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ps-Drazin Inverses in Banach Algebras

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Abstract. An element a in a Banach algebra $\mathcal{A}$ has ps-Drazin inverse if there exists $p^2 = p \in comm^2(a)$ such that $(a-p)^k \in J(\mathcal{A})$ for some $k \in \mathbb{N}$. Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have ps-Drazin inverses. If $a^2b = aba$ and $b^2a = bab$, we prove that

1. $ab \in \mathcal{A}$ has ps-Drazin inverse.
2. $a + b \in \mathcal{A}$ has ps-Drazin inverse if and only if $1 + a^d b \in \mathcal{A}$ has ps-Drazin inverse.

As applications, we present various conditions under which a 2×2 matrix over a Banach algebra has ps-Drazin inverse.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with an identity. The commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in \mathcal{A} \mid xa = ax\}$. The double commutant of $a \in \mathcal{A}$ is defined by $comm^2(a) = \{x \in \mathcal{A} \mid xy = yx \text{ for all } y \in comm(a)\}$. An element a in a Banach algebra $\mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $b \in comm^2(a)$ such that $b = bab$, $a - a^2b \in \mathcal{A}^{qnil}$. The preceding b is unique, if such element exists, and called the g-Drazin inverse of a and denote b by a^d. Also, $a^\pi = 1 - aa^d$ is called spectral idempotent of a. As is known, $a \in \mathcal{A}$ has g-Drazin inverse if and only if there exists $e^2 = e \in comm^2(a)$ such that $a + e \in U(\mathcal{A})$ and $ae \in \mathcal{A}^{qnil}$. Here, $R^{qnil} = \{x \mid 1 - xr \in U(\mathcal{A}) \text{ for any } r \in comm(x)\}$. Following [10], an element $a \in \mathcal{A}$ has p-Drazin inverse (i.e., pseudo Drazin inverse) if there exists $b \in \mathcal{A}$ such that

$$b = bab,\ b \in comm^2(a),\ a^k - a^{k+1}b \in J(\mathcal{A})$$

for some $k \in \mathbb{N}$. Evidently, $a \in \mathcal{A}$ has p-Drazin inverse if and only if there exists $e^2 = e \in comm^2(a)$ such that $a + e \in U(\mathcal{A})$ and $(ae)^k \in J(\mathcal{A})$ for some $k \in \mathbb{N}$, if and only if there exists $b \in \mathcal{A}$ such that

$$b = bab, b \in comm^2(a), (a - a^2b)^k \in J(\mathcal{A})$$

for some $k \in \mathbb{N}$. Following [4], an element $a \in \mathcal{A}$ has gs-Drazin inverse if there exists $b \in \mathcal{A}$ such that

$$b = bab, b \in comm^2(a), a - ab \in \mathcal{A}^{qnil}.$$

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These generalized inverses in a Banach algebra have extensively studied from different points of view, e.g., [1]-[8], [13] and [14].

Motivating by g-Drazin, p-Drazin and gs-Drazin inverses, we introduce a new kind of generalized inverses in a Banach algebra. An element a in a Banach algebra $\mathcal{A}$ has ps-Drazin inverse if there exists $p^2 = p \in comm^2(a)$ such that $(a-p)^k \in J(\mathcal{A})$ for some $k \in \mathbb{N}$. As in the proof of [8, Lemma 2.2], we easily prove that $a \in \mathcal{A}$ has ps-Drazin inverse if and only if there exists $b \in \mathcal{A}$ such that

$$b = bab, b \in comm^2(a), (a-ab)^k \in J(\mathcal{A})$$

for some $k \in \mathbb{N}$.

The purpose of this paper is to investigate further algebraic properties of ps-Drazin inverses. Let $a, b \in \mathcal{A}$ have ps-Drazin inverses. In Section 2, we investigate when the product of a and b has ps-Drazin inverse in a Banach algebra. If $a^2b = aba$ and $b^2a = bab$, we prove that $ab \in \mathcal{A}$ has ps-Drazin inverse. In Section 3, we determine when the sum of a and b has ps-Drazin inverse. We prove that $a + b \in \mathcal{A}$ has ps-Drazin inverse if and only if $1 + a^d b \in \mathcal{A}$ has ps-Drazin inverse. Finally, in the last section, we present various conditions under which a 2×2 matrix over a Banach algebra has ps-Drazin inverse.

