, for each $f \in \ell_0^\infty(\mathcal{R})$ there are pairwise disjoint sets $A_1, \ldots, A_m \in \mathcal{R}$ and nonzero $a_1, \ldots, a_m \in \mathbb{K}$, with $a_i \neq a_j$ if $i \neq j$ such that $f = \sum_{i=1}^m a_i \chi_{A_i}$, with $f = \chi_\emptyset$ if $f = 0$. Unless otherwise stated, we assume $\ell_0^\infty(\mathcal{R})$ endowed with the supremum norm $\|f\| = \sup\{|f(\omega)| : \omega \in \Omega\}$. If $Q = \operatorname{acx}\{\chi_A : A \in \mathcal{R}\}$ is the absolutely convex hull of $\{\chi_A : A \in \mathcal{R}\}$ another equivalent norm is defined on $\ell_0^\infty(\mathcal{R})$ by the *gauge* of $Q$, namely $\|f\|_Q = \inf\{\lambda > 0 : f \in \lambda Q\}$. For if $f \in \ell_0^\infty(\mathcal{R})$ with $\|f\| \leq 1$ it can be shown by induction on the number of non-vanishing different values of $f$ that $f \in 4Q$ (*cf.* [7, Proposition 5.1.1]), hence $\|\cdot\| \leq \|\cdot\|_Q \leq 4\|\cdot\|$. The dual of $\ell_0^\infty(\mathcal{R})$ is the Banach space $ba(\mathcal{R})$ of bounded finitely additive measures defined on $\mathcal{R}$ equipped with the supremum-norm, that is, with the dual norm of the gauge $\|\cdot\|_Q$. Each $A \in \mathcal{R}$ defines a continuous linear form on $ba(\mathcal{R})$ represented by $\delta_A$, named the

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evaluation on $A$, given by $\langle \delta_A, \mu \rangle = \mu(A)$ for each $\mu \in ba(\mathcal{R})$. The completion of $\ell_0^\infty(\mathcal{R})$ is the Banach space $\ell_\infty(\mathcal{R})$ of all bounded $\mathcal{R}$-measurable functions endowed with the supremum-norm. The ring $\mathcal{R}$ is an algebra of subsets of $\Omega$ if $\Omega \in \mathcal{R}$ and the ring (resp. algebra) $\mathcal{R}$ is a $\sigma$-ring ($\sigma$-algebra) if $\cup\{A_n : n \in \mathbb{N}\} \in \mathcal{R}$ whenever $A_n \in \mathcal{R}$ for all $n \in \mathbb{N}$.

We say that a subfamily $\Delta$ of a ring $\mathcal{R}$ is a *Nikodým set* for $ba(\mathcal{R})$, or that $\Delta$ has property $(N)$, if each set $\{\mu_\alpha : \alpha \in \Lambda\}$ in $ba(\mathcal{R})$ which is pointwise bounded on $\Delta$ is norm-bounded in $ba(\mathcal{R})$, i.e., if $\sup_{\alpha \in \Lambda} \left|\mu_\alpha(A)\right| < \infty$ for each $A \in \Delta$ implies that $\sup_{\alpha \in \Lambda} \sup_{A \in \mathcal{R}} \left|\mu_\alpha(A)\right| = \sup_{\alpha \in \Lambda} \left\|\mu_\alpha\right\| < \infty$. We say that a subfamily $\Delta$ of a ring $\mathcal{R}$ is a *strong Nikodým set* for $ba(\mathcal{R})$, or that it has property $(sN)$, if each increasing covering $\{\Delta_m : m \in \mathbb{N}\}$ of $\Delta$ contains a Nikodým set $\Delta_n$ for $ba(\mathcal{R})$.

The Nikodým-Grothendieck boundedness theorem asserts that every $\sigma$-algebra $\Sigma$ of subsets of a set $\Omega$ has property $(N)$. This result has been improved by some authors, in particular by Manuel Valdivia, who proved in [26, Theorem 2] that each $\sigma$-algebra $\Sigma$ has property $(sN)$. Valdivia obtained this result in order to prove that if $\mu$ is a bounded additive vector-measure defined in a $\sigma$-algebra $\Sigma$ with values in a inductive limit of Fréchet spaces $F(\tau) := \lim_n F_n(\tau_n)$, there exists $m \in \mathbb{N}$ such that $\mu$ is an $F_m(\tau_m)$-valued bounded finite additive measure [26, Theorem 4].

An *increasing web* $\left\{\Delta_{n_1,n_2,\ldots,n_p} : p, n_1, n_2, \ldots, n_p \in \mathbb{N}\right\}$ of subsets of a set $\Delta$ is a web on $\Delta$ such that $\Delta_{m_1} \subseteq \Delta_{n_1}$ whenever $m_1 \leq n_1$ and $\Delta_{n_1,n_2,\ldots,n_p,m_{p+1}} \subseteq \Delta_{n_1,n_2,\ldots,n_p,n_{p+1}}$ whenever $m_{p+1} \leq n_{p+1}$ for every $n_i \in \mathbb{N}$ and $i \leq p$. A subset $\Delta$ of a ring $\mathcal{R}$ is a *web Nikodým set* for $ba(\mathcal{R})$, or has property $(wN)$, if each *increasing web* $\left\{\Delta_{n_1,n_2,\ldots,n_p} : p, n_1, n_2, \ldots, n_p \in \mathbb{N}\right\}$ on $\Delta$ has a strand $\left\{\Delta_{m_1,m_2,\ldots,m_p} : p \in \mathbb{N}\right\}$ consisting of Nikodým sets. In particular, a ring $\mathcal{R}$ is called a $(wN)$-ring if each increasing web on $\mathcal{R}$ contains a *strand* $\left\{\mathcal{R}_{m_1,m_2,\ldots,m_p} : p \in \mathbb{N}\right\}$ consisting of Nikodým sets (see [15]). Valdivia's theorem concerning the $(sN)$ property for $\sigma$-algebras was improved in [16, Theorem 2.7], where it was shown that each $\sigma$-algebra $\Sigma$ of subsets of a set $\Omega$ has property $(wN)$. This result also extends other strong Nikodým properties involving finite strands of increasing webs (see [7] and references therein).

