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Some Fixed Point Results in a Strong Probabilistic Metric Spaces

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Abstract. In this paper we introduced the concept of strong probabilistic metric spaces (sPM spaces) and we show some of its basic properties. In this frame we present several fixed point results for mappings of contractive type. Our results generalize and unify several fixed point theorems in literature. Finally, we give some possible applications of our results.

1. Introduction

There are several abstract metric spaces ($b-$metric, metric-like, cone metric, fuzzy metric, probabilistic metric see [1]-[21]. Our structure, the structure of strong probabilistic metric spaces is one of them.

In 1942, K. Menger introduced the notion of a probabilistic metric space as a generalization of metric space in which the distance between p and q is replaced by the distribution function $\mathcal{F}_{p,q} \in \Delta^+$. $\mathcal{F}_{p,q}(x)$ can be interpreted as probability that the distance between p and q is less than x. In fact there are several definitions of probabilistic metric space. The following definition was given by Schweizer and Sklar [17]:

Definition 1.1. Menger probabilistic metric space *is an ordered triple* $(S, \mathcal{F}, T)$, *where* S *is a non-empty set and* $\mathcal{F} : S \times S \to \Delta^+$ *(*Δ^+ *is the set of all distribution functions* F, *with* $F(0) = 0$*) and* T *is a* $t-$*norm, such that the following hold*

1. $(\forall p, q \in S)\ \mathcal{F}_{p,q} = \mathcal{F}_{q,p}$;
2. $(\forall p, q \in S)\ (\forall u \in \mathbb{R}^+)\ \mathcal{F}_{p,q}(u) = 1 \Leftrightarrow p = q$;
3. $(\forall p, q, r \in S)(\forall u, v > 0)\ \ T(\mathcal{F}_{p,r}(u), \mathcal{F}_{r,q}(v)) \le \mathcal{F}_{p,q}(u + v)$.

Definition 1.2. *A function* $T : [0, 1]^2 \to [0, 1]$ *is called a* $t-$**norm** *if*

T_1) $(\forall a \in [0, 1])\ T(a, 1) = a$;

T_2) $(\forall a, b \in [0, 1])\ T(a, b) = T(b, a)$;

T_3) $(\forall a, b, c, d \in [0, 1])\ a \ge c \wedge b \ge d\ \Rightarrow\ T(a, b) \ge T(c, d)$;

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T_4) $(\forall a, b, c \in [0,1])\ T(a, T(b,c)) = T(T(a,b), c)$.

Sehgal [18], Sehgal and Bharucha-Reid [19] initiated the study of contraction mappings in probabilistic metric spaces. They introduced the notion of contraction (refer as B-contraction) and proved that contractive mapping on complete Menger probabilistic metric space $(S, \mathcal{F}, T)$, with $t-$norm $T = \min$ has a unique fixed point. Subsequently, Sherwood [20] improved their results and so the development of fixed point theory in probabilistic metric spaces began. Since then many papers in this direction have been published (for references see [4] for example). This article is our contribution to fixed point theory in probabilistic metric spaces.

2. Strong probabilistic metric spaces

Let Δ^+ be the set of all distribution functions $F : \mathbb{R} \to [0,1]$ (nondecreasing, leftcontinuous with $\sup_{u\in\mathbb{R}} F(u) = 1$ such that $F(0) = 0$).

Definition 2.1. *[17] The ordered pair $(S, \mathcal{F})$, where S is a nonempty set and $\mathcal{F} : S\times S \to \Delta^+$ is a* **weak probabilistic metric space** *(briefly wPM space) if the following conditions are satisfied:*

F_1) $(\forall p, q \in S)\ \mathcal{F}_{p,q} = \mathcal{F}_{q,p}$;

F_2) $(\forall u \in \mathbb{R}^+)\ \mathcal{F}_{p,q}(u) = 1 \Leftrightarrow p = q$;

F_3) $(\forall p, q, r \in S)(\forall u, v > 0)\ \ \mathcal{F}_{p,r}(u) = 1, \mathcal{F}_{r,q}(v) = 1 \Rightarrow \mathcal{F}_{p,q}(u+v) = 1$.

For $\mathcal{F}_{p,q}(u)$ we often use $\mathcal{F}(p, q, u)$, or $\mathcal{F}(p, q; u)$.