Throughout the paper, all Banach algebras are complex with an identity. We use $J(\mathcal{A})$ and $U(\mathcal{A})$ to denote the Jacobson radical of $\mathcal{A}$ and the set of all units in $\mathcal{A}$. $\mathcal{A}^{pd}$ and $\mathcal{A}^{ps}$ denote the sets of all elements having p-Drazin and ps-Drazin inverses in the Banach algebra $\mathcal{A}$, respectively. $\mathbb{N}$ stands for the set of all natural numbers.

2. Multiplicative property

In this section, we investigate multiplicative property of ps-Drazin inverses. We begin with the relation between ps-Drazin and p-Drazin inverse, which will be used frequently in the sequel.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. Then $a \in \mathcal{A}$ has ps-Drazin inverse if and only if*

(1) $a \in \mathcal{A}^{pd}$;
(2) $(a - a^2)^k \in J(\mathcal{A})$ *for some* $k \in \mathbb{N}$.

Proof. $\Longrightarrow$ Write $a = e + w$ with $e^2 = e \in comm^2(a), w^k \in J(\mathcal{A})$ for some $k \in \mathbb{N}$. Then $a + (1-e) = 1 + w \in U(\mathcal{A})$ and $(a(1-e))^k = (1-e)w^k \in J(\mathcal{A})$. Therefore, a has p-Drazin inverse. Moreover, $(a-a^2)^k = (1-2e-w)^k w^k \in J(\mathcal{A})$, as desired.

$\Longleftarrow$ Since $a \in \mathcal{A}$ has p-Drazin inverse, we can find some $b \in comm^2(a)$ such that $b = bab$ and $(a - a^2b)^k \in J(\mathcal{A})$. We check that $(a-1+ab)(b-1+ab) = 1-(a-a^2b) \in U(\mathcal{A})$. Hence, $a-1+ab \in U(\mathcal{A})$. Set $e = 1-ab$. Then $e^2 = e \in comm^2(a)$ and $u := a - e \in U(\mathcal{A})$. Hence, $a - a^2 = (e+u) - (e+u)^2 = -u(2e+u-1)$. This shows that $a-(1-e) = -u^{-1}(a-a^2)$. This implies that $(a-(1-e))^k \in J(\mathcal{A})$. This completes the proof. □

Corollary 2.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$. If $a \in \mathcal{A}$ has ps-Drazin inverse, then $a \in \mathcal{A}$ has p-Drazin inverse.*

We note that the converse of Corollary 2.2 is not true, in general. Let $\mathbb{C}$ be the field of all complex numbers. Then $2 \in \mathbb{C}$ has p-Drazin inverse. But it has no ps-Drazin inverse, as $(2^2-2)^k = 2^k \notin J(\mathbb{C})$ for all $k \in \mathbb{N}$.

Lemma 2.3. (see [12, Lemma 2.6]) *Let $\mathcal{A}$ be a Banach algebra with $a^2b = aba$ and $b^2a = bab$. Then, the following hold for any integer $k \in \mathbb{N}$.*

(1) $(ab)^k = a^k b^k$.
(2) $(a+b)^k = \sum_{i=0}^{k-1} C_{k-1}^i (a^{k-i}b^i + b^{k-i}a^i)$.