The situation of rings and algebras with respect to properties $(N)$, $(sN)$ and $(wN)$ is totally different. The algebra $\mathcal{A}$ of finite and cofinite subsets of $\mathbb{N}$ does not have property $(N)$, for if $\delta_n$ is the point mass at $\{n\}$ then the measures $\mu_n \in ba(\mathcal{A})$ such that $\mu_n(A) = n(\delta_{n+1}(A) - \delta_n(A))$ for $A$ finite and $\mu_n(A) = -n(\delta_{n+1}(A) - \delta_n(A))$ for $A$ cofinite and $n \in \mathbb{N}$ are pointwise bounded, but $\{\mu_n : n \in \mathbb{N}\}$ is unbounded in $ba(\mathcal{A})$.

Several important classes of algebras of sets have been shown to have property $(N)$, among them algebras with the following properties: *Interpolation Property* (Seever [24]), *Subsequential Interpolation Property* (Freniche [10]), *Weak Subsequential Interpolation Property* (Aizpuru [1]), *Property* $(f)$ (Moltó [17]), *Property* $(E)$ (Schachermayer [22]) and *Subsequential Completeness Property* (Haydon [11]). The last two properties are the same and they imply the well known Vitaly-Hans-Saks property, which is stronger than the Nikodým property. Koszmider and Shelah have shown in [13] that if an infinite algebra $\mathcal{A}$ has the so-called *Weak Subsequential Separation Property* then the cardinal of $\mathcal{A}$ is greater than or equal to the continuum $\mathfrak{c}$. Since all algebras considered here have the Weak Subsequential Separation Property, it arises the natural question whether there exist algebras with the Nikodým property with cardinality less than $\mathfrak{c}$. This question has been solved positively by Sobota in [25]. On the other hand, in [14, Theorem 1] it was proved that the algebra $\mathcal{J}(K)$ of Jordan measurable subsets of the compact interval $K = \prod_{i=1}^{k} [a_i, b_i]$ of $\mathbb{R}^k$, with $a_i < b_i$ for $1 \leq i \leq k$, has property $(wN)$, extending preliminary results due to Schachermayer [22, Proposition 3.3], Ferrando [4, Corollary] and Valdivia [27, Theorem 4]. Note that $|\mathcal{J}(K)| = 2^{\mathfrak{c}}$, where $|A|$ denotes the cardinality of the set $A$. Valdivia asked in [27, Problem 1] whether the equivalence $(N) \Leftrightarrow (sN)$ holds for an algebra $\mathcal{A}$ of sets which is not a $\sigma$-algebra. Concerning this question, the first named author showed in [5, theorem 2.5] that the ring $\mathcal{Z}$ of subsets of density zero of $\mathbb{N}$ has property $(wN)$, improving a previous result of Drewnowski, Florencio and Paúl [2] (see also [3] and [9]) stating that $\mathcal{Z}$ has property $(N)$.

Let us recall that a ring $\mathcal{R}$ of subsets of a set $\Omega$ has *property* $(G)$ if $\ell_\infty(\mathcal{R})$ is a Grothendieck space, i.e., if each weak* convergent sequence in $ba(\mathcal{R})$ is weak convergent in the Banach space $ba(\mathcal{R})$. In [22, equivalence $(G_1) \Leftrightarrow (G_2)$ of Definition 2.3] Schachermayer proved that an algebra $\mathcal{R}$ has property $(G)$ if and only if a bounded sequence $\{\mu_n : n \in \mathbb{N}\}$ in $ba(\mathcal{R})$ which converges pointwise on $\mathcal{R}$ is uniformly exhaustive, i.e., for each sequence $\{A_i : i \in \mathbb{N}\}$ of pairwise disjoint elements of $\mathcal{R}$ it happens that $\lim_{i\to\infty} \sup_{n\in\mathbb{N}} \left|\mu_n(A_i)\right| = 0$.

The algebra $\mathcal{J}$ of Jordan subsets of the interval $[0,1]$ was the first example, due to Schachermayer [22, Propositions 3.2 and 3.3], of an algebra of subsets with property $(N)$ that does not have property $(G)$, answering in the negative the question $(N) \Rightarrow (G)$? stated by Seever in [24] (see [9] for more details). Let us finally recall that a subset $X$ of the dual unit ball $B_{E^*}$ of a Banach space $E$ is called a *Rainwater set* for $E$ if every *bounded* sequence $\{x_n\}_{n=1}^{\infty}$ of $E$ that converges pointwise on $X$ converges weakly in $E$ (*cf.* [18]).

The rest of the paper is divided in three sections. In the second section we present a class of rings of sets with property $(N)$ that fail property $(G)$. In the third we get a partial solution to Valdivia's question with a class of rings without property $(G)$ for which the equivalence $(N) \Leftrightarrow (wN)$ holds. In the last section we show that a ring $\mathcal{R}$ has property $(G)$ if and only if the set of evaluations $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$.

## 2. A class of rings with property (N) that fail property (G)

If $\Sigma$ is a $\sigma$-algebra of subsets of a set $\Omega$ and $A \in \Sigma$, then $\Sigma_A := \{B \in \Sigma : B \subseteq A\}$ is a $\sigma$-algebra contained in $\Sigma$. A subfamily $\mathcal{H}$ of $\Sigma$ is called $\Sigma$-*hereditary* if $\mathcal{H} = \cup\{\Sigma_A : A \in \mathcal{H}\}$. Unless otherwise stated we shall always work with an underlying measurable space $(\Omega, \Sigma)$.