Remark. Obviously every Menger probabilistic metric space is a wPM space.

Now, for every $p \in S$ let us define the set

$$C(S, \mathcal{F}, p) = \{\{p_n\} \in S^{\mathbb{N}}\ :\ \lim_{n\to\infty} \mathcal{F}_{p_n,p}(u) = 1, \forall u > 0\}.$$

Remark. Sets $C(S, \mathcal{F}, p)$, $p \in S$ are nonempty since for $p_n = p$, $n \in \mathbb{N}$, $\{p_n\} \in C(S, \mathcal{F}, p)$.

Definition 2.2. *[5] The ordered pair $(S, \mathcal{F})$, is a* **strong probabilistic metric space** *(sPM space) if $(S, \mathcal{F})$ is a weak probabilistic metric space and $\mathcal{F}$ satisfied condition*

F_4) *there exists $C > 0$ such that*

$$(\forall p, q \in S)\ \{p_n\} \in C(S, \mathcal{F}, p)\ \Rightarrow\ \mathcal{F}_{p,q}(u) \geq \liminf_{n\to\infty} \mathcal{F}_{p_n,q}(\frac{u}{C}),\ \forall u > 0.$$

Example 2.3. *Every Menger probabilistic metric space $(S, \mathcal{F}, T)$ with continuous $t-$norm T is a strong probabilistic metric space.*

We only have to prove that F_4) is satisfied for some $C > 0$. Let $p, q \in S$ and $\{p_n\} \in C(S, \mathcal{F}, p)$. From $\mathcal{F}_{p,q}(u) \geq T(\mathcal{F}_{p,p_n}(\frac{u}{2}), \mathcal{F}_{p_n,q}(\frac{u}{2}))$, for every $n \in \mathbb{N}$, all $u > 0$, and continuity of T it follows that

$$\mathcal{F}_{p,q}(u) \geq \liminf_{n\to\infty} T(\mathcal{F}_{p,p_n}(\frac{u}{2}), \mathcal{F}_{p_n,q}(\frac{u}{2}))$$
$$\geq T(1, \liminf_{n\to\infty} \mathcal{F}_{p_n,q}(\frac{u}{2})) = \liminf_{n\to\infty} \mathcal{F}_{p_n,q}(\frac{u}{2}),$$

for all $u > 0$, so F_4) is satisfied for $C = 2$.

Example 2.4. *Every (Menger) probabilistic $b-$metric space is a strong probabilistic metric space.*

Let us recall that (Menger) probabilistic $b-$metric space [15] is a quadruple $(S, \mathcal{F}^b, T, s)$, where S is nonempty set, $\mathcal{F}^b$ is function from $S \times S$ into Δ^+, T is continuous $t-$norm, $s \geq 1$ is a real number, and the following conditions are satisfied for all $p, q, r \in S$.

F_1^b) $\mathcal{F}_{p,q}^b(u) = 1$, for all $u > 0 \Leftrightarrow p = q$;

F_2^b) $\mathcal{F}_{p,q}^b(u) = \mathcal{F}_{q,p}^b(u)$ for all $u > 0$;

F_3^b) $\mathcal{F}_{p,q}^b(s(u+v)) \geq T(\mathcal{F}_{p,r}^b(u), \mathcal{F}_{r,q}^b(v)$, for all $u, v > 0$.

We have just to prove that $\mathcal{F}^b$ satisfies condition F_4). Let $p \in S$ and $\{p_n\} \in C(S, \mathcal{F}, p)$. For every $q \in S$, by the property F_3^b), we have

$$\mathcal{F}_{p,q}^b(u) \geq T(\mathcal{F}_{p,p_n}^b(\frac{u}{2s}), \mathcal{F}_{p_n,q}^b(\frac{u}{2s})),$$

for every natural number n and all $u > 0$.

Hence, we have

$$\mathcal{F}_{p,q}^b(u) \geq T(1, \liminf_{n\to\infty} \mathcal{F}_{p_n,q}^b(\frac{u}{2s})) = \liminf_{n\to\infty} \mathcal{F}_{p_n,q}^b(\frac{u}{2s}).$$

The property F_4) is then satisfied with $C = 2s$.