Lemma 2.4. (see [12, Theorem 2.8]) *Let $\mathcal{A}$ be a Banach algebra and $a, b \in \mathcal{A}$ have p-Drazin inverse. If $a^2b = aba$ and $b^2a = bab$, then ab has p-Drazin inverse.*

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have ps-Drazin inverses. If $a^2b = aba$ and $b^2a = bab$, then $ab \in \mathcal{A}$ has ps-Drazin inverse.*

Proof. Let a and b have ps-Drazin inverses. Then there exists $m, n \in \mathbb{N}$ such that $(a - a^2)^m \in J(\mathcal{A})$ and $(b - b^2)^n \in J(\mathcal{A})$ by Theorem 2.1. Let $c = a - a^2$.

$$\begin{array}{rcl} c^2b & = & (a-a^2)^2b \\ & = & (a^2 - 2a^3 + a^4)b \\ & = & aba - 2aba^2 + aba^3 \\ & = & (a-a^2)b(a-a^2) \\ & = & cbc. \end{array}$$

Also, $b^2c = b^2(a - a^2) = b^2a - b^2aa = bab - baba = bab - ba^2b = b(a - a^2)b = bcb$. Thus, for any integer $k \geq 0$, $((a - a^2)b)^k = (a - a^2)^k b^k$ by Lemma 2.3. Similarly, $(a^2(b - b^2))^l = (a^2)^l(b - b^2)^l$ for any integer $l \geq 0$. Now, let $x = (a - a^2)b$ and $y = a^2(b - b^2)$. We show that $x^2y = xyx$.

$$\begin{array}{rcl} x^2y & = & (a-a^2)b(a-a^2)ba^2(b-b^2) \\ & = & (a-a^2)(bab-baba)a^2(b-b^2) \\ & = & (a-a^2)bab(a^2-a^3)(b-b^2) \\ & = & (a-a^2)bab(a^2b-a^2b^2-a^3b+a^3b^2) \\ & = & (a-a^2)bab(a^2-a^2b)(b-ab) \\ & = & (a-a^2)bab(a^2-a^2b)(1-a)b \\ & = & (a-a^2)ba^2(b-b^2)(a-a^2)b \\ & = & xyx. \end{array}$$

Also,

$$\begin{array}{rcl} y^2x & = & a^2(b-b^2)a^2(b-b^2)(a-a^2)b \\ & = & a^2(b-b^2)a^2b(ab-a^2b-bab+ba^2b) \\ & = & a^2(b-b^2)a^2(bab-baba-b^2ab+b^2aba) \\ & = & a^2(b-b^2)a^2(bab-baba-bab^2+b^2aba) \\ & = & a^2(b-b^2)a^2b(ab-a^2b-ab^2+abba) \\ & = & a^2(b-b^2)a^2b(ab-a^2b-ab^2+abab) \\ & = & a^2(b-b^2)a^2b(ab-a^2b-ab^2+a^2b^2) \\ & = & a^2(b-b^2)a^2b(1-a)a(b-b^2) \\ & = & a^2(b-b^2)(a-a^2)ba^2(b-b^2) \\ & = & yxy. \end{array}$$

Hence, $(ab - (ab)^2)^{m+n+1} = (x + y)^{m+n+1} = \sum_{i=0}^{m+n} C_{m+n}^i (x^{m+n+1-i}y^i + y^{m+n+1-i}x^i)$ by Lemma 2.3. As we proved, $x^k = ((a - a^2)b)^k = (a - a^2)^k b^k$ for any integer $k \geq 0$ and $y^l = (a^2(b - b^2))^l = (a^2)^l(b - b^2)^l$ for any integer $l \geq 0$. Also, $(a - a^2)^m \in J(\mathcal{A})$ and $(b - b^2)^n \in J(\mathcal{A})$ for some $m, n \in \mathbb{N}$, and so we have $(ab - (ab)^2)^{m+n+1} \in J(\mathcal{A})$. Therefore, ab has ps-Drazin inverse by Theorem 2.1 and Lemma 2.4. □

Corollary 2.6. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have ps-Drazin inverses. If $ab = ba$, then $ab \in \mathcal{A}$ has ps-Drazin inverse.*

Proof. It is clear by Theorem 2.5, since the condition $ab = ba$ implies that $a^2b = aba$ and $b^2a = bab$. □

3. Additive property

In this section, we concern on the additive properties of ps-Drazin inverses. For the convenience, we use $J^\#(\mathcal{A})$ do denote the set of all elements x with $x^n \in J(\mathcal{A})$ for some $n \in \mathbb{N}$. We now derive