**Definition 2.1.** *A subfamily $\mathcal{M}$ of a ring $\mathcal{R}$ of subsets of $\Omega$ is $\mathcal{R}$-singular if for each sequence $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{R}$ there is $\{M_n : n \in \mathbb{N}\} \subseteq \mathcal{M}$ with $\bigcup_{n=1}^{\infty} (A_n \setminus M_n) \in \mathcal{R}$.*

**Example 2.2.** *$\mathcal{M} = \{\emptyset\}$ is an $\mathcal{R}$-singular subfamily of every $\sigma$-ring $\mathcal{R}$ of subsets of $\Omega$. If $\mathcal{Z}$ stands for the $2^{\mathbb{N}}$-hereditary ring of subsets of density zero of $\mathbb{N}$, it is easy to prove that the ring $\mathcal{M}$ of finite subsets of $\mathbb{N}$ is $\mathcal{Z}$-singular. Obviously, the countable union $\bigcup\{M : M \in \mathcal{M}\} = \mathbb{N}$ does not belong to $\mathcal{Z}$.*

**Theorem 2.3.** *Let $\mathcal{R}$ be a $\Sigma$-hereditary subring of $\Sigma$ and $\mathcal{M}$ a $\Sigma$-hereditary and $\mathcal{R}$-singular subfamily of $\mathcal{R}$. If each subset $T$ of $ba(\mathcal{R})$ which is pointwise bounded on $\mathcal{R}$ is uniformly bounded on $\mathcal{M}$, then $\mathcal{R}$ has property $(N)$.*

*Proof.* Let $\{A_n : n \in \mathbb{N}\}$ be a sequence in $\mathcal{R}$. It suffices to show that $T$ is uniformly bounded on $\{A_n : n \in \mathbb{N}\}$, i.e., that there exists $k > 0$ such that $\sup_{n\in\mathbb{N}} \left|\mu(A_n)\right| < k$ for every $\mu \in T$. By the hypotheses on $\mathcal{M}$ there is a sequence $\{M_n : n \in \mathbb{N}\}$ in $\mathcal{M}$ satisfying that $A := \bigcup_{n=1}^{\infty} (A_n \setminus M_n) \in \mathcal{R}$ with $M_n \subseteq A_n$ for each $n \in \mathbb{N}$. Since $\mathcal{R}$ is $\Sigma$-hereditary, the $\sigma$-algebra $\Sigma_A$ is contained in $\mathcal{R}$. So, by the Nikodým-Grothendieck boundedness theorem, $T$ is uniformly bounded on $\Sigma_A$. By hypothesis $T$ is uniformly bounded on $\mathcal{M}$, hence $T$ is uniformly bounded on $\{A_n : n \in \mathbb{N}\}$. □

**Definition 2.4.** *Let $(\Omega, \Sigma)$ be a measurable space. If we have a sequence $\{\mu_n\}_{n=1}^{\infty}$ of $[0,1]$-valued finitely additive measures that are countably subadditive and a pairwise disjoint sequence $\{E_n : n \in \mathbb{N}\}$ in $\Sigma$ such that $\mu_n(E_n) = 1$ for each $n \in \mathbb{N}$, we shall call $\mathcal{R} = \{A \in \Sigma : \mu_n(A) \to 0\}$ the $\Sigma$-subring dominated by the sequence $\{(\mu_n, E_n) : n \in \mathbb{N}\}$.*

We shall also say that $\mathcal{R}$ is a *dominated $\Sigma$-subring*. Clearly, each dominated $\Sigma$-subring is $\Sigma$-hereditary and it does not have property $(G)$.

**Example 2.5.** *The ring $\mathcal{Z}$ of subsets of $\mathbb{N}$ of density zero is a dominated $2^{\mathbb{N}}$-subring.*

*Proof.* For each natural number $n$ let $E_n := \left\{2^{n-1}+1, 2^{n-1}+2, \ldots, 2^n\right\}$ and let $\mu_n$ be the $[0,1]$-valued positive measure defined on $2^{\mathbb{N}}$ by

$$\mu_n(A) = \frac{|A \cap E_n|}{2^{n-1}}.$$

The pairwise disjoint sets $E_n$ verify that $\mu_n(E_n) = 1$ and, since finite sets have density zero, we have that $E_n \in \mathcal{Z}$ for every $n \in \mathbb{N}$. Let's prove that $\mathcal{Z}$ is exactly the $2^{\mathbb{N}}$-subring dominated by the sequence $\{(\mu_n, E_n) : n \in \mathbb{N}\}$. In fact, if $A \in \mathcal{Z}$ one has

$$\lim_{n\to\infty} \mu_n(A) = \lim_{n\to\infty} \frac{\left|A \cap \left(2^{n-1}, 2^n\right]\right|}{2^{n-1}} = 2 \times \lim_{n\to\infty} \frac{|A \cap (0, 2^n]|}{2^n} - \lim_{n\to\infty} \frac{\left|A \cap \left(0, 2^{n-1}\right]\right|}{2^{n-1}} = 0,$$

and, conversely, if $A \subseteq \mathbb{N}$ verifies that $\mu_n(A) \to 0$, then $A$ is a set of density zero as a consequence of the Stolz convergence test. □

Consequently, the ring $\mathcal{Z}$ does not have property $(G)$ (a fact already observed in [2]).

