Filomat 33:8 (2019), 2237–2248
https://doi.org/10.2298/FIL1908237I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Class of Constacyclic Codes over the Ring $\mathbb{Z}_4[u,v]/\langle u^2, v^2, uv-vu\rangle$ and Their Gray Images

Habibul Islam[a], Om Prakash[a,*]

[a]*Department of Mathematics, Indian Institute of Technology Patna, Patna-801 106, India*

Abstract. In this paper, we study $(1+2u+2v)$-constacyclic and skew $(1+2u+2v)$-constacyclic codes over the ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4+uv\mathbb{Z}_4$ where $u^2=v^2=0, uv=vu$. We define some new Gray maps and show that the Gray images of $(1+2u+2v)$-constacyclic and skew $(1+2u+2v)$-constacyclic codes are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$. Further, we determine the structure of $(1+2u+2v)$-constacyclic codes of odd length n.

1. Introduction

In the study of the error-correcting, one of the primary targets is to find the codes with good error-correcting capability. It is well known that the error-correcting capability of the linear code varies proportionally with its minimum distance. Therefore, to obtain good codes, we have to find the codes with a larger minimum distance as much as possible. In this regard, cyclic codes play a crucial role in achieving the goal. The constacyclic code is one of the prominent generalizations of cyclic code, and it can be efficiently implemented by shift constant. Hence, the constacyclic code is the most popular in the area of science and technology. Many times it has been observed that the constacyclic code is an excellent choice instead of a cyclic code to obtain some codes with better parameters. In 2009, Abualrub and Siap [3] studied the constacyclic codes over the ring of 4 elements $\mathbb{F}_2+u\mathbb{F}_2$. In 2011, Karadeniz and Yildiz [14] investigated $(1+u)$-constacyclic codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$ while in 2012, Kai et al. [13] proved that the Gray images of the $(1+u)$-constacyclic codes of length n over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$ are distance preserving binary quasi-cyclic codes of length $4n$ and index 2. Further, they have obtained some optimal binary linear codes as the Gray images of $(1+u)$-constacyclic codes. Later on, in 2014, Yu et al. [22] presented several examples of p-ary linear optimal codes as the Gray images of $(1-uv)$-constacyclic codes over $\mathbb{F}_p+u\mathbb{F}_p+v\mathbb{F}_p+uv\mathbb{F}_p$. Recently, many mathematicians have been studying cyclic and constacyclic codes over the ring $\mathbb{Z}_4$ and its extensions to get some new techniques and optimal codes [1, 4–6, 10, 11, 17, 19–21].
On the other side, in 20107, Boucher et al. [7] introduced the concept of the skew cyclic codes into the coding theory. They characterized the skew cyclic codes of length n as the ideals of the skew polynomial ring $\mathbb{F}[x;\theta]/\langle x^n-1\rangle$, where θ is a non-trivial automorphism on the finite field $\mathbb{F}$. As an application, they

2010 *Mathematics Subject Classification*. Primary 94B15, 94B05, 94B60
Keywords. Constacyclic code, Gray map, Quasi-cyclic code, Skew polynomial ring, Skew constacyclic code.
Received: 19 July 2018; Revised: 20 November 2018; Accepted: 04 December 2018
Communicated by Paola Bonacini
Research supported by the University Grant Commission (UGC), Govt. of India
*Corresponding author: Om Prakash
Email addresses: habibul.pma17@iitp.ac.in (Habibul Islam), om@iitp.ac.in (Om Prakash)

have presented some linear codes which are better than the best known codes. Later, in 2011, Siap et al. [18] studied the skew cyclic codes of arbitrary length n over the finite field $\mathbb{F}$, where these codes are the left $\mathbb{F}[x;\theta]$-submodules of $\mathbb{F}[x;\theta]/\langle x^n-1\rangle$. Recently, many authors have been studied the properties of the skew cyclic and skew constacyclic codes over finite rings, refer [8, 9, 12].

In this paper, we consider the finite commutative ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4+uv\mathbb{Z}_4$ where $u^2=v^2=0, uv=vu$ and study a class of $(1+2u+2v)$-constacyclic codes. The motive of the study is to find some known codes like cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$ as the Gray images of the class of $(1+2u+2v)$-constacyclic codes over $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4+uv\mathbb{Z}_4$.

Presentation of the paper is organized as follows: In Section 2, we discuss some basic concepts. Section 3 includes some new Gray maps and investigates their $\mathbb{Z}_4$-images. The structures of $(1+2u+2v)$-constacyclic codes are obtain in Section 4 while Section 5 contains some results on constacyclic codes with Nechaev's permutation. Section 6 introduces skew $(1+2u+2v)$-constacyclic codes and Section 7 discusses their $\mathbb{Z}_4$-images. Further, in Section 8, we give some results on skew constacyclic codes with Nechaev's permutation and conclude the article in Section 9.

2. Preliminary

Through out the article, R denotes the ring $\mathbb{Z}_4+u\mathbb{Z}_4+v\mathbb{Z}_4+uv\mathbb{Z}_4$ where $u^2=v^2=0, uv=vu$. Note that R is a finite commutative non chain extension of $\mathbb{Z}_4$ and isomorphic to the ring $\mathbb{Z}_4[u,v]/\langle u^2,v^2,uv-vu\rangle$. Also, R is local with unique maximal ideal $\langle 2,u,v\rangle$ and quotient ring $R/\langle 2,u,v\rangle\cong\mathbb{Z}_2$. Recall that a non empty subset C of R^n is said to be a linear code of length n if C is an R-submodule of R^n and elements of C are called codewords. Let λ be a unit in R. A linear code C of length n over R is said to be a λ-constacyclic code if $\tau_\lambda(c)=(\lambda c_{n-1},c_0,\dots,c_{n-2})\in C$ whenever $c=(c_0,c_1,\dots,c_{n-1})\in C$. Note that the λ-constacyclic codes become cyclic if $\lambda=1$ and negacyclic if $\lambda=-1$. The operator τ_λ is known as λ-constacyclic shift operator. Observe that the units $(1+2u),(1+2v),(1+2uv),(1+2u+2v),(1+2u+2uv),(1+2v+2uv),(1+2u+2v+2uv)$ are satisfying $\lambda^n=1$ if n is an even integer and $\lambda^n=\lambda$ if n is an odd integer but in our further calculations, we use the operator τ_λ with $\lambda=(1+2u+2v)$. Similarly, one may obtain the results for any of the above other units represented by λ. Let C be a λ-constacyclic code of length n over R. We identify each codeword $c=(c_0,c_1,\dots,c_{n-1})\in C$ with a polynomial $c(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}$ in $R_{n,\lambda}=R[x]/\langle x^n-\lambda\rangle$ by the correspondence $c=(c_0,c_1,\dots,c_{n-1})\mapsto c(x)=c_0+c_1x+\cdots+c_{n-1}x^{n-1}$ $(mod\ \langle x^n-\lambda\rangle)$. Therefore, the code C can be considered as a subset of R^n as well as of R_n. In polynomial representation, one can easily verify the following result.