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have ps-Drazin inverses. If $ab = ba = 0$, then $a + b$ has ps-Drazin inverse.*

Proof. In view of [10, Theorem 5.4], $a + b$ has p-Drazin inverse. We easily checks that $a + b - (a + b)^2 = (a - a^2) + (b - b^2) \in J^{\#}(\mathcal{A})$. This completes the proof by Theorem 2.1. □

Lemma 3.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in J^{\#}(\mathcal{A})$. If $a^2b = aba$ and $b^2a = bab$, then $a + b \in J^{\#}(\mathcal{A})$.*

Proof. Write $a^m, b^n \in J(\mathcal{A})$ for some $m, n \in \mathbb{N}$. According to Lemma 2.3, we see that $(a + b)^{m+n} \in J(\mathcal{A})$, as desired. □

Lemma 3.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have ps-Drazin inverses. If $ab = ba$ and $1 + a^db \in \mathcal{A}$ has ps-Drazin inverse, then $a + b \in \mathcal{A}$ has ps-Drazin inverse.*

Proof. Clearly, $(1 + a^db) - (1 + a^db)^2 = -(1 + a^db)a^db \in J^{\#}(\mathcal{A})$ since $1 + a^db$ has ps-Drazin inverse. Hence, $a^db + (a^d)^2b^2 \in J^{\#}(\mathcal{A})$. Thus, $aa^db + a^db^2 \in J^{\#}(\mathcal{A})$. Also, $b - b^2 \in J^{\#}(\mathcal{A})$ since $b \in \mathcal{A}^{ps}$. So $a^d(b - b^2) \in J^{\#}(\mathcal{A})$ as well. Therefore, $aa^db + a^db = (aa^db + a^db^2) + a^d(b - b^2) \in J^{\#}(\mathcal{A})$. Then, $2ab = (a^2a^db + aa^db) - (a^2a^d - a)b - (aa^d - a)b \in J^{\#}(\mathcal{A})$. Consequently, $(a + b) - (a + b)^2 = (a - a^2) + (b - b^2) - 2ab \in J^{\#}(\mathcal{A})$. Furthermore, $a + b$ has p-Drazin inverse since $1 + a^db$ has p-Drazin inverse (see [12, Theorem 2.10]). So we have $a + b \in \mathcal{A}^{ps}$. □

Theorem 3.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ have ps-Drazin inverses. If $a^2b = aba$ and $b^2a = bab$, then $a + b \in \mathcal{A}$ has ps-Drazin inverse if and only if $1 + a^db \in \mathcal{A}$ has ps-Drazin inverse.*

Proof. $\Longrightarrow$ Let $a + b$ has ps-Drazin inverses. Write $1 + a^db = x + y$ where $x = 1 - aa^d$ and $y = a^d(a + b)$. Then $x \in \mathcal{A}^{ps}$ and $xy = 0$. Moreover, we see that $yx = a^d(a + b)(1 - aa^d) = a^db(1 - a^da) = (a^d)^2(ab)(1 - aa^d) = 0$, as $a \in comm(ab)$. WE easily check that $(a^d)^2(a + b) = a^d(a + b)a^d$, $(a + b)^2a^d = (a + b)a^d(a + b)$. Since a has ps-Drazin inverse, it has p-Drazin inverse and by we can find some $k \in \mathbb{N}$ such that $(a - a^2)^k \in J(\mathcal{A})$. In view of [12, Theorem 2.3], a^d has p-Drazin inverse. We easily check that $a^d - (a^d)^2 = -(a^d)^3(a - a^2)$, and so $(a^d - (a^d)^2)^k \in J((\mathcal{A}))$. In light of Theorem 2.1, a^d has ps-Drazin inverse. By hypothesis, $a + b \in \mathcal{A}^{ps}$, and so $y \in \mathcal{A}^{ps}$. Therefore $1 + a^db \in \mathcal{A}^{ps}$ by Lemma 3.1.