**Theorem 2.6.** *If $\mathcal{R}$ is the $\Sigma$-subring dominated by a sequence $\{(\mu_n, E_n) : n \in \mathbb{N}\}$, the countable family $\mathcal{M} := \left\{\bigcup_{p=1}^{n} E_p : n \in \mathbb{N}\right\}$ is $\mathcal{R}$-singular.*

*Proof.* If $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{R}$ there exists a strictly increasing sequence $\{n_s\}_{s=1}^{\infty}$ in $\mathbb{N}$ such that for $k \geqslant n_s$

$$0 \leq \mu_k(A_1) + \cdots + \mu_k(A_s) < s^{-1}.$$

Since $\mu_k(E_p) = \delta_{kp}$, for $k < n_{s+1}$ we have that

$$0 \leq \mu_k\left(\bigcup_{i=1}^{\infty}\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right)\right) \leq \sum_{i=1}^{s} \mu_k\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right) \leq \sum_{i=1}^{s} \mu_k(A_i).$$

Consequently, if $n_s \leq k < n_{s+1}$ then

$$0 \leq \mu_k\left(\bigcup_{i=1}^{\infty}\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right)\right) < s^{-1},$$

so that $\lim_{k\to\infty} \mu_k\left(\bigcup_{i=1}^{\infty}\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right)\right) = 0$. Hence $\bigcup_{i=1}^{\infty}\left(A_i \setminus \bigcup_{p=1}^{n_i} E_p\right) \in \mathcal{R}$. □

**Remark 2.7.** *Let $\mathcal{Z}$ be the ring of subsets of $\mathbb{N}$ of density zero. By the previous theorem and Example 2.5, the family $\mathcal{M} := \{[1, 2^n] : n \in \mathbb{N}\}$ is $\mathcal{Z}$-singular. Hence the family of finite subsets of $\mathbb{N}$ is $2^{\mathbb{N}}$-hereditary and $\mathcal{Z}$-singular.*

**Theorem 2.8.** *Assume that $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of atomless probability measures on $\Sigma$ and $\{E_n : n \in \mathbb{N}\}$ a pairwise disjoint sequence in $\Sigma$ with $\mu_n(E_m) = \delta_{n,m}$ for $n, m \in \mathbb{N}$. Then the $\Sigma$-subring $\mathcal{R}$ dominated by $\{(\mu_n, E_n) : n \in \mathbb{N}\}$ has property $(N)$.*

*Proof.* For each $s \in \mathbb{N}$ let $D_s := \bigcup_{p=1}^{s} E_p$. By Theorem 2.6 the family $\{D_s : s \in \mathbb{N}\}$ is $\mathcal{R}$-singular and hence $\mathcal{M} = \{\Sigma_{D_s} : s \in \mathbb{N}\}$ is $\Sigma$-hereditary and $\mathcal{R}$-singular. According to Theorem 2.3, it suffices to prove that each subset $H$ of $ba(\mathcal{R})$ pointwise bounded on $\mathcal{R}$ is uniformly bounded on $\mathcal{M}$.

Let us proceed by contradiction by supposing that $H$ is a subset of $ba(\mathcal{R})$ which it is pointwise bounded on $\mathcal{R}$ but not uniformly bounded on $\mathcal{M}$. Fix $n \in \mathbb{N}$ and for each $p \in \mathbb{N}$ let $\{E_{p,j}^n : 1 \leq j \leq n\}$ denote a partition of $E_p$ consisting of subsets of $\Sigma$ such that $\mu_p(E_{p,j}^n) = n^{-1}$ for $1 \leq j \leq n$. Then, for $s \in \mathbb{N}$ and $1 \leq j \leq n$ set $D_{s,j}^n := \bigcup_{p=1}^{s} E_{p,j}^n$ and

$$\mathcal{M}_j^n := \{\Sigma_{D_{s,j}^n} : s \in \mathbb{N}\}.$$

Since $H$ is not uniformly bounded on $\mathcal{M}$, for each $n \in \mathbb{N}$ there is $j_n$ with $1 \leq j_n \leq n$ such that $H$ is not uniformly bounded on $\mathcal{M}_{j_n}^n$. By the Nikodým-Grothendieck boundedness theorem we get that for each natural number $m_n$ the set $H$ is uniformly bounded on the $\sigma$-algebra $\Sigma_{D_{m_n}}$, hence for each pair of natural numbers $n$ and $m_n$ the set $H$ is not uniformly bounded on $\mathcal{M}_{j_n}^n \setminus \Sigma_{D_{m_n}}$. So, for each pair of natural numbers $n$ and $m_n$ there exist $v_n \in H$, $m_{n+1} > m_n$ and $A_n \subseteq \bigcup\{E_{p,j_n}^n : m_n < p \leq m_{n+1}\}$ with

$$|v_n(A_n)| > n, \text{ for each } n \in \mathbb{N}. \tag{1}$$

Let $A := \bigcup\{A_n : n \in \mathbb{N}\} \in \Sigma$. If $m_n < p \leq m_{n+1}$ we obtain by construction that $\mu_p(A) = \mu_p(A_n) \leq \mu_p(E_{p,j_n}^n) = n^{-1}$. Hence, $\lim_{p\to\infty} \mu_p(A) = 0$ and consequently $A \in \mathcal{R}$. Since the $\sigma$-algebra $\Sigma_A$ is contained in the $\Sigma$-hereditary ring $\mathcal{R}$, it turns out that $H$ must be uniformly bounded in $\Sigma_A$, which contradicts the inequalities (1). □