Lemma 2.1. *A linear code C of length n over R is λ-constacyclic code if and only if C is an ideal of $R_{n,\lambda}$.*

Definition 2.2. *Let C be a linear code of length $n=st$ over $\mathbb{Z}_4$. We define quasi-cyclic shift operator $\pi_s:\mathbb{Z}_4^n\longrightarrow\mathbb{Z}_4^n$ by*

$$\pi_s(e_0\mid e_1\mid\cdots\mid e_{s-1})=(\sigma(e_0)\mid\sigma(e_1)\mid\cdots\mid\sigma(e_{s-1})),\tag{1}$$

where $e_i\in\mathbb{Z}_4^t$ for all $i=0,1,\dots,(s-1)$ and σ is the cyclic shift operator. Then C is said to be a quasi-cyclic code of index s if C is invariant under the map π_s, i.e. $\pi_s(C)=C$.

3. Gray maps and $\mathbb{Z}_4$-images of λ-constacyclic codes

In this section, we define three distinct Gray maps and discuss the Gray images of λ-constacyclic code of R. Towards this, we first define

$$\phi_1:R\longrightarrow\mathbb{Z}_4^2$$

as

$$\phi_1(a+ub+vc+uvd)=(a+3b+3c+3d,a+b+c+3d),\tag{2}$$

for all $a, b, c, d \in \mathbb{Z}_4$. The map ϕ_1 is linear but not bijective. Also, it can be extended to R^n as follows:

$$\phi_1 : R^n \longrightarrow \mathbb{Z}_4^{2n}$$

defined by

$$\begin{aligned}\phi_1(r_0, r_1, \ldots, r_{n-1}) = (&a_0 + 3b_0 + 3c_0 + 3d_0, a_1 + 3b_1 + 3c_1 + 3d_1, \ldots, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1},\\ &a_0 + b_0 + c_0 + 3d_0, a_1 + b_1 + c_1 + 3d_1, \ldots, a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}),\end{aligned}$$

where $r_i = a_i + ub_i + vc_i + uvd_i \in R$ for all $i = 0, 1 \ldots, (n-1)$. Recall that the Lee weight of any $c \in \mathbb{Z}_4$ is given by $min\{| c |, | 4 - c |\}$, that is, the Lee weights of $0, 1, 2, 3$ are $0, 1, 2, 1$, receptively. The Lee weight for $r \in R$ is define as $w_L(r) = w_L(\phi_1(r))$ and for $\bar{r} = (r_0, r_1, \ldots, r_{n-1}) \in R^n$ is $w_L(\bar{r}) = \sum_{i=0}^{n-1} w_L(r_i)$. Then the Lee distance for the code C is define by $d(C) = min\{d_L(\bar{r_1}, \bar{r_2}) \mid \bar{r_1} \neq \bar{r_2}, \bar{r_1}, \bar{r_2} \in C\}$, where $d_L(\bar{r_1}, \bar{r_2}) = w_L(\bar{r_1} - \bar{r_2})$. Now, $d_L(\bar{r_1}, \bar{r_2}) = w_L(\bar{r_1} - \bar{r_2}) = w_L(\phi_1(\bar{r_1} - \bar{r_2})) = w_L(\phi_1(\bar{r_1}) - \phi_1(\bar{r_2})) = d_L(\phi_1(\bar{r_1}), \phi_1(\bar{r_2}))$, for all $\bar{r_1}, \bar{r_2} \in R^n$. Hence, ϕ_1 is a distance preserving map from R^n (Lee distance) to $\mathbb{Z}_4^{2n}$ (Lee distance).

Lemma 3.1. *Let ϕ_1 be the Gray map defined in equation (2), τ_λ be the λ-constacyclic shift operator and σ be the cyclic shift operator. Then $\phi_1\tau_\lambda = \sigma\phi_1$.*

Proof. Let $r = (r_0, r_1, \cdots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + vc_i + uvd_i$ for $i = 0, 1, \ldots, n-1$. Now,

$$\begin{aligned}\phi_1\tau_\lambda(r) &= \phi_1(\lambda r_{n-1}, r_0, \ldots, r_{n-2})\\ &= (a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}, a_0 + 3b_0 + 3c_0 + 3d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2},\\ &\quad a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}, a_0 + b_0 + c_0 + 3d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}).\end{aligned}$$

On the other side,

$$\begin{aligned}\sigma\phi_1(r) &= \sigma(a_0 + 3b_0 + 3c_0 + 3d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2}, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1},\\ &\quad a_0 + b_0 + c_0 + 3d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}, a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1})\\ &= (a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}, a_0 + 3b_0 + 3c_0 + 3d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2},\\ &\quad a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}, a_0 + b_0 + c_0 + 3d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}).\end{aligned}$$

Hence, $\phi_1\tau_\lambda = \sigma\phi_1$. □

Theorem 3.2. *Let C be a λ-constacyclic code of length n over R. Then $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Let C be a λ-constacyclic code of length n over R. Then $\tau_\lambda(C) = C$. By Lemma 3.1, we have $\phi_1\tau_\lambda(C) = \phi_1(C) = \sigma(\phi_1(C))$. This shows that $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Again, we define a map

$$\phi_2 : R \longrightarrow \mathbb{Z}_4^2$$

by

$$\phi_2(a + ub + vc + uvd) = (a + 2b + 2c + 2d, 2a + 2b + 2c + 2d), \tag{3}$$

for all $a, b, c, d \in \mathbb{Z}_4$.