Example 2.5. *Let $S = \mathbb{R}_0^+$ and for all $p, q \in S$ the functions*

$$\mathcal{F}_{p,q}^{(1)}(u) = \begin{cases} \frac{\min\{p,q\}+u}{\max\{p,q\}+u}, & u > 0 \\ 0, & u \leq 0 \end{cases},$$

$$\mathcal{F}_{p,q}^{(2)}(u) = \begin{cases} \frac{\sqrt[m]{\frac{p^m+q^m}{2}}+u}{\max\{p,q\}+u}, & u > 0 \\ 0, & u \leq 0 \end{cases} \quad (m > 0),$$

are in Δ^+ and obviously conditions F_1), F_2) and F_3) are satisfied.

Now, let sequence $\{p_n\} \in C(S, \mathcal{F}^{(1)}, p)$. It means that

$$\mathcal{F}_{p_n,p}^{(1)}(u) \to 1,\ n \to \infty,\ \textit{for all } u > 0, \textit{so}$$

$$1 - \mathcal{F}_{p_n,p}^{(1)}(u) = \frac{\max\{p_n, p\} - \min\{p_n, p\}}{\max\{p_n, p\} + u} \to 0,\ n \to \infty,\ \textit{for all } u > 0.$$

But, then $p_n \to p$, $n \to \infty$, in usual topology on $\mathbb{R}$ so for any $q \in S$

$$\lim_{n\to\infty} \mathcal{F}_{p_n,q}^{(1)}(u) = \mathcal{F}_{p,q}^{(1)}(u),\ ,\ \forall u > 0,$$

and F_4) is satisfied for $C = 1$.

We just prove that $(S, \mathcal{F}^{(1)})$ is a strong probabilistic metric space.

Similarly, one can prove that $(S, \mathcal{F})$ for $\mathcal{F}_{p,q}(u) = \mathcal{F}_{p,q}^{(2)}(u)$, $u \in \mathbb{R}$, is a strong probabilistic metric space.

Remark. In general case, in any nontrivial sPM space constant $C \geq 1$.

Definition 2.6. *Let $(S, \mathcal{F})$ be a strong probabilistic metric space. Let $\{p_n\}$ be a sequence in S and $p \in S$. We say that $\{p_n\}$ $\mathcal{F}-$**converges** to p if*

$$\{p_n\} \in C(S, \mathcal{F}, p).$$

Proposition 2.7. *Let $(S, \mathcal{F})$ be a strong probabilistic metric space and $(p, q) \in S^2$. If $\{p_n\}$ $\mathcal{F}-$converges to p and $\{p_n\}$ $\mathcal{F}-$converges to q, then $p = q$.*

Proof. Let $\{p_n\} \in C(S, \mathcal{F}, p)$ and $\{p_n\} \in C(S, \mathcal{F}, q)$. By property F_4)

$$\mathcal{F}_{p,q}(u) \geq \liminf_{n\to\infty} \mathcal{F}_{p_n,q}(\frac{u}{C}) = 1, \ \forall u > 0,$$

i.e. $\mathcal{F}_{p,q}(u) = 1$, for all $u > 0$, which implies, by F_1) property, that $p = q$. □

Definition 2.8. *Let $(S, \mathcal{F})$ be a PM space. We say that $\{p_n\}$ is a $\mathcal{F}-$***Cauchy sequence** *if*

$$\lim_{n,m\to\infty} \mathcal{F}_{p_n,p_m}(u) = 1, \ \forall u > 0.$$

*Also, we say that $(S, \mathcal{F})$ is $\mathcal{F}-$***complete PM space** *if every $\mathcal{F}-$Cauchy sequence in S is $\mathcal{F}-$convergent to some element in S.*

Let $(S, \mathcal{F})$ be a probabilistic metric space and $f : S \to S$. Let for any $p_0 \in S$

$$O(p_0; f) \in \{f^n p_0 : n \in \mathbb{N} \cup \{0\}\},$$

where $f^n = f \circ f \circ ... \circ f$. The set $O(p_0; f)$ is the **orbit** of f at p_0.