$\Longleftarrow$ Step 1. Clearly, $1 + (a^2a^d)^d(aa^d)b = 1 + (aaa^d)^d(aa^db) = 1 + a^daa^daa^db = 1 + a(a^d)^2b = 1 + a^db \in \mathcal{A}^{ps}$. Also, $(a^2a^d)(aa^db) = (aa^db)(a^2a^d)$. Since $a(aa^d) = (aa^d)a$ and $(aa^d)b = b(aa^d)$, it follows by Corollary 2.6 that a^2a^d and aa^db have ps-Drazin inverses. Hence, we have $a^2a^d + aa^db = aa^d(a + b) \in \mathcal{A}^{ps}$ by applying Lemma 3.3 to a^2a^d and aa^db.

Step 2. Assume that $b \in J^{\#}(\mathcal{A})$. Then $(1 - aa^d)(a + b) = x + y$ where $x = (a - a^2a^d)$ and $y = (1 - aa^d)b$. Then $x^2y = xyx$ and $y^2x = yxy$. Also, $x = a - a^2a^d \in J^{\#}(\mathcal{A})$ and $y = (1 - aa^d)b \in J^{\#}(\mathcal{A})$ since $b \in J^{\#}(\mathcal{A})$. Hence, $(1 - aa^d)(a + b) = x + y \in J^{\#}(\mathcal{A})$. Choose $p = aa^d$. Then

$$a + b = \begin{pmatrix} p(a + b)p & 0 \\ (1 - p)(a + b)p & (1 - p)(a + b)(1 - p) \end{pmatrix}_p.$$

Since $p(a + b)p = p(a + b) \in \mathcal{A}^{ps}$ and $(1 - p)(a + b)(1 - p) = (1 - p)(a + b) \in \mathcal{A}^{ps}$, $a + b \in \mathcal{A}^{ps}$.

Step 3. Choose $p = bb^d$. Then $a = \begin{pmatrix} a_1 & 0 \\ * & a_2 \end{pmatrix}_p$ and $b = \begin{pmatrix} b_1 & 0 \\ * & b_2 \end{pmatrix}_p$ where $a_1 = pap$, $a_2 = (1 - p)a(1 - p)$, $b_1 = pbp$ and $b_2 = (1 - p)b(1 - p)$. Hence,

$$a + b = \begin{pmatrix} a_1 + b_1 & 0 \\ * & a_2 + b_2 \end{pmatrix}_p.$$

Obviously, $a_1, b_1 \in \mathcal{A}^{ps}$ and $a_1b_1 = b_1a_1$. As in Step 1, we see that $1 + a_1^db_1 \in \mathcal{A}^{ps}$. Therefore $a_1 + b_1 \in \mathcal{A}^{ps}$ by Lemma 3.3. As $b^2a = bab$, we see that $ba \in comm(b)$, and so $b^d(ba) = (ba)b^d$. This implies that $a_2 = (1 - bb^d)a(1 - bb^d) = (a - bb^da)(1 - bb^d) = a(1 - bb^d) = a - a(bb^d)$. It is easy to verify that $a^2(bb^d) = a(bb^d)a$ and $(bb^d)^2a = (bb^d)a(bb^d)$. It follows by Theorem 2.5 that abb^d has ps-Drazin inverse. Moreover, we see that $a^2(bb^d) = (aba)b^d = a(bb^d)a$. We verify that $1 - a^da(bb^d) = 1 - a^d(ab)b^d = 1 - (ab)(ab)^d$ has ps-Drazin inverse by Theorem 2.5. As in the proof of [15, Theorem 3.3], we see that $a_2 \in \mathcal{A}^{ps}$. Clearly, $b_2 \in J^{\#}(\mathcal{A})$, $a_2^2b_2 = a_2b_2a_2$ and $b_2^2a_2 = b_2a_2b_2$. By Step 2, $a_2 + b_2 \in \mathcal{A}^{ps}$. Consequently, $a + b \in \mathcal{A}^{ps}$. □

We see that the condition in Theorem 3.4 is a generalization of the commutativity of a and b. But we have,

Example 3.5. *Let* $a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)$. *Then* $a^2b = aba$, $b^2a = bab$. *In this case* $a, b, 1 + a^d b \in M_2(\mathbb{Z}_2)$ *has ps-Drazin inverse and* $ab \neq ba$.