Lemma 3.3. *Let ϕ_2 be the Gray map defined in equation (3), τ_λ be the λ-constacyclic shift and π_2 be the quasi-cyclic shift operator defined in equation (1). Then $\phi_2\tau_\lambda = \pi_2\phi_2$.*

Proof. Let $r = (r_0, r_1, \cdots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + vc_i + uvd_i$ for $i = 0, 1, \ldots, n-1$. Now,

$$\begin{aligned}\phi_2\tau_\lambda(r) &= \phi_1(\lambda r_{n-1}, r_0, \ldots, r_{n-2})\\ &= (a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1}, a_0 + 2b_0 + 2c_0 + 2d_0, \ldots, a_{n-2} + 2b_{n-2} + 2c_{n-2} + 2d_{n-2},\\ &\quad 2a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1}, 2a_0 + 2b_0 + 2c_0 + 2d_0, \ldots, 2a_{n-2} + 2b_{n-2} + 2c_{n-2} + 2d_{n-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\pi_2\phi_2(r) &= \pi_2(a_0 + 2b_0 + 2c_0 + 2d_0, a_1 + 2b_1 + 2c_1 + 2d_1, \ldots, a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1},\\ &\quad 2a_0 + 2b_0 + 2c_0 + 2d_0, 2a_1 + 2b_1 + 2c_1 + 2d_1, \ldots, 2a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1})\\ &= (a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1}, a_0 + 2b_0 + 2c_0 + 2d_0, \ldots, a_{n-2} + 2b_{n-2} + 2c_{n-2} + 2d_{n-2},\\ &\quad 2a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1}, 2a_0 + 2b_0 + 2c_0 + 2d_0, \ldots, 2a_{n-2} + 2b_{n-2} + 2c_{n-2} + 2d_{n-2}).\end{aligned}$$

Hence, $\phi_2\tau_\lambda = \pi_2\phi_2$. □

Theorem 3.4. *Let C be a λ-constacyclic code of length n over R. Then $\phi_2(C)$ is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Let C be a λ-constacyclic code of length n over R. Then $\tau_\lambda(C) = C$. By Lemma 3.3, we have $\phi_2(\tau_\lambda(C)) = \phi_2(C) = \pi_2(\phi_2(C))$. This implies $\phi_2(C)$ is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$. □

Further, we define a map

$$\phi_3 : R \longrightarrow \mathbb{Z}_4^2$$

by

$$\phi_3(a + ub + vc + uvd) = (a + b + c + d, a + 3b + 3c + d) \tag{4}$$

for all $a, b, c, d \in \mathbb{Z}_4$.

Lemma 3.5. *Let ϕ_3 be the Gray map defined in equation (4), τ_λ be the λ-constacyclic shift and π_2 be the quasi-cyclic shift operator defined in equation (1). Then $\phi_3\tau_\lambda = \xi\pi_2\phi_3$ where ξ is the permutation on $\mathbb{Z}_4^{2n}$ define by $\xi(s_1, s_2, \ldots, s_{2n}) = (s_{\epsilon(1)}, s_{\epsilon(2)}, \ldots, s_{\epsilon(2n)})$ with $\epsilon = (1, n+1)$ of $\{1, 2, \ldots, 2n\}$.*

Proof. Let $r = (r_0, r_1, \cdots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + vc_i + uvd_i$ for $i = 0, 1, \ldots, n-1$. Now,

$$\begin{aligned}\phi_3\tau_\lambda(r) &= \phi_1(\lambda r_{n-1}, r_0, \ldots, r_{n-2})\\ &= (a_{n-1} + 3b_{n-1} + 3c_{n-1} + d_{n-1}, a_0 + b_0 + c_0 + d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2},\\ &\quad a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, a_0 + 3b_0 + 3c_0 + d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + d_{n-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\pi_2\phi_3(r) &= \pi_2(a_0 + b_0 + c_0 + d_0, a_1 + b_1 + c_1 + d_1, \ldots, a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1},\\ &\quad a_0 + 3b_0 + 3c_0 + d_0, a_1 + 3b_1 + 3c_1 + d_1, \ldots, a_{n-1} + 3b_{n-1} + 3c_{n-1} + d_{n-1})\\ &= (a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, a_0 + b_0 + c_0 + d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2},\\ &\quad a_{n-1} + 3b_{n-1} + 3c_{n-1} + d_{n-1}, a_0 + 3b_0 + 3c_0 + d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + d_{n-2}).\end{aligned}$$

Now, applying the permutation ξ on the both sides, we have $\phi_3\tau_\lambda = \xi\pi_2\phi_3$. □

Theorem 3.6. *Let C be a λ-constacyclic code of length n over R. Then $\phi_3(C)$ is a permutation equivalent to a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Let C be a λ-constacyclic code of length n over R. Then $\tau_\lambda(C) = C$. By Lemma 3.5, we have $\phi_3\tau_\lambda(C) = \phi_3(C) = \xi\pi_2(\phi_3(C))$. This shows that $\phi_3(C)$ is permutation equivalent to a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$. □

We denote the permutation version of the map ϕ_1 by $(\phi_1)_\pi$ and define as follows:

$$\begin{aligned}(\phi_1)_\pi(r) &= (\phi_1)_\pi(r_0, r_1, \dots, r_{n-1})\\ &= (a_0 + 3b_0 + 3c_0 + 3d_0, a_0 + b_0 + c_0 + 3d_0, \dots, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1},\\ &\quad a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}),\end{aligned} \tag{5}$$

where $r_i = a_i + ub_i + vc_i + uvd_i \in R$ for all $i = 0, 1, \dots, (n-1)$. We know that the codes obtained by ϕ_1 and $(\phi_1)_\pi$ are permutation equivalent and have the same parameters.

Lemma 3.7. *Let $(\phi_1)_\pi$ be the map defined in equation (5) and σ be the cyclic shift operator. Then $(\phi_1)_\pi\sigma = \sigma^2(\phi_1)_\pi$.*

Proof. Let $r = (r_0, r_1, \dots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + vc_i + uvd_i \in R$ for all $i = 0, 1, \dots, n-1$. Now,

$$\begin{aligned}(\phi_1)_\pi\sigma(r) &= (\phi_1)_\pi(r_{n-1}, r_0, \dots, r_{n-2})\\ &= (a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}, a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}, \dots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2},\\ &\quad a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}).\end{aligned}$$

On the other side,

$$\begin{aligned}\sigma^2(\phi_1)_\pi(r) &= \sigma^2(a_0 + 3b_0 + 3c_0 + 3d_0, a_0 + b_0 + c_0 + 3d_0, \dots, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1},\\ &\quad a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1})\\ &= (a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}, a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}, \dots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2},\\ &\quad a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}).\end{aligned}$$

Hence, $(\phi_1)_\pi\sigma = \sigma^2(\phi_1)_\pi$. □

Theorem 3.8. *Let C be a cyclic code of length n over R. Then $\phi_1(C)$ is permutation equivalent to a 2-quasicyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Let C be a cyclic code of length n over R. Then $\sigma(C) = C$, and therefore by Lemma 3.7, $(\phi_1)_\pi\sigma(C) = (\phi_1)_\pi(C) = \sigma^2((\phi_1)_\pi(C))$. This shows that $\phi_1(C)$ is permutation equivalent to a 2-quasicyclic code of length $2n$ over $\mathbb{Z}_4$. □