**Example 2.9.** *Dominated Σ-subrings with property (N). Let $\Omega = [0,1]$ and $\Sigma$ be the $\sigma$-algebra of Lebesgue measurable subsets of the interval $[0,1]$. Define the atomless measures*

$$\mu_n(A) = \int_A f_n(t)\, d\lambda(t)$$

*on $\Sigma$, where $f_n : [0,1] \to \mathbb{R}$ is the function whose graph consists of a flat peak of height $2^n$ over the segment $\left(2^{-n}, 2^{-n+1}\right]$ along with the segments $\{(x,0) : x \in [0,2^{-n}] \cup \left(2^{-n+1}, 1\right]\}$ and $\lambda$ stands for the Lebesgue probability measure of $[0,1]$. Set $E_n := \left(2^{-n}, 2^{-n+1}\right]$ for each $n \in \mathbb{N}$. The $\Sigma$-subring $\mathcal{R}$ of subsets of $[0,1]$ dominated by $\{(\mu_n, E_n) : n \in \mathbb{N}\}$ has property $(N)$ by virtue of the previous theorem and $\mathcal{R}$, as every dominated $\Sigma$-subring, does not have property $(G)$. Each Lebesgue measurable set that meets only finitely many sets $E_n$ belongs to $\mathcal{R}$. Moreover $M = \bigcup_{n=1}^{\infty} \left(\frac{2^{n+1}-1}{4^n}, \frac{1}{2^{n-1}}\right] \in \mathcal{R}$ since $\mu_n(M) = 2^{-n} \to 0$, and $M$ meets each $E_n$.*

## 3. Rings for which (*N*) ⇔ (*wN*)

We exhibit a class of rings for which properties $(N)$ and $(wN)$ are equivalent. This provides a partial positive solution of the still open problem for algebras of sets concerning whether $(N) \Rightarrow (sN)$ [27, Problem 1].

If a subfamily $\Delta$ of a ring $\mathcal{R}$ is not a *Nikodým set* for $ba(\mathcal{R})$ there exists an unbounded sequence $\{\mu_n\}_{n=1}^{\infty}$ in $ba(\mathcal{R})$ which is pointwise bounded on $\Delta$. Consequently $\Delta$ is the union of the sets $\Delta_m := \cup\{A \in \Delta : \left|\mu_n(\chi_A)\right| \leq m, \forall n \in \mathbb{N}\}$ for $m \in \mathbb{N}$. Since $\{m^{-1}\mu_n : n \in \mathbb{N}\} \subseteq \{\chi_A : A \in \Delta_m\}^0$, it follows that $\{\chi_A : A \in \Delta_m\}^0$ is an unbounded subset of $ba(\mathcal{R})$ for every $m \in \mathbb{N}$. Conversely, if $\Delta$ is the union of an increasing sequence $\{\Delta_m\}_{m=1}^{\infty}$ and each $\{\chi_A : A \in \Delta_m\}^0$ is unbounded, there is $\mu_m \in \{\chi_A : A \in \Delta_m\}^0$ with $\left\|\mu_m\right\| > m$ for each $m \in \mathbb{N}$. Since $\{\mu_n : n \in \mathbb{N}\}$ is $\Delta$-pointwise bounded, $\Delta$ is not a Nikodým set.

Therefore a subfamily $\Delta$ of a ring $\mathcal{R}$ is a *Nikodým set* for $ba(\mathcal{R})$ if and only if for each increasing covering $\{\Delta_m\}_{m=1}^{\infty}$ of $\Delta$ there exists $\Delta_n$ such that $\{\chi_A : A \in \Delta_n\}^0$ is a bounded subset of $ba(\mathcal{R})$ or, equivalently, if the closed absolutely convex hull of $\{\chi_A : A \in \Delta_n\}$ is a neighborhood of zero in $\ell_0^{\infty}(\mathcal{R})$. This result also follows from the Amemiya-Kōmura property (see [21]).

If a subfamily $\Delta$ of a ring $\mathcal{R}$ is a Nikodým set for $ba(\mathcal{R})$ then $F := \operatorname{span}\{\chi_A : A \in \Delta\}$ is a subspace of $\ell_0^{\infty}(\mathcal{R})$ dense and barrelled (i.e., each subset $\{\mu_\alpha : \alpha \in \Lambda\}$ of $ba(\mathcal{R})$ which is pointwise bounded on $F$ verifies that $\sup_{\alpha\in\Lambda} \left\|\mu_\alpha|_F\right\| < \infty$). The converse is obvious because a subset $\{\mu_\alpha : \alpha \in \Lambda\}$ of $ba(\mathcal{R})$ is pointwise bounded on $F$ if and only if is pointwise bounded in $\Delta$ and, by density $\left\|\mu_\alpha|_F\right\| = \left\|\mu_\alpha\right\|$. Therefore $\Delta$ is a Nikodým set if and only if $\operatorname{span}\{\chi_A : A \in \Delta\}$ is a subspace of $\ell_0^{\infty}(\mathcal{R})$ dense and barrelled. In particular, the barrelledness of $\ell_0^{\infty}(\mathcal{R})$ is equivalent to the fact that $\mathcal{R}$ has property $(N)$.