Let $\mathcal{D}_{O(p_0;f)} : \mathbb{R} \to [0, 1]$ (the **diameter** of $O(p_0; f)$) be defined by

$$\mathcal{D}_{O(p_0;f)}(u) = \sup_{v<u} \inf_{p,q\in O(p_0;f)} F_{p,q}(v).$$

If $\sup_{u\in\mathbb{R}} \mathcal{D}_{O(p_0;f)}(u) = 1$, we say that the orbit $O(p_0; f)$ is **probabilistic bounded**. Hence, $O(p_0; f)$ is probabilistic bounded if and only if $\mathcal{D}_{O(p_0;f)} \in \Delta^+$, i.e. $\mathcal{D}_{O(p_0;f)}$ is a probability distribution function.

3. Contraction principle in strong probabilistic metric space and some generalizations

V. H. Sehgal introduced the notion of contraction mapping ($B-$contraction) in probabilistic metric space [18].

Definition 3.1. *Let $(S, \mathcal{F})$ be a probabilistic metric space and $f : S \to S$. The mapping f is a $\lambda-$***contraction***, $\lambda \in (0, 1)$, if*

$$\mathcal{F}_{fp_1,fp_2}(u) \geq \mathcal{F}_{p_1,p_2}(\frac{u}{\lambda}),$$

for all $p_1, p_2 \in S$ and $u > 0$.

Now we are going to prove a generalization of fixed point result proved in Menger probabilistic metric space by Sherwood [20].

Theorem 3.2. *Let $(S, \mathcal{F})$ be a $\mathcal{F}-$complete strong probabilistic metric space and $f : S \to S$ a $\lambda-$contraction. If $\mathcal{D}_{O(p_0;f)} \in \Delta^+$, for some $p_0 \in S$, then there exists a unique fixed point p of f and sequence $\{f^n p_0\}$ $\mathcal{F}-$converges to p. Moreover, $\{f^n q\}$ $\mathcal{F}-$converges to p for any $q \in X$.*

Proof. Let $p_n = f^n p_0$, $n \in \mathbb{N}$. We shall prove that $\{p_n\}$ is an $\mathcal{F}-$Cauchy sequence. For $m, n \in \mathbb{N}$, $m > n$,

$$\mathcal{F}_{p_n,p_m}(u) = \mathcal{F}_{f^n p_0, f^m p_0}(u) \geq \mathcal{F}_{f^{n-1}p_0,f^{m-1}p_0}(\frac{u}{\lambda}) \geq ... \geq \mathcal{F}_{p_0,f^{m-n}p_0}(\frac{u}{\lambda^n}) \geq \mathcal{D}_{O(p_0;f)}(\frac{u}{\lambda^n}) \to 1, \ n \to \infty, \ \forall u > 0.$$

Hence, $\{p_n\}$ is $\mathcal{F}-$Cauchy sequence in S, and since S is $\mathcal{F}-$complete, it follows the existence of $p \in S$ such that $\{p_n\}$ $\mathcal{F}-$converges to p. Let us prove that $fp = p$. Since

$$\mathcal{F}_{f^{n+1}p_0,fp}(u) \geq \mathcal{F}_{f^n p_0,p}(\frac{u}{\lambda}) \to 1, \ n \to \infty, \ \forall u > 0,$$

$\{p_n\}$ is $\mathcal{F}-$convergent to fp. Now by Proposition 2.7: $fp = p$. Let us prove that p is a unique fixed point of f If we suppose that $fq = q$, for some $q \in S$, then

$$\mathcal{F}_{p,q}(u) = \mathcal{F}_{fp,fq}(u) \geq \mathcal{F}_{p,q}(\frac{u}{\lambda}) \geq ... \geq \mathcal{F}_{p,q}(\frac{u}{\lambda^n}) \to 1,\ n \to \infty, \forall u > 0,$$

so $\mathcal{F}_{p,q}(u) = 1$, for all $u > 0$, and consequently $q = p$.

Now for any $q \in X$ and all $n \in \mathbb{N}$

$$\mathcal{F}_{f^n q,p}(u) = \mathcal{F}_{f^n q,f^n p}(u) \geq ... \geq \mathcal{F}_{q,p}(\frac{u}{\lambda^n})$$

so $\mathcal{F}_{f^n q,p}(u) \to 1$, $n \to \infty$, for all $u > 0$ which means that $\{f^n q\}$ also $\mathcal{F}-$converges to p. □

Over the years, various extensions and generalizations of contraction principle have appeared in literature. Lj. Ćirić introduced the notion of a quasicontractions as one of the most general contractive type mappings. He proved that quasicontraction on complete metric space possesses a unique fixed point.

