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Approximation by a Generalization of the Jakimovski-Leviatan Operators

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Abstract. In this paper, we introduce a Kantorovich type generalization of Jakimovski-Leviatan operators constructed by A. Jakimovski and D. Leviatan (1969) and the theorems on convergence and the degree of convergence are established. Furthermore, we study the convergence of these operators in a weighted space of functions on $[0, \infty)$.

1. Introduction

In approximation theory, Szász type operators and Chlodovsky operators have been studied intensively [see [1], [2], [9], [11], [12], [13], [16], [17], [18] and many others]. Also orthogonal polynomials are important area of mathematical analysis, mathematical and theoretical physics. In mathematical analysis and in the positive approximation processes, the notion of orthogonal polynomials seldomly appears. Cheney and Sharma [8] established an operator

$$P_n(f;x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t) x^k, \tag{1}$$

where $t \leq 0$ and $L_k^{(n)}$ denotes the Laguerre polynomials. For the special case $t = 0$, the operators given by (1) reduce to the well-known Meyer-König and Zeller operators [15].

In view of the relation between orthogonal polynomials and positive linear operators have been investigated by many researchers (see [18],[9]). One of them is Jakimovski and Leviatan 's study. In 1969, the authors introduced Favard-Szász type operators P_n, by using Appell polynomials are given by $g(u) = \sum\limits_{n=0}^{\infty} a_n u^n, g(1) \neq 0$ be an analytic function in the disk $|u| < r$ $(r > 1)$ and $p_k(x) = \sum\limits_{i=0}^{k} a_i \frac{x^{k-i}}{(k-i)!}$, $(k \in \mathbb{N})$ be the Appell polynomials defined by the identity

$$g(u)e^{ux} \equiv \sum_{k=0}^{\infty} p_k(x) u^k. \tag{2}$$

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Let $E[0,\infty)$ denote the space of exponential type functions on $[0,\infty)$ which satisfy the property $\left|f(x)\right| \leq \beta e^{\alpha x}$ for some finite constants $\alpha, \beta > 0$.

In [1], the authors considered the operator P_n, with

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f(\frac{k}{n}) \tag{3}$$

for $f \in E[0,\infty)$ and studied approximation properties of these operators, as well as the analogue to Szász's results. These operators are called as the Jakimovski-Leviatan operators.

If $g(u) \equiv 1,$ from (2) we obtain $p_k(x) = \frac{x^k}{k!}$ and we obtain classical Szász-Mirakjan operator which is given by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(\frac{k}{n}).$$

In 1969, Wood [7] showed that the operators P_n are positive if and only if $\frac{a_k}{g(1)} \geq 0$, $(k = 0, 1, ...)$. In 1996, Ciupa [2] was studied the rate of convergence of these operators. In 1999, Abel and Ivan [19] showed an asimptotic expansion of the operators given by (3) and their derivatives. In 2003, İspir [16] showed the approximation of continuous functions having polynomial growth at infinity by the sequence of the operator in (3). In 2007, Ciupa [3] defined Modified Jakimovski-Leviatan operators and studied rate of convergence, order of approximation and Voronovskaya type theorem. Recently, Büyükyazıcı et al, [13] studied approximation properties of Chlodovsky type Jakimovski -Leviatan operators. They proved Voronovskaya-type theorem and studied the convergence of these operators in a weighted space by using a new type of weighted modulus of continuity. In 2015, Aktaş et al [20] studied approximation properties and also proved a Voronovskaya- type theorem for Kantorovich-Stancu type operators including Gould-Hopper polynomials.

In this paper, we consider the following Kantorovich generalization of the Chlodovsky form of the Jakimovski-Leviatan operators given by

$$L_n^*(f;x) = \frac{e^{-\frac{n}{b_n}x}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k(\frac{n}{b_n}x) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} f(t)dt, \tag{4}$$

where b_n is a positive increasing sequence with the properties

$$\lim_{n\to\infty} b_n = \infty, \ \lim_{n\to\infty} \frac{b_n}{n} = 0 \tag{5}$$

and p_k are Appell polynomials defined by (2). Recently, some generalizations of the Jakimovski-Leviatan operators given by (3) have been considered in [2], [3] and [10].