**Lemma 3.1.** *Let $\mathcal{R}$ be a ring. If $\mathcal{N}$ is a Nikodým set for $ba(\mathcal{R})$ and $\{\mathcal{N}_n : n \in \mathbb{N}\}$ is an increasing covering of $\mathcal{N}$, there exists $m \in \mathbb{N}$ such that $\operatorname{span}\{\chi_A : A \in \mathcal{N}_m\}$ is dense in $\ell_0^{\infty}(\mathcal{R})$. If $\mathcal{N}$ is not a Nikodým set for $ba(\mathcal{R})$ and $\operatorname{span}\{\chi_A : A \in \mathcal{N}\}$ is dense in $\ell_0^{\infty}(\mathcal{R})$, for each countable subfamily $\mathcal{M}$ of $\mathcal{R}$ it holds that $\mathcal{N} \cup \mathcal{M}$ is not a Nikodým set for $ba(\mathcal{R})$.*

*Proof.* If $\mathcal{N}$ is a Nikodým set then there exists $\mathcal{N}_m$ such that the closed absolutely convex hull of $\{\chi_A : A \in \mathcal{N}_m\}$ is a neighborhood of 0 in $\ell_0^{\infty}(\mathcal{R})$. When $\mathcal{N}$ is not a Nikodým set for $ba(\mathcal{R})$ and $\operatorname{span}\{\chi_A : A \in \mathcal{N}\}$ is dense in $\ell_0^{\infty}(\mathcal{R})$, it turns out that $\operatorname{span}\{\chi_A : A \in \mathcal{N}\}$ is a non barrelled subspace of $\ell_0^{\infty}(\mathcal{R})$. So, if $\mathcal{M}$ is countable, the countable dimension of $\ell_0^{\infty}(\mathcal{M})$ implies that $\operatorname{span}\{\chi_A : A \in \mathcal{N} \cup \mathcal{M}\}$ is also non barrelled (*cf.* [19, Theorem 4.3.6]). Hence $\mathcal{N} \cup \mathcal{M}$ is not a Nikodým set for $ba(\mathcal{R})$. □

**Lemma 3.2.** *Let $\mathcal{R}$ be a ring with property $(N)$ which fails to have property $(wN)$. Then there exists an increasing web $\{\mathcal{R}_t : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ in $\mathcal{R}$ such that for each countable subfamily $\mathcal{M}$ of $\mathcal{R}$ the increasing web $\{\mathcal{R}_t \cup \mathcal{M} : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ does not contain any strand consisting entirely of Nikodým sets for $ba(\mathcal{R})$.*

*Proof.* Let $\{\mathcal{R}'_t : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ be an increasing web in $\mathcal{R}$ without any strand consisting of Nikodým sets and let $J$ be the subset of $\bigcup_{s\in\mathbb{N}} \mathbb{N}^s$ such that $t \in J$ whenever both $\mathcal{R}'_t$ is a Nikodým set and $\operatorname{span}\{\chi_A : A \in \mathcal{R}'_t\}$ is dense in $\ell_0^\infty(\mathcal{R})$. Since $\mathcal{R}$ has property $(N)$, by Lemma 3.1 there exists $m \in \mathbb{N}_0$ such that $\operatorname{span}\{\chi_A : A \in \mathcal{R}'_{t_1}\}$ is dense in $\ell_0^\infty(\mathcal{R})$ for each $t_1 \geqslant m$. If $J$ were the empty set, no $\mathcal{R}'_{t_1}$ would be a Nikodým set for $ba(\mathcal{R})$. Hence, due to Lemma 3.1 and the increasing web condition, no $\mathcal{R}'_{t_1} \cup \mathcal{M}$ is a Nikodým set for each countable subset $\mathcal{M}$ of $\mathcal{R}$ and each $t_1 \in \mathbb{N}$. So, the web formed by the sets $\mathcal{R}_t = \mathcal{R}'_{t_1}$ for $t = (t_1, t_2, \cdots, t_p) \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s$ verifies that $\mathcal{R}_t \cup \mathcal{M}$ is not a Nikodým set for each countable subset $\mathcal{M}$ of $\mathcal{R}$.

If $t = (t_1, t_2, \cdots, t_p) \in J$ and $(t_1, t_2, \cdots, t_p, t_{p+1}) \notin J$, for each $t_{p+1} \in \mathbb{N}$, then it is obvious that

$$\{(t'_1) : t'_1 \geqslant t_1\} \cup \{(t_1, t'_2) : t'_2 \geqslant t_2\} \cup \cdots \cup \{(t_1, t_2, \cdots, t'_p) : t'_p \geqslant t_p\} \subseteq J$$

and $\mathcal{R}'_{t_1,t_2,\cdots,t_p}$ is a Nikodým set. Applying Lemma 3.1 with $\mathcal{N} = \mathcal{R}'_{t_1,t_2,\cdots,t_p}$ and $\mathcal{N}_n = \mathcal{R}'_{t_1,t_2,\cdots,t_p,n}$ we get $m = m_{t_1,t_2,\cdots,t_p} \in \mathbb{N}_0$ such that $\operatorname{span}\{\chi_A : A \in \mathcal{R}'_{t_1,t_2,\cdots,t_p,t_{p+1}}\}$ is dense in $\ell_0^\infty(\mathcal{R})$ for each $t_{p+1} \geqslant m$. So, $\mathcal{R}'_{t_1,t_2,\cdots,t_p,t_{p+1}}$ is not a Nikodým set for every $t_{p+1} \geqslant m$. Consequently, Lemma 3.1 implies that $\mathcal{R}'_{t_1,t_2,\cdots,t_p,t_{p+1}} \cup \mathcal{M}$ is not a Nikodým set for $ba(\mathcal{R})$ for each $t_{p+1} \in \mathbb{N}$ and each countable subset $\mathcal{M}$ of $\mathcal{R}$. We establish the lemma by means of the increasing web determined by the sets $\mathcal{R}'_t$ with $t \in J$ together with the sets $\mathcal{R}_{t_1,t_2,\cdots,t_p,t_{p+1},\cdots t_{p+s}} = \mathcal{R}'_{t_1,t_2,\cdots,t_p,t_{p+1}}$ when $(t_1, t_2, \cdots, t_p) \in J$, $(t_1, t_2, \cdots, t_p, t_{p+1}) \notin J$ for each $t_{p+1} \in \mathbb{N}$, and $(t_{p+1}, \cdots t_{p+s}) \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s$. □