Remark 3.9. *Note that to get the Theorem 3.8, we have used the permutation version of the Gray map ϕ_1. Analogously, we can use the permutation version of Gray maps ϕ_2 and ϕ_3 defined in equation (3) and (4), respectively for similar results.*

4. Structures of λ-constacyclic codes over R

Here every element $r = a + ub + vc + uvd \in R$ can be written as $r = (a + ub) + v(c + ud) = s + vt$ where $s = (a + ub), t = (c + ud) \in \mathbb{Z}_4 + u\mathbb{Z}_4$. Now, we define the map $\Psi_1 : R \longrightarrow \mathbb{Z}_4 + u\mathbb{Z}_4$ by $\Psi_1(s + vt) = s$ mod v. The map Ψ_1 is a ring homomorphism and can be extended to the polynomial ring $R[x]/\langle x^n - 1\rangle$ as $\Psi_1(\sum_{i=0}^{n-1} c_i x^i) = \sum_{i=0}^{n-1} \Psi_1(c_i)x^i$. Let C be a cyclic code of length n over R. Consider the restriction of Ψ_1 to the ideal C. Then $ker(\Psi_1) = vI$, where I is an ideal of $(\mathbb{Z}_4 + u\mathbb{Z}_4)[x]/\langle x^n - 1\rangle$. Further, $\Psi_1(C)$ is an ideal of $(\mathbb{Z}_4 + u\mathbb{Z}_4)[x]/\langle x^n - 1\rangle$. Let $\mathcal{D}$ be an ideal of $(\mathbb{Z}_4 + u\mathbb{Z}_4)[x]/\langle x^n - 1\rangle$. Again, we define the map $\Psi_2 : \mathbb{Z}_4 + u\mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$ by $\Psi_2(a + ub) = a$ mod u. The map Ψ_2 is also a ring homomorphism and can be extended to the polynomial ring $(\mathbb{Z}_4 + u\mathbb{Z}_4)[x]/\langle x^n - 1\rangle$ as previously. Moreover, $ker(\Psi_2) = uJ$, where J is an ideal of $\mathbb{Z}_4[x]/\langle x^n - 1\rangle$ and also $\Psi_2(\mathcal{D})$ is an ideal of $\mathbb{Z}_4[x]/\langle x^n - 1\rangle$. Therefore, for an odd integer n, by Theorem 1 of [1], $\Psi_2(\mathcal{D}) = \langle g_1 + 2a_1\rangle$ where $g_1, a_1 \in \mathbb{Z}_4[x]$ such that $a_1 \mid g_1 \mid (x^n - 1)$ mod 4. Thus, $\mathcal{D} = \langle g_1 + 2a_1 + up_1, u(g_2 + 2a_2)\rangle$ where $g_i, a_i \in \mathbb{Z}_4[x]$ such that $a_i \mid g_i \mid (x^n - 1)$ mod 4 for $i = 1, 2$. Hence, by the above discussion, we can characterize the cyclic codes of odd length over R as follows.

Theorem 4.1. *Let n be an odd integer and C be a cyclic code of length n over R. Then C is an ideal of $R[x]/\langle x^n - 1\rangle$ given by*

$$C = \langle g_1 + 2a_1 + up_1 + vp_2 + uvp_3, u(g_2 + 2a_2) + vp_4 + uvp_5, v(g_3 + 2a_3) + uvp_6, uv(g_4 + 2a_4)\rangle,$$

where $a_i, g_i, p_i \in \mathbb{Z}_4[x]$ such that $a_i \mid g_i \mid (x^n - 1) \bmod 4$ for $i = 1, 2, 3, 4$.

Theorem 4.2. *Let n be an odd integer. Then the map $\alpha : R[x]/\langle x^n - 1\rangle \longrightarrow R[x]/\langle x^n - \lambda\rangle$ define by $\alpha(f(x)) = f(\lambda x)$ is a ring homomorphism.*

Proof. Let $s_1(x) = s_2(x) \bmod \langle x^n - 1\rangle$. Then $s_1(x) - s_2(x) = h(x)(x^n - 1) \iff s_1(\lambda x) - s_2(\lambda x) = h(\lambda x)(\lambda^n x^n - 1) \iff s_1(\lambda x) - s_2(\lambda x) = \lambda h(\lambda x)(x^n - \lambda) \iff s_1(\lambda x) = s_2(\lambda x) \bmod \langle x^n - \lambda\rangle$. This shows that α is well define and one-to-one. Rest part is easy to verify. □

Corollary 4.3. *A linear code C of odd length n is cyclic over R if and only if $\alpha(C)$ is a λ-constacyclic code over R.*

Proof. Simple consequence of Theorem 4.2. □

Corollary 4.4. *Let ω be a map define by $\omega : R^n \longrightarrow R^n$ as $\omega(c_0, c_1, \ldots, c_{n-1}) = (c_0, \lambda c_1, \ldots, \lambda^{n-1} c_{n-1})$. Then a linear code C of odd length n over R is cyclic if and only if $\omega(C)$ is λ-constacyclic code of length n over R.*

Proof. Simple consequence of Theorem 4.2. □

Theorem 4.5. *Let C be a λ-constacyclic code of odd length n over R. Then C is an ideal of $R[x]/\langle x^n - \lambda\rangle$ given by*

$$\begin{aligned} C = \langle & g_1(y) + 2a_1(y) + up_1(y) + vp_2(y) + uvp_3(y), u(g_2(y) + 2a_2(y)) + vp_4(y) + uvp_5(y), \\ & v(g_3(y) + 2a_3(y)) + uvp_6(y), uv(g_4(y) + 2a_4(y))\rangle, \end{aligned}$$

where $a_i(x), g_i(x), p_i(x) \in \mathbb{Z}_4[x]$ such that $a_i(x) \mid g_i(x) \mid (x^n - 1) \bmod 4$ for $i = 1, 2, 3, 4$ and $y = \lambda x$.