4. Splitting in Banach algebras

The goal of this section is to use splitting approach to determine when an element in a Banach algebra has ps-Drazin inverse. We derive

Lemma 4.1. *Let* $\mathcal{A}$ *be a Banach algebra. If* $a, d \in \mathcal{A}$ *have ps-Drazin inverses, then* $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathcal{A})$ *has ps-Drazin inverse.*

Proof. In view of Theorem 2.1, $a, d \in \mathcal{A}$ have p-Drazin inverse and $(a - a^2)^k, (b - b^2)^k \in J(\mathcal{A})$ for some $k \in \mathbb{N}$. In view of [10, Theorem 5.3], $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathcal{A})$ has p-Drazin inverse. On the other hand, we have some $z \in \mathcal{A}$ such that

$$\begin{aligned} \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^2 \right)^{2k} &= \begin{pmatrix} (a - a^2)^k & z \\ 0 & (d - d^2)^k \end{pmatrix}^2 \\ &= \begin{pmatrix} (a - a^2)^{2k} & (a - a^2)^k z + z(d - d^2)^k \\ 0 & (d - d^2)^{2k} \end{pmatrix}^2 \\ &\in J(M_2(\mathcal{A})). \end{aligned}$$

According to Theorem 2.1, we complete the proof. □

Theorem 4.2. *Let* $\mathcal{A}$ *be a Banach algebra, and let* $a, d \in \mathcal{A}$ *have ps-Drazin inverses. If* $bc = dc = 0$, *then* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ *has ps-Drazin inverse.*

Proof. Clearly, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = p + q$, where

$$p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

In view of Lemma 4.1, $p \in M_2(\mathcal{A})$ has ps-Drazin inverse. As $q^2 = 0$, we easily see that $q \in M_2(\mathcal{A})$ has ps-Drazin inverse. Moreover,

$$q^2 p = 0 = \begin{pmatrix} 0 & 0 \\ cbc & 0 \end{pmatrix} = qpq,$$

and

$$p^2 q = \begin{pmatrix} abc + bdc & 0 \\ d^2 c & 0 \end{pmatrix} = 0 = \begin{pmatrix} bca & bcb \\ dca & dcb \end{pmatrix} = pqp.$$

Clearly, $q^d = 0$, and so $1 + q^d p = 1$ has ps-Drazin inverse. Therefore, $p + q \in \mathcal{A}$ has ps-Drazin inverse, by Theorem 3.4. This completes the proof. □

Corollary 4.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, d \in \mathcal{A}$ have ps-Drazin inverses. If $bc = 0$ and $dc = c$, then* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ *has ps-Drazin inverse.*

Proof. Since $dc = c$, $-(1-d)c = 0$. So in light of Theorem 4.2, $I_2 - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ has ps-Drazin inverse since $bc = 0$ and $-(1-d)c = 0$. Thus, we can find an idempotent $E \in comm^2\left(I_2 - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ such that

$$\left(I_2 - \begin{pmatrix} a & b \\ c & d \end{pmatrix} - E\right)^k \in J(M_2(\mathcal{A})) \text{ for some } k \in \mathbb{N},$$

and so

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (I_2 - E)\right)^k \in J(M_2(\mathcal{A})).$$

Clearly, $I_2 - E \in comm^2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$. This completes proof. □

Next we consider another splitting of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and get the alternative results.