**Theorem 3.3.** *Let $\mathcal{M}$ be a $\Sigma$-hereditary, countable and singular subfamily of the $\Sigma$-hereditary ring $\mathcal{R}$. If $\mathcal{R}$ has property $(N)$, then $\mathcal{R}$ has property $(wN)$.*

*Proof.* Assume by way of contradiction that $\mathcal{R}$ has property $(N)$ but does not have property $(wN)$. By Lemma 3.2 there exists an increasing web $\{\mathcal{R}_t : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ in $\mathcal{R}$ such that the increasing web $\{\mathcal{R}_t \cup \mathcal{M} : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ does not contain any strand formed entirely by Nikodým sets for $ba(\mathcal{R})$. Let

$$J := \{t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s : \mathcal{R}_t \cup \mathcal{M} \text{ is not a Nikodým set for } ba(\mathcal{R})\}.$$

Then for each $t \in J$ there exists in $ba(\mathcal{R})$ a subset $T_t$ which is pointwise bounded on $\mathcal{R}_t \cup \mathcal{M}$ but is not uniformly bounded on $\mathcal{R}$. Since $\mathcal{R}$ is a Nikodým set, $T_t$ cannot be pointwise bounded on $\mathcal{R}$ so that there exists $A_t \in \mathcal{R}$ such that $T_t$ is unbounded in $A_t$ for each $t \in J$. On the other hand, since $\mathcal{M}$ is a $\Sigma$-hereditary and singular, for each $t \in J$ the set $A_t$ contains a subset $M_t \in \mathcal{M}$ such that $A := \bigcup\{A_t \setminus M_t : t \in J\} \in \mathcal{R}$. As in addition the $\Sigma$-hereditary ring $\mathcal{R}$ contains the $\sigma$-algebra $\Sigma_A$, which has property $(wN)$, there exists a sequence $\left\{m_p\right\}_{p=1}^\infty \subseteq \mathbb{N}$ such that, for each $p \in \mathbb{N}$ one has that

$$\{\mathcal{R}_{m_1 m_2 \cdots m_p} \cup \mathcal{M}\} \cap \Sigma_A \text{ is a Nikodým set for } ba(\Sigma_A). \tag{2}$$

Given that $\{\mathcal{R}_t \cup \mathcal{M} : t \in \bigcup_{s\in\mathbb{N}} \mathbb{N}^s\}$ does not contain any strand formed entirely by Nikodým sets for $ba(\mathcal{R})$, there exists $q \in \mathbb{N}$ such that $t_q := (m_1, m_2, \cdots, m_q) \in J$ and then $T_{t_q}$ is unbounded in $A_{t_q}$. By construction $T_{t_q}$ is pointwise bounded on $\mathcal{R}_{t_q} \cup \mathcal{M}$, in particular $T_{t_q}$ is bounded in $M_{t_q}$, and by (2) with $p = q$ it follows that $T_{t_q}$ is also uniformly bounded in $\Sigma_A$. In particular, since $A_{t_q} \setminus M_{t_q} \in \Sigma_A$, it turns out that $T_{t_q}$ is bounded in $M_{t_q} \cup \left(A_{t_q} \setminus M_{t_q}\right) = A_{t_q}$, a contradiction. □

**Remark 3.4.** *By [2] (see also [3] and [9]), the ring $\mathcal{Z}$ of subsets of density zero of $\mathbb{N}$ has property $(N)$, hence Remark 2.7 and Theorem 3.3 imply that $\mathcal{Z}$ has property $(wN)$.*

## 4. Rainwater sets for *ba* $(\mathcal{R})$

As mentioned in the preliminaries a subset $X$ of the closed dual unit ball $B_{E^*}$ of a Banach space $E$ is called a *Rainwater set* for $E$ if every *bounded* sequence $\{x_n\}_{n=1}^\infty$ of $E$ that converges pointwise on $X$, i.e., such that $x^* x_n \to x^* x$ for each $x^* \in X$, converges weakly in $E$. By [23, Corollary 11] each James boundary $J$ for $B_{E^*}$ is a Rainwater set for $E$ (the converse is not true). In particular $\operatorname{Ext} B_{E^*}$ is a Rainwater set for $E$, [20]. This latter fact also follows from Choquet's integral representation theorem, and implies that for each compact space $K$ the set of evaluations $\{\delta_a : a \in K\}$ is a Rainwater set for $C(K)$. Recently, Rainwater sets for the Banach space $C^b(X)$ of continuous and bounded real-valued functions defined on a completely regular space $X$ have been investigated in [6]. The next proposition provides a relation between Rainwater sets and property $(G)$.

**Proposition 4.1.** *Let $\mathcal{R}$ be a ring of subsets of a set $\Omega$. The following are equivalent*

1. *$\mathcal{R}$ has property $(G)$.*
2. *The set of evaluations $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$.*

*Proof.* First we suppose that $\mathcal{R}$ is an algebra. Assume that $\mathcal{R}$ has property $(G)$ and that $\{\mu_n\}_{n=1}^{\infty}$ is a bounded sequence in $ba(\mathcal{R})$ such that $\langle \mu_n, \delta_A \rangle \to \langle \mu, \delta_A \rangle$ for every $A \in \mathcal{R}$, i.e., such that $\mu_n(A) \to \mu(A)$ for each $A \in \mathcal{R}$. Since $\{\mu_n\}_{n=1}^{\infty}$ is a bounded sequence in $ba(\mathcal{R})$ that converges pointwise on $\mathcal{R}$, according to [22, condition $G_1$ in Definition 2.3] the sequence $M = \{\mu_n : n \in \mathbb{N}\}$ is uniformly exhaustive on $\mathcal{R}$. Given that $M$ is uniformly exhaustive and bounded on the members of $\mathcal{R}$, [22, Proposition 1.2] ensures that $M$ is a relatively weakly compact set of $ba(\mathcal{R})$. Thus, by Eberlein's theorem, $M$ is weakly sequentially compact. Then, as $\mu_n(A) \to \mu(A)$ for each $A \in \mathcal{R}$, we get that $\mu$ is the only possible weakly adherent point of the sequence $\{\mu_n\}_{n=1}^{\infty}$. So, $\mu_n \to \mu$ weakly in $ba(\mathcal{R})$ and $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$.