Proof. Follows from Theorem 4.1 and Corollary 4.4. □

Theorem 4.6. *Let C be a λ-constacyclic code of length n over R given by $C = \langle a(x) + ub(x) + vc(x) + uvd(x)\rangle$, where $a(x), b(x), c(x), d(x) \in \mathbb{Z}_4[x]$ with degree less than n. Then $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$ generated by the polynomials $[a(x) + 3b(x) + 3c(x) + 3d(x)] + x^n[a(x) + b(x) + c(x) + 3d(x)]$, $[3a(x) + 3c(x)] + x^n[a(x) + 3c(x)]$, $[3a(x) + 3b(x)] + x^n[a(x) + 3b(x)]$ and $[3a(x)] + x^n[3a(x)]$.*

Proof. First, we define the polynomial version of Gray map ϕ_1 of equation (2) as

$$\phi_1 : R[x]/\langle x^n - \lambda\rangle \longrightarrow \mathbb{Z}_4[x]/\langle x^n - 1\rangle \times \mathbb{Z}_4[x]/\langle x^n - 1\rangle$$

$$\phi_1[a(x) + ub(x) + vc(x) + uvd(x)] = [a(x) + 3b(x) + 3c(x) + 3d(x), a(x) + b(x) + c(x) + 3d(x)].$$

Here, for $r_i \in \mathbb{Z}_4[x]$, we have

$$\begin{aligned} & \phi_1[(r_1 + ur_2 + vr_3 + uvr_4)(a + ub + vc + uvd)] \\ & = r_1(a + 3b + 3c + 3d, a + b + c + 3d) + r_2(3a + 3c, a + 3c) + r_3(3a + 3b, a + 3b) + r_4(3a, 3a). \end{aligned}$$

Result follows from the fact that the vector $(a, b) \in \mathbb{Z}_4[x]/\langle x^n - 1\rangle \times \mathbb{Z}_4[x]/\langle x^n - 1\rangle$ represents the vector $(a + bx^n)$ in $\mathbb{Z}_4[x]/\langle x^{2n} - 1\rangle$. □

Theorem 4.7. *Let C be a λ-constacyclic code of length n over R given by $C = \langle a(x) + ub(x) + vc(x) + uvd(x)\rangle$, where $a(x), b(x), c(x), d(x) \in \mathbb{Z}_4[x]$ with degree less than n. Then $\phi_2(C)$ is a quasi-cyclic code of length $2n$ over $\mathbb{Z}_4$ generated by the polynomials $[a(x) + b(x) + c(x) + d(x)] + x^n[a(x) + 3b(x) + 3c(x) + d(x)]$, $[a(x) + c(x)] + x^n[3a(x) + c(x)]$, $[a(x) + b(x)] + x^n[3a(x) + b(x)]$ and $[a(x)] + x^n[a(x)]$.*

Proof. Same as the proof of Theorem 4.6. □

Theorem 4.8. *Let C be a λ-constacyclic code of length n over R given by $C = \langle a(x) + ub(x) + vc(x) + uvd(x)\rangle$, where $a(x), b(x), c(x), d(x) \in \mathbb{Z}_4[x]$ with degree less than n. Then $\phi_3(C)$ is a permutation equivalent to quasi-cyclic code of length $2n$ over $\mathbb{Z}_4$ generated by the polynomials $[a(x) + 2b(x) + 2c(x) + 2d(x)] + x^n[2a(x) + 2b(x) + 2c(x) + 2d(x)]$, $2[a(x) + c(x)][1 + x^n]$, $2[a(x) + b(x)][1 + x^n]$ and $2a(x)[1 + x^n]$.*

Proof. Same as the proof of Theorem 4.6. □

Example 4.9. *In reference to the Theorem 4.5, let $n = 9$, $g_1(x) = (x^3 + 3), g_2(x) = g_3(x) = (x^2 + x + 1), g_4(x) = (x + 3), a_1(x) = (x + 3), a_2(x) = a_3(x) = a_4(x) = 1$ and $p_i(x) = 0$ for $i = 1, 2, \dots, 6$. Then $C = \langle\lambda(x^3 + 2x + 1), u(x^2 + \lambda x + 3\lambda), v(x^2 + \lambda x + 3\lambda), uv\lambda(x + 1)\rangle$ is a λ-constacyclic code of length 9 over R. Further, $\phi_1(C)$ is a linear code with parameter $[18, 4^9 2^7, 2]$.*

5. Constacyclic codes with Nechaev's permutation

Definition 5.1. *Let n be an odd positive integer and $\epsilon = (1, n+1)(3, n+3)\dots,(2i+1, n+2i+1)\dots,(n-2, 2n-2)$ of $\{0, 1, 2, \dots, 2n-1\}$. The Nachaev's permutation π is defined by $\pi(c_0, c_1, \dots, c_{2n-1}) = (c_{\epsilon(0)}, c_{\epsilon(1)}, \dots, c_{\epsilon(2n-1)})$.*

Lemma 5.2. *Let ϕ_1 be the Gray map defined in equation (2), π be the Nechaev's permutation and ω be the map defined in Corollary 4.4. Then $\phi_1\omega = \pi\phi_1$.*

Proof. Let $r = (r_0, r_1, \dots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + vc_i + uvd_i$ for $i = 0, 1, \dots, n-1$. Then

$$\begin{aligned}\phi_1\omega(r) &= \phi_1(r_0, \lambda r_1, r_2, \dots, \lambda r_{n-2}, r_{n-1})\\ &= (a_0 + 3b_0 + 3c_0 + 3d_0, a_1 + b_1 + c_1 + 3d_1, a_2 + 3b_2 + 3c_2 + 3d_2, \dots, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\pi\phi_1(r) = \pi(&a_0 + 3b_0 + 3c_0 + 3d_0, a_1 + 3b_1 + 3c_1 + 3d_1, \dots, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1},\\ &a_0 + b_0 + c_0 + 3d_0, a_1 + b_1 + c_1 + 3d_1, \dots, a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1})\\ = (&a_0 + 3b_0 + 3c_0 + 3d_0, a_1 + b_1 + c_1 + 3d_1, a_2 + 3b_2 + 3c_2 + 3d_2, \dots, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}).\end{aligned}$$

Hence, $\phi_1\omega = \pi\phi_1$. □

Theorem 5.3. *Let C be a cyclic code of odd length n over R and $G = \phi_1(C)$. Then $\pi(G)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code, by Corollary 4.4, $\omega(C)$ is a λ-constacyclic code of length n over R. Further, by Theorem 3.2, $\phi_1(\omega(C))$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. Also, by Lemma 5.2, we have $\phi_1(\omega(C)) = \pi\phi_1(C) = \pi(G)$. Hence, $\pi(G)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Lemma 5.4. *Let ϕ_2 be the Gray map defined in equation (3), π be the Nechaev's permutation and ω be the map defined in Corollary 4.4. Then $\phi_2\omega = \pi\phi_2$.*

Proof. Same as the proof of Lemma 5.2. □

Theorem 5.5. *Let C be a cyclic code of odd length n over R and $G = \phi_2(C)$. Then $\pi(G)$ is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Same procedure follows as the proof of Theorem 5.3. Here, we use the Corollary 4.4, Lemma 5.4 and Theorem 3.4. □