Theorem 4.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, d \in \mathcal{A}$ have ps-Drazin inverses. If $bc = cb = 0$ and $dc = ca$, then* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ *has ps-Drazin inverse.*

Proof. We see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = p + q,$$

where

$$p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

In view of Lemma 4.1, p has ps-Drazin inverse. Since $q - q^2 \in J(M_2(\mathcal{A}))$ and $q \in \mathcal{A}^{pd}$, q has ps-Drazin inverse by Theorem 2.1. Clearly, $q^d = 0$, and so $1 + q^d p$ has ps-Drazin inverse. From $bc = cb = 0$ and $dc = ca$, we see that

$$pq = \begin{pmatrix} bc & 0 \\ dc & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ca & cb \end{pmatrix} = qp.$$

In light of Lemma 3.3, $p + q$ has ps-Drazin inverse, as asserted. □

Example 4.5. *Let A, B, C be operators, acting on separable Hilbert space $l_2(\mathbb{N})$, defined as follows respectively:*

$$\begin{array}{rcl} A(x_1, x_2, x_3, x_4, \cdots) & = & (x_1, x_2, x_3, x_4, \cdots), \\ B(x_1, x_2, x_3, x_4, \cdots) & = & (x_1, -x_1, 0, 0, \cdots), \\ C(x_1, x_2, x_3, x_4, \cdots) & = & (0, x_1 + x_2, x_3, x_4, \cdots), \\ D(x_1, x_2, x_3, x_4, \cdots) & = & (-x_1, x_2, x_3, x_4, \cdots). \end{array}$$

Then we easily check that $BC = CB = 0$ and $DC = CA$. In light of Theorem 4.4, the operator matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *has ps-Drazin inverse. In this case, $DC \neq 0$.*

Lemma 4.6. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$ have ps-Drazin inverse. If $e^2 = e \in comm(a)$, then $ea \in \mathcal{A}$ has ps-Drazin inverse.*

Proof. Since $e \in \mathcal{A}^{ps}$, we easily obtain the result by Theorem 2.5. □

Let $a \in \mathcal{A}$ have ps-Drazin inverse. Then it has g-Drazin inverse. We use a^{π} to denote the spectral idempotent of a, i.e., $a^{\pi} = 1 - aa^d$. We now derive

Theorem 4.7. *Let $\mathcal{A}$ be a Banach algebra, and let $a, d \in \mathcal{A}$ have ps-Drazin inverses. If $bc = cb = 0$, $ca(1-a^{\pi}) = d^{\pi}dc$ and $a^{\pi}ab = bd(1-d^{\pi})$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ has ps-Drazin inverse.*

Proof. Let

$$p = \begin{pmatrix} a(1-a^{\pi}) & b \\ 0 & dd^{\pi} \end{pmatrix}, \ q = \begin{pmatrix} aa^{\pi} & 0 \\ c & d(1-d^{\pi}) \end{pmatrix}.$$

Then $M = p + q$. In view of Lemma 4.1, p has ps-Drazin inverse. Likewise, q has ps-Drazin inverse. It is easy to verify that

$$pq = \begin{pmatrix} 0 & bd(1-d^{\pi}) \\ dd^{\pi}c & 0 \end{pmatrix} = \begin{pmatrix} 0 & aa^{\pi}b \\ ca(1-a^{\pi}) & 0 \end{pmatrix} = qp.$$

One easily checks that

$$p^d = \begin{pmatrix} (a(1-a^{\pi}))^d & x \\ 0 & d^d d^{\pi} \end{pmatrix} = \begin{pmatrix} a^d & x \\ 0 & 0 \end{pmatrix}$$

where $x = (a^d)^2 \sum_{n=0}^{\infty} (a^d)^n b (dd^{\pi})^n$. Hence,

$$p^d q = \begin{pmatrix} a^d & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} aa^{\pi} & 0 \\ c & d(1-d^{\pi}) \end{pmatrix} = \begin{pmatrix} xc & xd(1-d^{\pi}) \\ 0 & 0 \end{pmatrix}$$

where $xc = (a^d)^2(b + \sum_{n=1}^{\infty} (a^d)^n b (dd^{\pi})^n)c = 0$ as $bc = 0$, $b(dd^{\pi})^n c = 0$. Moreover, we have