2. Some approximation properties of $L_n^*(f;x)$

In approximation theory, the positive approximation processes discovered by Korovkin plays a central role and arise in a natural way in many problems connected with functional analysis, harmonic analysis, measure theory, partial differential equations and probability theory.

Now we give some results.

Lemma 2.1. *The operators $L_n^*(f;x)$ defined by (4) satisfy the following equalities.*

$$L_n^*(1;x) = 1, \tag{6}$$

$$L_n^*(t;x) = x + \frac{g'(1)}{g(1)}\frac{b_n}{n} + \frac{1}{2}\frac{b_n}{n}, \tag{7}$$

$$L_n^*(t^2;x) = x^2 + \frac{b_n}{n}x\left(\frac{2g(1)+2g'(1)}{g(1)}\right) + \frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3}\right). \tag{8}$$

Lemma 2.2. *The central moments of the operators* $L_n^*(f;x)$ *are given by*

$$L_n^*(t-x;x) = \frac{g'(1)}{g(1)}\frac{b_n}{n} + \frac{1}{2}\frac{b_n}{n}$$

$$L_n^*((t-x)^2;x) = \frac{b_n}{n}x + \frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3}\right). \tag{9}$$

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions on $[0,\infty)$ and $\widetilde{C_B}[0,\infty)$ denote the space of real valued uniformly continuous and bounded functions on $[0,\infty)$, with the usual sup −norm.

Theorem 2.3. *If* $f \in C_B[0,\infty)$, *then* $\lim L_n^*(f) = f$ *uniformly on* $[0,a]$, $a > 0$.

Proof. *From (6)-(8), we have*

$$\lim_{n\to\infty} L_n^*(e_i;x) = e_i(x), \quad i \in \{0,1,2\},$$

where $e_i(t) = t^i$. *Applying the Korovkin theorem (see, e.g., [10]), we obtain the desired result.* □

In this section, we deal with the rate of convergence of the $L_n^*(f;x)$ to f by means of a classical approach, the second modulus of smoothness and Peetre's K-functional.

For $f \in C_B[0,\infty)$, the modulus of continuity of f is defined by

$$\omega(f,\delta) = \sup_{\substack{x,y\in[0,\infty)\\ 0<|x-y|\le\delta}} \left|f(x)-f(y)\right|, \ \delta > 0.$$

It is also well known that, for any $\delta > 0$ and each $x, y \in [0,\infty)$,

$$\left|f(x)-f(y)\right| \le \omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right).$$

The next result gives the rate of convergence of the sequence $L_n^*(f)$ to f by means of the modulus of continuity for a bounded interval.

Theorem 2.4. *Let* $f \in C_B[0,\infty)$ *.Then for any* $x \in [0,a]$ *we have*

$$\left|L_n^*(f;x) - f(x)\right| \le \left\{1 + \frac{1}{\delta}\left(\sqrt{\theta_n}\right)\right\}\omega_a(f,\delta).$$

where $\omega_a(f,\delta) = \sup_{\substack{x,y\in[0,a]\\ 0<|x-y|\le\delta}} \left|f(x)-f(y)\right|$ *is the modulus of continuity of* f *on* $[0,a]$ *and* $\delta = \sqrt{\theta_n}$ *with*

$$\theta_n = \frac{b_n}{n}a + \frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3}\right).$$

Proof.

$$\begin{aligned}\left|L_n^*(f;x)-f(x)\right| &\le \frac{e^{-nx}}{g(1)}\frac{n}{b_n}\sum_{k=0}^{\infty}p_k(\frac{n}{b_n}x)\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}\left|f(s)-f(x)\right|ds\\ &\le \frac{e^{-nx}}{g(1)}\frac{n}{b_n}\sum_{k=0}^{\infty}p_k(\frac{n}{b_n}x)\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}\omega_a(f,\delta)\left(\frac{|s-x|}{\delta}+1\right)ds\\ &\le \left\{1+\frac{e^{-nx}}{g(1)}\frac{n}{b_n}\sum_{k=0}^{\infty}p_k(\frac{n}{b_n}x)\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}|s-x|\,ds\right\}\omega_a(f,\delta).\end{aligned}$$