We are going to prove fixed point theorem for quasicontraction mappings on sPM spaces.

Definition 3.3. *Let $(S, \mathcal{F})$ be a (strong) probabilistic metric space and $k \in (0, 1)$. A mapping $f : S \to S$ is said to be a probabilistic Ćirić quasicontraction if for every $p, q \in S$ and every $t > 0$, the following is satisfied:*

$$\mathcal{F}_{fp,fq}(kt) \geq \min\{\mathcal{F}_{p,q}(t), \mathcal{F}_{p,fp}(t), \mathcal{F}_{q,fq}(t), \mathcal{F}_{p,fq}(t), \mathcal{F}_{fp,q}(t)\}.$$

Theorem 3.4. *Let $(S, \mathcal{F})$ be a $\mathcal{F}-$complete strong probabilistic metric space and $f : S \to S$ a probabilistic Ćirić quasicontraction for some $k \in (0, 1)$. If there exists $p_0 \in S$ such that $\mathcal{D}_{O(p_0;f)} \in \Delta^+$ and $Ck < 1$, then $\{f^n p_0\}$ $\mathcal{F}-$converges to unique fixed point ω of f on S.*

Proof. Let $n \in \mathbb{N}$. Since f is a probabilistic Ćirić quasicontraction for all $i, j \in \mathbb{N}_0$ and $t > 0$

$$\mathcal{F}_{f^{n+i}p_0,f^{n+j}p_0}(t) \geq \min\{\mathcal{F}_{f^{n+i-1}p_0,f^{n+j-1}p_0}(\frac{t}{k}), \mathcal{F}_{f^{n+i-1}p_0,f^{n+i}p_0}(\frac{t}{k}),$$

$$\mathcal{F}_{f^{n+j-1}p_0,f^{n+j}p_0}(\frac{t}{k}), \mathcal{F}_{f^{n+i-1}p_0,f^{n+j}p_0}(\frac{t}{k}), \mathcal{F}_{f^{n+i}p_0,f^{n+j-1}p_0}(\frac{t}{k})\}$$

which implies that

$$\mathcal{D}_{O(f^n p_0;f)}(t) \geq \mathcal{D}_{O(f^{n-1}p_0;f)}(\frac{t}{k}) \geq ... \geq \mathcal{D}_{O(p_0;f)}(\frac{t}{k^n}), \quad \forall t > 0.$$

Using the above inequality, we have that

$$\mathcal{F}_{f^n p_0,f^{n+m}p_0}(t) \geq \mathcal{D}_{O(f^n p_0;f)}(t) \geq \mathcal{D}_{O(p_0;f)}(\frac{t}{k^n})$$

for every $m, n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \mathcal{F}_{f^n p_0,f^{n+m}p_0}(t) = 1,$$

for all $t > 0$ which means that $\{f^n p_0\}$ is a $\mathcal{F}-$Cauchy sequence in $\mathcal{F}-$complete probabilistic metric space. Let $\{f^n p_0\}$ $\mathcal{F}-$converges to some $\omega \in S$. Moreover, by property F_4)

$$\mathcal{F}_{f^n p_0,\omega}(t) \geq \liminf_{m\to\infty} \mathcal{F}_{f^n p_0,f^{n+m}p_0}(\frac{t}{C}) \geq \mathcal{D}_{O(p_0;f)}(\frac{t}{Ck^n})$$

for all $n \in \mathbb{N}_0$ and $t > 0$. Now

$$\begin{aligned}\mathcal{F}_{fp_0,f\omega}(t) &\geq \min\{\mathcal{F}_{p_0,\omega}(\frac{t}{k}), \mathcal{F}_{p_0,fp_0}(\frac{t}{k}), \mathcal{F}_{\omega,f\omega}(\frac{t}{k}), \mathcal{F}_{p_0,f\omega}(\frac{t}{k}), \mathcal{F}_{fp_0,\omega}(\frac{t}{k})\} \\ &\geq \min\{\mathcal{D}_{O(p_0;f)}(\frac{t}{Ck}), \mathcal{D}_{O(p_0;f)}(\frac{t}{k}), \mathcal{F}_{\omega,f\omega}(\frac{t}{k}), \mathcal{F}_{p_0,f\omega}(\frac{t}{k})\}\end{aligned}$$

$$\mathcal{F}_{f^2p_0,f\omega}(t) \geq \min\{\mathcal{D}_{O(p_0;f)}(\frac{t}{Ck^2}), \mathcal{D}_{O(p_0;f)}(\frac{t}{k^2}), \mathcal{F}_{\omega,f\omega}(\frac{t}{k}), \mathcal{F}_{p_0,f\omega}(\frac{t}{k^2})\}.$$