Lemma 5.6. *Let ϕ_3 be the Gray map defined in equation (4), π be the Nechaev's permutation and ω be the map defined in Corollary 4.4. Then $\phi_3\omega = \pi\phi_3$.*

Proof. Same as the proof of Lemma 5.2. □

Theorem 5.7. *Let C be a cyclic code of odd length n over R and $G = \phi_2(C)$. Then $\pi(G)$ is permutation equivalent to a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Same procedure follows as the proof of Theorem 5.3. Here, we use the Corollary 4.4, Lemma 5.6 and Theorem 3.6. □

6. Skew λ-constacyclic codes

In order to start discussion of the skew λ-constacyclic codes over R, first we define an automorphism θ on R by $\theta(u) = v$, $\theta(v) = u$, $\theta(a) = a$ for all $a \in \mathbb{Z}_4$. It is clear that θ is an automorphism on R of order 2. It is easy to check that $R[x;\theta] = \{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in R\ \forall\ i = 1,2,\ldots,n\}$ is a skew polynomial ring under usual addition of polynomials and multiplication of polynomials, denoted by $*$, is define with respect to $(ax^s) * (bx^t) = a\theta^s(b)x^{s+t}$. Clearly $R[x;\theta]$ is a non-commutative ring and $\langle x^n - \lambda\rangle$ is a two sided ideal in $R[x;\theta]$ if n is an even integer. Therefore, for any even integer n, $R_{n,\lambda,\theta} = R[x;\theta]/\langle x^n - \lambda\rangle$ is a non-commutative ring. Moreover, $R_{n,\lambda,\theta}$ is a left $R[x;\theta]$-module with respect to the left multiplication define by $r(x)(g(x) + \langle x^n - \lambda\rangle) = r(x) * g(x) + \langle x^n - \lambda\rangle$, where $r(x), g(x) \in R[x;\theta]$. We identify each vector $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$ with a polynomial $r(x)$ in $R_{n,\lambda,\theta}$ by the following correspondence

$$r = (r_0, r_1, \ldots, r_{n-1}) \longmapsto r_0 + r_1x + \cdots + r_{n-1}x^{n-1}\ (mod\ \langle x^n - \lambda\rangle) = r(x).$$

Definition 6.1. *A non-empty subset C of R^n is said to be a skew λ-constacyclic code if*

1. *C is an R-submodule of R^n, and*
2. *for any $c = (c_0, c_1, \ldots, c_{n-1}) \in C$,*
   $\tau_{\lambda,\theta}(c) = (\theta(\lambda c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C$.

Theorem 6.2. *Let C be a linear code of length n over R. Then C is a skew λ-constacyclic code if and only if C is a left $R[x;\theta]$-submodule of $R_{n,\lambda,\theta}$.*

Proof. Straightforward. □

Theorem 6.3. *Let n be an odd integer. Then the map $\beta : R[x;\theta]/\langle x^n - 1\rangle \longrightarrow R[x;\theta]/\langle x^n - \lambda\rangle$ define by $\beta(f(x)) = f(\lambda x)$ is a module homomorphism.*

Proof. Same as the proof of Theorem 4.2. □

Corollary 6.4. *A linear code C of odd length n over R is skew cyclic if and only if $\beta(C)$ is a skew λ-constacyclic code of length n over R.*

Corollary 6.5. *Let ω be the map as defined in Corollary 4.4. Then C is a skew cyclic code of odd length n over R if and only if $\omega(C)$ is a skew λ-constacyclic code of length n over R.*

Theorem 6.6. *Let C be a skew λ-constacyclic code of odd length n. Then C is a λ-constacyclic code of length n over R.*

Proof. Same as the proof of Theorem 22 of [9]. □

Theorem 6.7. *Let C be a skew λ-constacyclic code of even length n over R. Then C is a λ-quasi-twisted code of index 2 over R.*

Proof. Same as the proof of Theorem 23 of [9]. □

7. $\mathbb{Z}_4$-images of skew λ-constacyclic codes

Lemma 7.1. *Let $\tau_{\lambda,\theta}$ be the skew λ-constacyclic shift, ϕ_1 be the Gray map defined in equation (2) and σ be the cyclic shift operator. Then $\phi_1\tau_{\lambda,\theta} = \sigma\phi_1$.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a + ub_i + vc_i + uvd_i$ for $i = 0, 1, \ldots, (n-1)$. Now, we have $\lambda r_{n-1} = (1+2u+2v)(a_{n-1}+ub_{n-1}+vc_{n-1}+uvd_{n-1}) = a_{n-1}+(2a_{n-1}+b_{n-1})u+(2a_{n-1}+c_{n-1})v+(2b_{n-1}+2c_{n-1}+d_{n-1})uv$ and hence, $\theta(\lambda r_{n-1}) = a_{n-1} + (2a_{n-1} + c_{n-1})u + (2a_{n-1} + b_{n-1})v + (2b_{n-1} + 2c_{n-1} + d_{n-1})uv$. Therefore,

$$\begin{aligned}\phi_1\tau_{\lambda,\theta}(r) &= \phi_1(\theta(\lambda r_{n-1}), \theta(r_0), \ldots, \theta(r_{n-2}))\\ &= (a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}, a_0 + 3b_0 + 3c_0 + 3d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2},\\ &\quad a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}, a_0 + b_0 + c_0 + 3d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma\phi_1(r) &= \sigma(a_0 + 3b_0 + 3c_0 + 3d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2}, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1},\\ &\quad a_0 + b_0 + c_0 + 3d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}, a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1})\\ &= (a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}, a_0 + 3b_0 + 3c_0 + 3d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2},\\ &\quad a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}, a_0 + b_0 + c_0 + 3d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}).\end{aligned}$$

Thus, $\phi_1\tau_{\lambda,\theta} = \sigma\phi_1$. □

Theorem 7.2. *Let C be a skew λ-constacyclic code of length n over R. Then $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Let C be a skew λ-constacyclic code of length n over R. Then $\tau_{\lambda,\theta}(C) = C$ and hence by Lemma 7.1, we have $\phi_1\tau_{\lambda,\theta}(C) = \phi_1(C) = \sigma(\phi_1(C))$. This shows that $\phi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Lemma 7.3. *Let $\tau_{\lambda,\theta}$ be the skew λ-constacyclic shift, ϕ_2 be the Gray map defined in equation (3) and π_2 be the quasi-cyclic shift operator defined in equation (1). Then $\phi_2\tau_{\lambda,\theta} = \pi_2\phi_2$.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a + ub_i + vc_i + uvd_i$ for $i = 0, 1, \ldots, n-1$. Now,

$$\begin{aligned}\phi_2\tau_{\lambda,\theta}(r) &= \phi_2(\theta(\lambda r_{n-1}), \theta(r_0), \ldots, \theta(r_{n-2}))\\ &= (a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1}, a_0 + 2b_0 + 2c_0 + 2d_0, \ldots, a_{n-2} + 2b_{n-2} + 2c_{n-2} + 2d_{n-2},\\ &\quad 2a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1}, 2a_0 + 2b_0 + 2c_0 + 2d_0, \ldots, 2a_{n-2} + 2b_{n-2} + 2c_{n-2} + 2d_{n-2}).\end{aligned}$$