By using the Cauchy-Schwarz inequality for integration, we get

$$\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}|s-x|\,ds \le \sqrt{\frac{b_n}{n}}\left(\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}(s-x)^2\,ds\right)^{1/2}$$

which holds that

$$\sum_{k=0}^{\infty}p_k(\frac{n}{b_n}x)\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}|s-x|\,ds \le \sqrt{\frac{b_n}{n}}\sum_{k=0}^{\infty}p_k(\frac{n}{b_n}x)\left(\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}(s-x)^2\,ds\right)^{1/2}.$$

If we apply the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|L_n^*(f;x)-f(x)\right| &\le \left\{1+\frac{1}{\delta}\left(\frac{e^{-nx}}{g(1)}\frac{n}{b_n}\sum_{k=0}^{\infty}p_k(\frac{n}{b_n}x)\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}(s-x)^2\,ds\right)\right\}\omega_a(f,\delta)\\ &= \left\{1+\frac{1}{\delta}\left(\sqrt{L_n^*((s-x)^2;x)}\right)\right\}\omega_a(f,\delta)\\ &\le \left\{1+\frac{1}{\delta}\left(\sqrt{\theta_n}\right)\right\}\omega_a(f,\delta).\end{aligned}$$

Now if we choose $\delta=\sqrt{\theta_n}$, it completes the proof. □

Now, we remember the second order modulus of smoothness of $f\in C_B[0,\infty)$ which is defined by

$$\omega_2(f;\delta)=\sup_{0<t<\delta}\left\|f(.+2t)-2f(.+t)+f(.)\right\|_{C_B},\ \delta>0.$$

Peetre's K−functional of the function $f\in C_B[0,\infty)$ is defined by

$$K(f,\delta)=\inf_{g\in C_B^2[0,\infty)}\left\{\left\|f-g\right\|+\delta\left\|g''\right\|\right\}, \tag{10}$$

where

$$C_B^2[0,\infty):=\left\{g\in C_B[0,\infty): g',g''\in C_B[0,\infty)\right\},$$

with the norm $\left\|g\right\|_{C_B^2}=\left\|g\right\|_{C_B^2}+\left\|g'\right\|_{C_B}+\left\|g''\right\|_{C_B}$. The following inequality

$$K(f;\delta)\leq M\left\{\omega_2\left(f;\sqrt{\delta}\right)+\min\left(1,\delta\right)\left\|f\right\|_{C_B}\right\}, \tag{11}$$

holds for all $\delta>0$, where the constant M is independent of f and δ (see in **[4])**.

The following theorem will be useful in the subsequent result.

Theorem 2.5. *Let $f\in C_B^2[0,\infty)$. Then we have*

$$\left|L_n^*(f;x)-f\left(x\right)\right|\leq\xi\left\|f\right\|_{C_B^2},$$

where

$$\xi:=\xi_n\left(x\right)=\left\{\frac{g'(1)}{g(1)}\frac{b_n}{n}+\frac{1}{2}\frac{b_n}{n}+\frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)}+\frac{g''(1)}{g(1)}+\frac{1}{3}\right)+\frac{b_n}{n}x\right\}.$$

Proof. From the Taylor expansion of f, the linearity of the operators L_n^* and (6), we have

$$L_n^*(f;x)-f\left(x\right)=f'\left(x\right)L_n^*(s-x;x)+\frac{1}{2}f''\left(\eta\right)L_n^*((s-x)^2;x),\eta\in\left(x,s\right). \tag{12}$$