$$\begin{array}{rcl} xd(1-d^{\pi}) & = & (a^d)^2(b + \sum_{n=1}^{\infty} (a^d)^n b (dd^{\pi})^n) d(1-d^{\pi}) \\ & = & (a^d)^2(b + bd(1-d^{\pi})) \\ & = & (a^d)^2(b + a^{\pi}ab) \\ & = & (a^d)^2 b \end{array}$$

and so $p^d q = \begin{pmatrix} 0 & (a^d)^2 b \\ 0 & 0 \end{pmatrix}$. Thus, $1 + p^d q$ is invertible. So, it has p-Drazin inverse. Further, we have

$$\begin{array}{rcl} (1 + p^d q) - (1 + p^d q)^2 & = & -p^d q(1 + p^d q) \\ & = & \begin{pmatrix} 0 & -(a^d)^2 b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & (a^d)^2 b \\ 0 & 1 \end{pmatrix} \\ & = & \begin{pmatrix} 0 & -(a^d)^2 b \\ 0 & 0 \end{pmatrix} \\ & = & \in J^{\#}(\mathcal{A}). \end{array}$$

In light of Theorem 2.1, $1 + p^d q \in \mathcal{A}^{ps}$. Therefore, we complete the proof by Theorem 3.4. □

Finally, we concern on the ps-Drazin inverse for a operator matrix M has ps-Drazin inverse. Here,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (1)$$

where $A, D \in L(X)$ has ps-Drazin inverses and X is a complex Banach space. Then M is a bounded linear operator on $X \oplus X$.

Lemma 4.8. *Let $\mathcal{A}$ be a Banach algebra, and let $A \in M_{m\times n}(\mathcal{A}), B \in M_{n\times m}(\mathcal{A})$ and $k \in \mathbb{N}$. Then $AB \in M_m(\mathcal{A})$ has ps-Drazin inverse if and only if $BA \in M_n(\mathcal{A})$ has ps-Drazin inverse.*

Proof. Suppose that $AB \in M_m(\mathcal{A})$ has ps-Drazin inverse. Then $AB \in M_m(\mathcal{A})$ has p-Drazin inverse and $(AB - (AB)^2)^k \in M_m(J(\mathcal{A}))$. In light of [10, Theorem 3.6], BA has p-Drazin inverse. One easily checks that

$$(BA - (BA)^2)^{k+1} = B(AB - (AB)^2)^k(A - ABA) \in M_n(J(\mathcal{A})).$$

According to Theorem 2.1, $BA \in M_n(\mathcal{A})$ has ps-Drazin inverse, as asserted. □

Lemma 4.9. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$. If a, b have ps-Drazin inverses and $ab = 0$, then $a + b \in \mathcal{A}$ has ps-Drazin inverse.*

Proof. Let $A = (1, b)$ and $B = \begin{pmatrix} a \\ 1 \end{pmatrix}$. By the similar technique to the Lemma 4.1, $BA = \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}$ has ps-Drazin inverse. By virtue of Lemma 4.8, $AB = a + b \in \mathcal{A}$ has ps-Drazin inverse, as asserted. □

Theorem 4.10. *Let $A \in L(X)$ has ps-Drazin inverse, $D \in L(X)$ and M be given by (4.1). Let $W = AA^d + A^dBCA^d$. If AW has ps-Drazin inverse,*

$$A^{\pi}BC = 0, D = CA^dB,$$

then M has ps-Drazin inverse.

Proof. We easily see that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, Q = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}.$$

By hypothesis, we verify that $QP = 0$. Clearly, Q has ps-Drazin inverse. Furthermore, we have

$$P = P_1 + P_2,\ P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix},\ P_2 = \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}$$

and $P_2P_1 = 0$. By virtue of Theorem 4.2, P_2 has ps-Drazin inverse. Obviously, we have

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = AW$$

has ps-Drazin inverse. In light of Lemma 4.8, P_1 has ps-Drazin inverse. Thus, P has ps-Drazin inverse by Lemma 4.9. According to Lemma 4.9, M has ps-Drazin inverse. Therefore, we complete the proof. □

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