Also, from the proof of Lemma 3.3, we have

$$\begin{aligned}\pi_2\phi_2(r) &= (a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1}, a_0 + 2b_0 + 2c_0 + 2d_0, \ldots, a_{n-2} + 2b_{n-2} + 2c_{n-2} + 2d_{n-2},\\ &\quad 2a_{n-1} + 2b_{n-1} + 2c_{n-1} + 2d_{n-1}, 2a_0 + 2b_0 + 2c_0 + 2d_0, \ldots, 2a_{n-2} + 2b_{n-2} + 2c_{n-2} + 2d_{n-2}).\end{aligned}$$

Hence, $\phi_2\tau_{\lambda,\theta} = \pi_2\phi_2$. □

Theorem 7.4. *Let C be a skew λ-constacyclic code of length n over R. Then $\phi_2(C)$ is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Let C be a skew λ-constacyclic code of length n over R. Then $\tau_{\lambda,\theta}(C) = C$. By Lemma 7.3, we have $\phi_2\tau_{\lambda,\theta}(C) = \phi_2(C) = \pi_2(\phi_2(C))$. Hence, $\phi_2(C)$ is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$. □

Lemma 7.5. *Let ϕ_3 be the Gray map defined in equation (4), $\tau_{\lambda,\theta}$ be the skew λ-constacyclic shift and π_2 be the quasi-cyclic shift operator defined in equation (1). Then $\phi_3\tau_{\lambda,\theta} = \xi\pi_2\phi_3$ where ξ is the permutation on $\mathbb{Z}_4^{2n}$ define by $\xi(s_1, s_2, \ldots, s_{2n}) = (s_{\epsilon(1)}, s_{\epsilon(2)}, \ldots, s_{\epsilon(2n)})$ with $\epsilon = (1, n+1)$ of $\{1, 2, \ldots, 2n\}$.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a + ub_i + vc_i + uvd_i$ for $i = 0, 1, \ldots, (n-1)$. Now,

$$\begin{aligned}\phi_3\tau_{\lambda,\theta}(r) &= \phi_3(\theta(\lambda r_{n-1}), \theta(r_0), \ldots, \theta(r_{n-2}))\\ &= (a_{n-1} + 3b_{n-1} + 3c_{n-1} + d_{n-1}, a_0 + b_0 + c_0 + d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2},\\ &\quad a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, a_0 + 3b_0 + 3c_0 + d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + d_{n-2}).\end{aligned}$$

On the other hand, from the proof of Lemma 3.3, we have

$$\begin{aligned}\pi_2\phi_3(r) = (&a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}, a_0 + b_0 + c_0 + d_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2} + d_{n-2},\\ &a_{n-1} + 3b_{n-1} + 3c_{n-1} + d_{n-1}, a_0 + 3b_0 + 3c_0 + d_0, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + d_{n-2}).\end{aligned}$$

Hence, by applying the permutation ξ on the both side, we get $\phi_3\tau_{\lambda,\theta} = \xi\pi_2\phi_3$. □

Theorem 7.6. *Let C be a skew λ-constacyclic code of length n over R. Then $\phi_3(C)$ is permutation equivalent to a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Let C be a skew λ-constacyclic code of length n over R. Then $\tau_{\lambda,\theta}(C) = C$. Also, by Lemma 7.5, we have $\phi_3\tau_{\lambda,\theta}(C) = \phi_3(C) = \xi\pi_2(\phi_3(C))$. This shows that $\phi_3(C)$ is permutation equivalent to a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$. □

Lemma 7.7. *Let $(\phi_1)_\pi$ be the map defined in equation (5) and ρ be a skew cyclic shift operator. Then $(\phi_1)_\pi\rho = \rho^2(\phi_1)_\pi$.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + vc_i + uvd_i \in R$ for all $i = 0, 1, \ldots, n-1$. Now,

$$\begin{aligned}(\phi_1)_\pi\rho(r) &= (\phi_1)_\pi(\theta(r_{n-1}), \theta(r_0), \ldots, \theta(r_{n-2}))\\ &= (a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}, a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2},\\ &\quad a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}).\end{aligned}$$

On the other side,

$$\begin{aligned}\rho^2(\phi_1)_\pi(r) &= \rho^2(a_0 + 3b_0 + 3c_0 + 3d_0, a_0 + b_0 + c_0 + 3d_0, \ldots, a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1},\\ &\quad a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1})\\ &= (\theta^2(a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}), \theta^2(a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}), \theta^2(a_0 + 3b_0 + 3c_0 + 3d_0),\\ &\quad \theta^2(a_0 + b_0 + c_0 + 3d_0), \ldots, \theta^2(a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2}), \theta^2(a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2})).\end{aligned}$$

Since, the order of the automorphism is 2, so $\theta^2(a) = a$ for all $a \in R$. Therefore, we have

$$\begin{aligned}\rho^2(\phi_1)_\pi(r) = (&a_{n-1} + 3b_{n-1} + 3c_{n-1} + 3d_{n-1}, a_{n-1} + b_{n-1} + c_{n-1} + 3d_{n-1}, \ldots, a_{n-2} + 3b_{n-2} + 3c_{n-2} + 3d_{n-2},\\ &a_{n-2} + b_{n-2} + c_{n-2} + 3d_{n-2}).\end{aligned}$$

Hence, $(\phi_1)_\pi\rho = \rho^2(\phi_1)_\pi$. □

Theorem 7.8. *Let C be a skew cyclic code of length n over R. Then $\phi_1(C)$ is permutation equivalent to a skew 2-quasicyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Let C be a skew cyclic code of length n over R. Then $\rho(C) = C$. By Lemma 7.7, we have $(\phi_1)_\pi\rho(C) = (\phi_1)_\pi(C) = \rho^2((\phi_1)_\pi(C))$. This implies that $\phi_1(C)$ is permutation equivalent to a skew 2-quasicyclic code of length $2n$ over $\mathbb{Z}_4$. □

Remark 7.9. *It is noted that if we use the permutation version of Gray maps ϕ_1 and ϕ_2, then the similar results as of Theorem 7.8 can be obtained.*

8. Skew constacyclic codes with Nechaev's permutation

Based on the results of the section 5, we have the following results for the skew constacyclic codes together with the Nechaev's permutation.