Since

$$L_n^*(s-x;x)=\frac{g'(1)}{g(1)}\frac{b_n}{n}+\frac{1}{2}\frac{b_n}{n}\geq 0$$

for $s\geq x$, by considering Lemma 2 and (12), we can write

$$\begin{aligned}\left|L_n^*(f;x)-f\left(x\right)\right|&\leq\left\{\frac{g'(1)}{g(1)}\frac{b_n}{n}+\frac{1}{2}\frac{b_n}{n}\right\}\left\|f'\right\|_{C_B}+\left\{\frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)}+\frac{g''(1)}{g(1)}+\frac{1}{3}\right)+\frac{b_n}{n}x\right\}\left\|f''\right\|_{C_B}\\&\leq\left\{\frac{g'(1)}{g(1)}\frac{b_n}{n}+\frac{1}{2}\frac{b_n}{n}+\frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)}+\frac{g''(1)}{g(1)}+\frac{1}{3}\right)+\frac{b_n}{n}x\right\}\left\|f\right\|_{C_B^2},\end{aligned}$$

which completes the proof. □

Theorem 2.6. *Let $f\in\widetilde{C_B}[0,\infty)$. Then*

$$\left|L_n^*(f;x)-f\left(x\right)\right|\leq 2M\left\{\omega_2\left(f;\sqrt{\delta}\right)+\min\left(1,\delta\right)\left\|f\right\|_{C_B}\right\},$$

where

$$\delta:=\delta_n\left(x\right)=\frac{1}{2}\xi_n\left(x\right)$$

and $M>0$ is a constant which is independent of the functions f and δ. Also, $\xi_n\left(x\right)$ is the same as in the Theorem 3.

Proof. Suppose that $g\in C_B^2[0,\infty)$. From Theorem 3, we can write

$$\begin{aligned}\left|L_n^*(f;x)-f\left(x\right)\right|&\leq\left|L_n^*(f-g;x)\right|+\left|L_n^*(g;x)-g\left(x\right)\right|+\left|g(x)-f\left(x\right)\right|\\&\leq 2\left\|f-g\right\|_{C_B}+\xi\left\|g\right\|_{C_B^2}\\&=2\left[\left\|f-g\right\|_{C_B}+\delta\left\|g\right\|_{C_B^2}\right]\end{aligned} \tag{13}$$

The left-hand side of inequality (13) does not depend on the function $g\in C_B^2[0,\infty)$, so passing to infimum over $g\in C_B^2[0,\infty)$ we have

$$\left|L_n^*(f;x)-f\left(x\right)\right|\leq 2K(f,\delta)$$

holds where $K\left(f,\delta\right)$ is Peetre's $K-$functional defined by (10). By the relation between Peetre's K functional and the second modulus of smoothness given by (11), we reach to the desired result. □

Now, let us consider the Lipschitz type space with two parameters (see [14]).

$$Lip_M^{(\alpha_1,\alpha_2)}(\alpha) := \left\{ f \in C_B[0,\infty) : \left| f(t) - f(x) \right| \leq M \frac{|t-x|^{\alpha}}{(t+\alpha_1 x^2+\alpha_2 x)^{\frac{\alpha}{2}}} ; x,t \in [0,\infty) \right\}$$

for $\alpha_1, \alpha_2 > 0$, M is a positive constant and $\alpha \in (0,1]$.

Theorem 2.7. *Let* $f \in Lip_M^{(\alpha_1,\alpha_2)}(\alpha)$. *For all* $x > 0$, *we have*

$$\left| L_n^*(f;x) - f(x) \right| \leq M \left(\frac{\frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3} \right) + \frac{b_n}{n} x}{\alpha_1 x^2 + \alpha_2 x} \right)^{\frac{\alpha}{2}}.$$

Proof. Let $\alpha = 1$.

$$\begin{aligned}
&\left| L_n^{\alpha,*}(f;x) - f(x) \right| \\
&\leq \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} \left| f(t) - f(x) \right| dt \\
&\leq M \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |t-x|\, dt \\
&\leq \frac{M}{\sqrt{\alpha_1 x^2 + \alpha_2 x}} L_n^{\alpha,*}(|t-x|;x) \\
&\leq \frac{M}{\sqrt{\alpha_1 x^2 + \alpha_2 x}} \sqrt{L_n^{\alpha,*}((t-x)^2;x)} \\
&= \frac{M}{\sqrt{\alpha_1 x^2 + \alpha_2 x}} \sqrt{\frac{b_n}{n}x + \frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3} \right)}.
\end{aligned}$$