By induction we get

$$\mathcal{F}_{f^np_0,f\omega}(t) \geq \min\{\mathcal{D}_{O(p_0;f)}(\frac{t}{Ck^n}), \mathcal{D}_{O(p_0;f)}(\frac{t}{k^n}), \mathcal{F}_{\omega,f\omega}(\frac{t}{k}), \mathcal{F}_{p_0,f\omega}(\frac{t}{k^n})\}$$

for each $n \in \mathbb{N}$, so

$$\mathcal{F}_{\omega,f\omega}(t) \geq \liminf_{n\to\infty} \mathcal{F}_{f^np_0,f\omega}(\frac{t}{C}) \geq \mathcal{F}_{\omega,f\omega}(\frac{t}{Ck}) \geq \ldots \geq \mathcal{F}_{\omega,f\omega}(\frac{t}{(Ck)^j}),$$

for all $j \in \mathbb{N}$. Since $Ck < 1$, we have $\mathcal{F}_{\omega,f\omega}(t) = 1$, for all $t > 0$. Therefore $\omega = f\omega$. Suppose that $fu = u$ for some $u \in S$. Then

$$\mathcal{F}_{u,\omega}(t) = \mathcal{F}_{fu,f\omega}(t) \geq \min\{\mathcal{F}_{u,\omega}(\frac{t}{k}), \mathcal{F}_{u,fu}(\frac{t}{k}), \mathcal{F}_{\omega,f\omega}(\frac{t}{k}), \mathcal{F}_{u,f\omega}(\frac{t}{k}), \mathcal{F}_{\omega,fu}(\frac{t}{k})\}$$

$$= \mathcal{F}_{u,\omega}(\frac{t}{k}) \geq \ldots \geq \mathcal{F}_{u,\omega}(\frac{t}{k^m})$$

for every $m \in \mathbb{N}$. It implies that $\mathcal{F}_{u,\omega}(t) = 1$, for all $t > 0$ so $u = \omega$, and ω is an unique fixed point of f on S. □

Remark. It is easy now to prove that if

$$\mathcal{F}_{fp,fq}(kt) \geq \min\{\mathcal{F}_{p,q}(t), \mathcal{F}_{p,fq}(t), \mathcal{F}_{fp,q}(t)\}$$

for all $p, q \in S$ and $t > 0$, the condition $Ck < 1$ in Theorem 3.4 can be relaxed with $k < 1$.

Now, we are going to introduce an interesting subclass of strong PM spaces.

Definition 3.5. *Ordered pair $(S, \mathcal{F})$ is a $m-$***strong probabilistic metric space** *(msPM space) if $\mathcal{F}$ satisfied conditions F_1), F_2), F_3) and*

F_4^*) *there exists $C > 0$ such that for all $p, q \in S$,*

$$\{p_n\} \in C(S, \mathcal{F}, p),\ \{q_n\} \in C(S, \mathcal{F}, q) \ \Rightarrow\ \mathcal{F}_{p,q}(t) \geq \liminf_{n\to\infty} \mathcal{F}_{p_n,q_n}(\frac{t}{C}),$$

for all $t > 0$.

Remark. Since sequence $p_n = p$, for all $n \in \mathbb{N}$, $\mathcal{F}-$converges to p every msPM space is sPM space.