Theorem 8.1. *Let C be a skew cyclic code of odd length n over R and $G = \phi_1(C)$. Then $\pi(G)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Let C be a skew cyclic code of odd length n over R. Then by Corollary 6.5, $\omega(C)$ is a skew λ-constacyclic code of length n over R. Also, by Theorem 7.2, $\phi_1(\omega(C))$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. Moreover, by Lemma 5.2, we have $\phi_1(\omega(C)) = \pi(\phi_1(G)) = \pi(G)$. Hence the result. □

Theorem 8.2. *Let C be a skew cyclic code of odd length n over R and $G = \phi_2(C)$. Then $\pi(G)$ is a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Procedure is same as the proof of Theorem 8.1. □

Theorem 8.3. *Let C be a skew cyclic code of odd length n over R and $G = \phi_3(C)$. Then $\pi(G)$ is permutation equivalent to a quasi-cyclic code of length $2n$ with index 2 over $\mathbb{Z}_4$.*

Proof. Procedure is same as the proof of Theorem 8.1. □

9. Conclusion

In this paper, we consider the $(1+2u+2v)$-constacyclic and skew $(1+2u+2v)$-constacyclic codes over R. We have obtained cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$ as the Gray images of $(1+2u+2v)$-constacyclic as well as of skew $(1+2u+2v)$-constacyclic codes over R, respectively. We have incorporated an example in Section 4. It is our believe that some better linear codes over $\mathbb{Z}_4$ can be obtained as the Gray images of these class of $(1+2u+2v)$-constacyclic codes over R in future.

Acknowledgement

We are thankful to the University Grant Commission (UGC), Govt. of India for financial support and Indian Institute of Technology Patna for providing research facility. Authors are also thankful to the anonymous referee(s) and the editor for their constructive comments to improve the presentation of the article.

References

[1] T. Abualrub and I. Siap, *Reversible cyclic codes over $\mathbb{Z}_4$*, Australas. J. Comb. 88 (2007) 195-205.
[2] T. Abualrub and I. Siap, *Cyclic codes over the rings $\mathbb{Z}_2 + u\mathbb{Z}_2$ and $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$*, Design Code Cryptogr. 42 (2007) 273-287.
[3] T. Abualrub and I. Siap, *Constacyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$*, J. Franklin Inst. 346(5) (2009) 520-529.
[4] M. Ashraf and G. Mohammad, *$(1+2u)$-constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$*, arXiv:1504.03445v1. (2015) 14 April.
[5] N. Aydin, Y. Cengellenmis and A. Dertli, *On some constacyclic codes over $\mathbb{Z}_4[u]/\langle u^2-1\rangle$, their $\mathbb{Z}_4$ images, and new codes*, Des. Codes Cryptogr. 86 (2018) 1249-1255.
[6] T. Bag, H. Islam, O. Prakash and A. K. Upadhyay, *A study of constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^2-3\rangle$*, Discrete Math. Algorithms Appl. 10(4) (2018) 1850056.
[7] D. Boucher, W. Geiselmann and F. Ulmer, *Skew cyclic codes*, Appl. Algebra Eng. Comm. 18 (2007) 379-389.
[8] J. Gao, F. Ma and F. Fu, *Skew constacyclic codes over the ring $\mathbb{F}_q + u\mathbb{F}_q$*, Appl. Comput. Math. 6(3) (2017) 286-295.
[9] H. Islam and O. Prakash, *Skew cyclic and skew $(\alpha_1 + u\alpha_2 + v\alpha_3 + uv\alpha)$ codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$*, Int. J. Inf. Coding Theory 5(2) (2018) 101-116.
[10] H. Islam and O. Prakash, *A study of cyclic and constacyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$*, Int. J. Inf. Coding Theory 5(2) (2018) 155-168.
[11] H. Islam, T. Bag and O. Prakash, *A class of constacyclic codes over $\mathbb{Z}_4[u]/\langle u^k\rangle$*, J. Appl. Math. Comput. 60(1-2) (2019) 237–251.
[12] S. Jitman, S. Ling and P. Udomkavanich, *Skew constacyclic codes over finite chain ring*, Adv. Math. Commun. 6 (2012) 39-63.
[13] X. S. Kai and S. X. Zhu, *A family of constacyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$*, J. Syst. Sci. Complex. 25(5) (2012) 1032-1040.
[14] S. Karadeniz and B. Yildiz, *$(1+v)$-constacyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$*, J. Franklin Inst. 348 (2011) 2625-2632.
[15] P.K. Kewat, B. Ghosh and S. Pattanayak, *Cyclic codes over the ring $\mathbb{Z}_p[u,v]/\langle u^2, v^2, uv-vu\rangle$*, Finite Fields Appl. 34 (2015) 161-175.

[16] B.R. McDonald, *Finite Rings With Identity*, Marcel Dekker Inc., New York, 1974.
[17] M. Ozen, F.Z. Uzekmek, N. Aydin and N.T. Ozzaim, *Cyclic and some constacyclic codes over the ring* $\mathbb{Z}_4[u]/\langle u^2-1\rangle$, Finite Fields Appl. 38 (2016) 27-39.
[18] I. Siap, T. Abualrub, N. Aydin and P. Seneviratne, *Skew cyclic codes of arbitrary length*, Int. J. Inf. Coding Theory 2 (2011) 10-20.
[19] M. Shi, L. Qian, L. Sok, N. Aydin and P. Sole, *On constacyclic codes over* $\mathbb{Z}_4[u]/\langle u^2-1\rangle$ *and their Gray images*, Finite Fields Appl. 45 (2017) 86-95.
[20] B. Yildiz and N. Aydin, *On cyclic codes over* $\mathbb{Z}_4 + u\mathbb{Z}_4$ *and thier* $\mathbb{Z}_4$*-images*, Int. J. Inf. Coding Theory 2 (2014) 226-237.
[21] H. Yu, Y. Wang and M. Shi, $(1+u)$*-Constacyclic codes over* $\mathbb{Z}_4 + u\mathbb{Z}_4$, Springer Plus, DOI 10.1186/s40064-016-2717-0.
[22] H. Yu, S. Zhu and X. Kai, $(1-uv)$*-constacyclic codes over* $\mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p$, J. Syst. Sci. Complex. 27(5) (2014) 811-816.