Let $\alpha \in (0,1)$. By applying Hölder inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$

$$\begin{aligned}
\left| L_n^{\alpha,*}(f;x) - f(x) \right| &\leq \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} \left| f(t) - f(x) \right| dt \\
&\leq \left\{ \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \left(\frac{n}{b_n} \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} \left| f(t) - f(x) \right| dt \right)^{\frac{1}{\alpha}} \right\}^{\alpha} \\
&\leq \left\{ \frac{n}{b_n} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} \left| f(t) - f(x) \right|^{\frac{1}{\alpha}} dt \right\}^{\alpha} \\
&\leq M \left\{ \frac{n}{b_n} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} \frac{|t-x|}{\sqrt{t + \alpha_1 x^2 + \alpha_2 x}} dt \right\}^{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{M}{(\alpha_1 x^2+\alpha_2 x)^{\frac{\alpha}{2}}}\left\{\frac{n}{b_n}\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(\frac{n}{b_n}x)\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n}|t-x|\,dt\right\}^{\alpha}\\
&\leq \frac{M}{(\alpha_1 x^2+\alpha_2 x)^{\frac{\alpha}{2}}}\left(L_n^{\alpha,*}(|t-x|\,;x)\right)^{\alpha}\\
&\leq M\left(\frac{L_n^{\alpha,*}((t-x)^2;x)}{\alpha_1 x^2+\alpha_2 x}\right)^{\frac{\alpha}{2}} = M\left(\frac{\frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)}+\frac{g''(1)}{g(1)}+\frac{1}{3}\right)+\frac{b_n}{n}x}{\alpha_1 x^2+\alpha_2 x}\right)^{\frac{\alpha}{2}}.
\end{aligned}
$$

□

3. Approximation properties in weighted spaces

In this section, we study weighted uniform approximation by the sequence $\{L_n^*\}_{n\geq 1}$ with the help of weighted Korovkin type theorem proved by Gadjiev in [5], [6].

Denoting $\mathbb{R}_0^+ = [0,\infty)$ and recall that

$$B_\rho\left(\mathbb{R}_0^+\right) = \left\{f : \mathbb{R}_0^+ \to \mathbb{R} : \left|f(x)\right| \leq M_f\rho(x)\right\},$$

where $\rho(x)$ is a weight function,

$$C_\rho\left(\mathbb{R}_0^+\right) = \left\{f \in B\left(\mathbb{R}_0^+\right) : f \text{ is continuous on } \mathbb{R}_0^+\right\},$$

$$C_\rho^k\left(\mathbb{R}_0^+\right) = \left\{f \in C\left(\mathbb{R}_0^+\right) : \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = K_f < \infty\right\},$$

where K_f is a constant depending on f. It is obvious that $C_\rho^k\left(\mathbb{R}_0^+\right) \subset C_\rho\left(\mathbb{R}_0^+\right) \subset B_\rho\left(\mathbb{R}_0^+\right)$. $B_\rho\left(\mathbb{R}_0^+\right)$ is a linear normed space with the norm

$$\left\|f\right\|_\rho = \sup_{x\in\mathbb{R}_0^+}\frac{\left|f(x)\right|}{\rho(x)}.$$

The following results on the sequence of positive linear operators in these weighted spaces are given by Gadjiev in [5], [6].

Lemma 3.1. *The sequence of positive linear operators $(L_n)_{n\geq 1}$ acts from $C_\rho\left(\mathbb{R}_0^+\right)$ to $B_\rho\left(\mathbb{R}_0^+\right)$ if and only if there exists a positive constant k such that $L_n(\rho;x) \leq k\rho(x)$, i.e. $\left\|L_n(\rho;x)\right\|_\rho \leq k$.*

Theorem 3.2. *Let $\{L_n\}_{n\geq 1}$ be the sequence of positive linear operators acting from $C_\rho\left(\mathbb{R}_0^+\right)$ to $B_\rho\left(\mathbb{R}_0^+\right)$ satisfying the conditions*

$$\lim_{n\to\infty}\|L_n(e_v;x) - e_v(x)\|_\rho = 0,\ v = 0,1,2.$$

Then for any function $f \in C_\rho^k\left(\mathbb{R}_0^+\right)$

$$\lim_{n\to\infty}\left\|L_n\left(f\right) - f\right\|_\rho = 0.$$

In the light of Lemma 3, we have the following result.