Example 3.6. *Every Menger probabilistic metric space (with continuous $t-$norm T) is msPM space. Namely, for $\{p_n\} \in C(S, \mathcal{F}, p)$ and $\{q_n\} \in C(S, \mathcal{F}, q)$ in Menger probabilistic metric space*

$$\mathcal{F}_{p,q}(t) \geq T(\mathcal{F}_{p,p_n}(\frac{t}{2}), \mathcal{F}_{p_n,q}(\frac{t}{2})) \geq T(\mathcal{F}_{p,p_n}(\frac{t}{2}), T(\mathcal{F}_{p_n,q_n}(\frac{t}{4}), \mathcal{F}_{q_n,q}(\frac{t}{4}))),$$

for all $t > 0$, so

$$\mathcal{F}_{p,q}(t) \geq T(1, T(\liminf_{n\to\infty} \mathcal{F}_{p_n,q_n}(\frac{t}{4}), 1)) = \liminf_{n\to\infty} \mathcal{F}_{p_n,q_n}(\frac{t}{4}),\ t > 0,$$

and for $C = 4$ condition F_4^*) is satisfied.

Example 3.7. *Similarly, one can prove that every (Menger) probabilistic $b-$metric space is a ms PM space.*

Example 3.8. *Spaces from Example 2.5 are $m-$strong probabilistic metric spaces too.*

In this class of spaces we can prove the next generalization of probabilistic contraction principle.

Definition 3.9. *Let $(S, \mathcal{F})$ be a PM space and $f : S \to S$. The f is a* **mapping with contractive iterate at a point** *if for some $\lambda \in (0, 1)$ and every $p \in S$, there exists $n(p) \in \mathbb{N}$ such that for any $q \in S$ and all $t > 0$*

$$\mathcal{F}_{f^{n(p)}p, f^{n(p)}q}(\lambda t) \geq \mathcal{F}_{p,q}(t).$$

Theorem 3.10. *Let $(S, \mathcal{F})$ be a $\mathcal{F}-$complete $m-$strong probabilistic metric space and $f : S \to S$ a mapping with contractive iterate at a point. If for some $p_0 \in S$, $\mathcal{D}_{O(p_0;f)} \in \Delta^+$, then there exists a unique fixed point u of f and $u = \lim\limits_{k\to\infty} f^k p_0$.*

Proof. Let $p_1 = f^{n(p_0)}p_0$, $p_2 = f^{n(p_1)}p_1$, ... $p_{k+1} = f^{n(p_k)}p_k$, $k \in \mathbb{N}$. For all $m, k \in \mathbb{N}$ and $j = n(p_{k+m-1}) + n(p_{k+m-2}) + \cdots + n(p_k)$, hold

$$\mathcal{F}_{p_k,p_{k+m}}(t) = \mathcal{F}_{f^{n(p_{k-1})}p_{k-1},f^{j+n(p_{k-1})}p_{k-1}}(t) \geq \mathcal{F}_{p_{k-1},f^j p_{k-1}}(\frac{t}{\lambda}) \geq \ldots \mathcal{F}_{p_0,f^j p_0}(\frac{t}{\lambda^k}) \geq \mathcal{D}_{O(p_0;f)}(\frac{t}{\lambda^k}) \to 1,\ k \to \infty,$$

for all $t > 0$ so $\{p_k\}$ is $\mathcal{F}-$Cauchy in $\mathcal{F}-$complete $m-$strong probabilistic metric space. Let $\{p_k\}$ $\mathcal{F}-$converges to $u \in S$. We are going to prove that $f^{n(u)}u = u$. From the inequality

$$\mathcal{F}_{f^{n(u)}u,f^{n(u)}p_k}(t) \geq \mathcal{F}_{u,p_k}(\frac{t}{\lambda}) \to 1,\ k \to \infty,$$

for all $t > 0$, it follows that $\{f^{n(u)}p_k\}$ $\mathcal{F}-$converges to $f^{n(u)}u$. On the other side

$$\mathcal{F}_{p_k,f^{n(u)}p_k}(t) \geq \mathcal{F}_{p_{k-1},f^{n(u)}p_{k-1}}(\frac{t}{\lambda}) \geq \ldots \mathcal{F}_{p_0,f^{n(u)}p_0}(\frac{t}{\lambda^k}) \geq ... \geq \mathcal{D}_{O(p_0;f)}(\frac{t}{\lambda^k}) \to 1,\ k \to \infty,$$

for all $t > 0$.