Assume conversely that the set of evaluations $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$. Let $\{\mu_n\}_{n=1}^{\infty}$ be any bounded sequence in $ba(\mathcal{R})$ that converges pointwise on $\mathcal{R}$ to some $\mu \in ba(\mathcal{R})$. The latter means that $\mu_n(A) \to \mu(A)$ for each $A \in \mathcal{R}$, so that

$$\langle \mu_n, \delta_A \rangle \to \langle \mu, \delta_A \rangle$$

for all $A \in \mathcal{R}$. Hence $\mu_n \to \mu$ weakly in $ba(\mathcal{R})$, so that $\{\mu_n : n \in \mathbb{N}\}$ is a relatively weakly compact subset of $ba(\mathcal{R})$. Again by [22, Proposition 1.2] we have that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is uniformly exhaustive, which according to [22, equivalence $(G_1) \Leftrightarrow (G_2)$ of Definition 2.3] means that $\mathcal{R}$ has property $(G)$.

To get the proof for a ring $\mathcal{R}$, notice that as the algebra $\mathcal{F}$ generated by $\mathcal{R}$ and $\{\Omega\}$ verifies that the codimension of $l_0^{\infty}(\mathcal{R})$ in $l_0^{\infty}(\mathcal{F})$ is 1, then Proposition 1.2. in [22] as well as the equivalence $(G_1) \Leftrightarrow (G_2)$ of Definiton 2.3. in [22] hold for the ring $\mathcal{R}$. □

**Corollary 4.2.** *In the $(wN)$-ring $\mathcal{Z}$ of subsets of density zero of $\mathbb{N}$ the set of evaluations $\{\delta_A : A \in \mathcal{Z}\}$ is not a Rainwater set for $ba(\mathcal{Z})$.*

*Proof.* Since no dominated subring has property $(G)$, this is consequence of Example 2.5 and Proposition 4.1. □

**Corollary 4.3.** *Let $\mathcal{N}$ be a Nikodým set for $ba(\mathcal{R})$ such that $\{\delta_A : A \in \mathcal{N}\}$ is a Rainwater set for $ba(\mathcal{R})$. Then each sequence $\{\mu_n : n \in \mathbb{N}\}$ in $ba(\mathcal{R})$ pointwise convergent on $\mathcal{N}$ is weakly convergent in $ba(\mathcal{R})$.*

*Proof.* Since $\mu_n(A) \to \mu(A)$ for every $A \in \mathcal{N}$, the sequence $\{\mu_n : n \in \mathbb{N}\}$ is pointwise bounded on $\mathcal{N}$, hence norm bounded in $ba(\mathcal{R})$ due to $\mathcal{N}$ is a Nikodým set. As in addition $\{\delta_A : A \in \mathcal{N}\}$ is a Rainwater set for $ba(\mathcal{R})$, then $\mu_n \to \mu$ weakly in $ba(\mathcal{R})$. □

**Corollary 4.4.** *If a ring $\mathcal{R}$ of subsets of $\Omega$ has both properties $(N)$ and $(G)$, i.e., $\mathcal{R}$ is a so-called ring with the Vitali-Hahn-Saks property, or property $(VHS)$, then each sequence in $ba(\mathcal{R})$ pointwise convergent on $\mathcal{R}$ is weakly convergent in $ba(\mathcal{R})$.*

**Remark 4.5.** *There have been several attempts of introducing boundedness properties stronger than property $(wN)$ defined in terms of increasing webs, as properties $(w\text{-}sN)$ or $\left(w^2N\right)$ (see [12] and [15]), but all them have shown to be equivalent to property $(wN)$ (this follows from [15, Proposition 1]). It is easy to prove that a ring $\mathcal{R}$ has $(wN)$-property if and only $\ell_0^{\infty}(\mathcal{R})$ is baireled, i.e. if each increasing web $\left\{E_{n_1,n_2,\dots,n_p} : p, n_1, n_2, \dots, n_p \in \mathbb{N}\right\}$ on $\ell_0^{\infty}(\mathcal{R})$ formed by linear subspaces contains a strand $\{E_{m_1,m_2,\dots,m_p} : p \in \mathbb{N}\}$ formed by subspaces both dense and barrelled [8]. Other classic barrelledness properties stronger than baireledness fail for the space $\ell_0^{\infty}(\mathcal{R})$ even if $\mathcal{R}$ is a $\sigma$-algebra (see [8, 9] for details).*

Summarizing, if $(\Omega, \Sigma)$ is a measurable space and $\mathcal{R}$ is a $\Sigma$-hereditary ring of subsets of $\Omega$ that contains a $\Sigma$-hereditary, countable and singular subfamily $\mathcal{M}$, then $\mathcal{R}$ has property $(wN)$ if and only if it has property $(N)$, which provides a partial solution to Valdivia's question. We have also shown that a ring of sets $\mathcal{R}$ has property $(G)$ if and only if the family of evaluations $\{\delta_A : A \in \mathcal{R}\}$ is a Rainwater set for $ba(\mathcal{R})$.

**Problem 4.6.** *Characterize those rings $\mathcal{R}$ of subsets of a set $\Omega$ for which $(N) \Leftrightarrow (wN)$.*

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