Lemma 3.3. *Let $\rho(x) = 1 + x^2$ and $f \in C_\rho\left(\mathbb{R}_0^+\right)$. Then*

$$\left\|L_n^*(\rho;x)\right\|_\rho \leq 1 + M,$$

where $M > 0$ is a constant.

Proof. Using (6) and (7), we have

$$L_n^*(\rho;x) = 1 + x^2 + \frac{b_n}{n}x\left(\frac{2g(1)+2g'(1)}{g(1)}\right) + \frac{2b_n^2}{n^2}\frac{g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{g''(1)}{g(1)} + \frac{b_n^2}{3n^2}$$

$$\begin{aligned}\left\|L_n^*(\rho;x)\right\|_\rho &= \sup_{x\in\mathbb{R}_0^+}\frac{1}{1+x^2}\left[1 + x^2 + \frac{b_n}{n}x\left(\frac{2g(1)+2g'(1)}{g(1)}\right) + \frac{2b_n^2}{n^2}\frac{g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{g''(1)}{g(1)} + \frac{b_n^2}{3n^2}\right] \\ &\leq 1 + \frac{b_n}{n}\left(\frac{2g(1)+2g'(1)}{g(1)}\right) + \frac{2b_n^2}{n^2}\frac{g'(1)}{g(1)} + \frac{b_n^2}{n^2}\frac{g''(1)}{g(1)} + \frac{b_n^2}{3n^2}.\end{aligned}$$

Since $\lim\limits_{n\to\infty}\frac{b_n}{n} = 0$, we have

$$\left\|L_n^*(\rho;x)\right\|_\rho \leq 1 + M.$$

□

Theorem 3.4. *Let $\{L_n^*\}_{n\geq 1}$ be the sequence of linear positive operators defined by (4) and $\rho(x) = 1 + x^2$. Then for each $f \in C_\rho^k\left(\mathbb{R}_0^+\right)$*

$$\lim_{n\to\infty}\left\|L_n^*(f;x) - f(x)\right\|_\rho = 0.$$

Proof. It is enough to show that the conditions of the weighted Korovkin type theorem given by Theorem 6. From (6), we can write

$$\lim_{n\to\infty}\left\|L_n^*(1;x) - 1\right\|_\rho = 0. \tag{14}$$

Using (7), we have

$$\left\|L_n^*(e_1;x) - e_1(x)\right\|_\rho = \frac{b_n}{n}\frac{g'(1)}{g(1)} + \frac{1}{2}\frac{b_n}{n}.$$

This implies that

$$\lim_{n\to\infty}\left\|L_n^*(e_1;x) - e_1(x)\right\|_\rho = 0. \tag{15}$$

From (8),

$$\begin{aligned}\left\|L_n^*(e_2;x) - e_2(x)\right\|_\rho &= \sup_{x\in R_0^+}\frac{1}{1+x^2}\left\{\frac{b_n}{n}x\left(\frac{2g(1)+2g'(1)}{g(1)}\right) + \frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3}\right)\right\} \\ &\leq \frac{b_n}{n}\left(\frac{2g(1)+2g'(1)}{g(1)}\right) + \frac{b_n^2}{n^2}\left(\frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3}\right).\end{aligned}$$

Using the conditions (5), it follows that

$$\lim_{n\to\infty}\left\|L_n^*(e_2;x) - e_2(x)\right\|_\rho = 0. \tag{16}$$

From (14), (15) and (16) for $v = 0, 1, 2$, we have

$$\lim_{n\to\infty}\left\|L_n^*(e_v;x) - e_v(x)\right\|_\rho = 0.$$

If we apply Theorem 6, we obtain desired result. □

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