Now, by condition F_4^*):

$$\mathcal{F}_{u,f^{n(u)}u}(t) \geq \liminf_{k\to\infty} \mathcal{F}_{p_k,f^{n(u)}p_k}(\frac{t}{C}) = 1,$$

for all $t > 0$. This implies that $u = f^{n(u)}u$. Let us prove that $f^{n(u)}\omega = \omega$, for some $\omega \in S$, implies $\omega = u$. Since

$$\mathcal{F}_{u,\omega}(t) = \mathcal{F}_{f^{n(u)}u,f^{n(u)}\omega}(t) \geq \mathcal{F}_{u,\omega}(\frac{t}{\lambda}) \geq ... \geq \mathcal{F}_{u,\omega}(\frac{t}{\lambda^k})$$

for all $k \in \mathbb{N}$ and all $t > 0$, we have $\mathcal{F}_{u,\omega}(t) = 1$, for all $t > 0$, so by property F_1) $u = \omega$. Now $fu = ff^{n(u)}u = f^{n(u)}fu$ implies that $fu = u$.

At the end let us prove that $u = \lim\limits_{k\to\infty} f^k p_0$. For any $k \in \mathbb{N}, k \geq n(u)$, there exist $m \in \mathbb{N}$ and $0 \leq r < n(u)$ such that $k = m \cdot n(u) + r$. Then, we have that

$$\mathcal{F}_{f^k p_0,u}(t) = \mathcal{F}_{f^{m\cdot n(u)+r}p_0,f^{n(u)}u}(t) \geq ... \geq \mathcal{F}_{f^r p_0,u}(\frac{t}{\lambda^m}),$$

for all $t > 0$. If $k \to \infty$, then $m \to \infty$ so

$$\mathcal{F}_{f^k p_0,u}(t) \to 1,\ k \to \infty,$$

for all $t > 0$, and $\{f^k p_0\}$ $\mathcal{F}-$converges to u. The proof is completed. Moreover, it is easy now to see that $\{f^k q\}$ $\mathcal{F}-$converges to u for every $q \in S$. □

4. Applications

The essence of filtering the image is to choose a window with odd number of pixels that belong to image. The middle pixel is going to be replaced with a pixel that is the most similar to all the others pixels in that window. Different algorithms for image filtering give different criteria for choosing this pixel that is going to replace middle pixel. This window slides through whole image, process of selecting a pixel is the same for all windows.

In this paper we use UIQI (Universal Image Quality Index) quality index, introduced in [22]. This index isn't using error summation method like other image quality measures. It models any image distortion as a combination of three factors: loss of correlation, luminance distortion and contrast distortion. This measure method of image quality considers human visual system characteristics. It is lot more complicated image quality measure than PSNR that was usually used to measure image quality. PSNR is mathematically defined measure that is easy to calculate and it is independent of viewing conditions and human perception of image quality. For RGB images we will get three indexes, calculated for each plane of image. One of the criteria for determining image quality can be sharpness of the image. Sharpness is defined by the boundaries between zones of different tones or colors.

In this paper we are representing the application of metric-like functions from the Example 2.5, for $m = 1$. Noise that we applied on original image is 10% of salt&pepper noise. The image was filtered by using two metrics like in paper of Valentin, Morillas and Sapena [21]. For the metric that considers similarity in colors we applied metric-like function that we defined in paper, setting for parameter m value 1. For the metric that considers spatial closeness we've chosen the same metric s as in [21], for Euclidean norm we've taken max norm.

In the paper we compared image filtered with our metric-like function to image filtered by VMF (vector median filter). The result was that our image has slightly lower values for corresponding UIQI image quality, but much higher sharpness. This is very important in cases where details in image are needed to be reproduced. We have used for measuring sharpness image quality metrics introduced in [16].

For image filtered by VMF with window size 3, UIQI is equal to vector (calculated for all three colors) [0.546475813084152, 0.673989789093080, 0.525819221430506].

For image filtered by VMF with window size 3, UIQI is equal to vector (calculated for all three colors) [0.546475813084152, 0.673989789093080, 0.525819221430506].

The sharpness for image filtered by VMF is 0.690730837789661.

The sharpness for image filtered by our metric is 0.927492447129909.

Figure 1. Lena, $256 * 256$, $K = 384$, $t = 2.8$, window_size=3

Figure 2. Lena, $256 * 256$, VMF filtered, window_size=3

Figure 3. Lena, 256 ∗ 256

Figure 4. Lena, 256 ∗ 256, 10% salt&pepper noise

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