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Operator Inequalities Related to p-Angular Distances

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Abstract. For any nonzero elements x, y in a normed space X, the angular and skew-angular distance is respectively defined by $\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$ and $\beta[x,y] = \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|$. Also inequality $\alpha \leq \beta$ characterizes inner product spaces. Operator version of α_p has been studied by Pečarić, Rajić, and Saito, Tominaga, and Zou et al.
In this paper, we study the operator version of p-angular distance β_p by using Douglas' lemma. We also prove that the operator version of inequality $\alpha_p \leq \beta_p$ holds for normal and double commute operators. Some examples are presented to show essentiality of these conditions.

1. Introduction

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. For $T \in B(\mathcal{H})$, we denote by $|T|$ the absolute value operator of T, that is, $|T| = (T^*T)^{\frac{1}{2}}$, where T^* stands for the adjoint operator of T. A self-adjoint operator $T \in B(\mathcal{H})$ is said to be positive if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$. For self-adjoint operators A and B in $B(\mathcal{H})$, we write $A \leq B$ if $B - A$ is positive.

For $A, B \in B(\mathcal{H})$, let $A = U|A|$ and $B = V|B|$ be polar decompositions of A and B, respectively. By using a simple method Zou et al. [11, Theorem 2.1] obtained an inequality for absolute value operators as follows:

$$|(U-V)|A||^2 \quad \leq \quad |A-B|^2 + (|A|-|B|)^2 - (T+T^*), \tag{1}$$

where $T = (|A|-|B|)V^*(A-B)$. It is a refinement of the following inequality due to Saito and Tominaga [10, Theorem 2.3]:

$$|(U-V)|A||^2 \quad \leq \quad p|A-B|^2 + q(|A|-|B|)^2. \tag{2}$$

Inequality (2) is a generalization of the following inequality without invertibility condition on $|A|$ and $|B|$:

$$\left|A|A|^{-1} - B|B|^{-1}\right|^2 \leq |A|^{-1}\left(p|A-B|^2 + q(|A|-|B|)^2\right)|A|^{-1}. \tag{3}$$

To illustrate the problem we need to mention several lines about the origin of the above inequalities.

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Let $\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$ be the *angular distance* between two nonzero elements x and y in a normed linear space X, which introduced by Clarkson in [1]. Over the years, the following interesting estimations of $\alpha[x,y]$ have been obtained:

$$\alpha[x,y] \leq \frac{\|x-y\| + |\,\|x\| - \|y\|\,|}{\max\{\|x\|,\|y\|\}} \leq \frac{\sqrt{2\|x-y\|^2 + 2(\|x\|-\|y\|)^2}}{\max\{\|x\|,\|y\|\}} \tag{4}$$

$$\leq \frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}} \leq \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{5}$$

The first and second bound in (4), obtained respectively by Maligranda [6] and Pečarić and Rajić [9], are refinements of the Massera-Schaffer inequality (first bound in (5)) proved in 1958 [8], which is stronger than the Dunkl-Williams inequality (second bound in (5)) proved in [4].

In fact, inequality (3) for $p = q = 2$ is operator version of the second bound in (4).

On the other hand, Dehghan [2] introduced the concept of *skew-angular distance* $\beta[x,y] = \left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|$ and proved that $\alpha[x,y] \leqslant \beta[x,y]$ if and only if X is an inner product space. Moreover, he obtained the following inequalities:

$$\beta[x,y] \leq \frac{\|x-y\|}{\max\{\|x\|,\|y\|\}} + \frac{|\,\|x\| - \|y\|\,|}{\min\{\|x\|,\|y\|\}} \tag{6}$$

$$\leq \sqrt{\frac{2\|x-y\|^2}{\max^2\{\|x\|,\|y\|\}} + \frac{2(\|x\|-\|y\|)^2}{\min^2\{\|x\|,\|y\|\}}}. \tag{7}$$

After then p-angular distance $\alpha_p[x,y]$ and skew p-angular distance $\beta_p[x,y]$ have been studied extensively(see [5] and references therein).

The main aim of this paper is to compare the operator version of $\alpha_p[x,y]$ and $\beta_p[x,y]$. To proceed in this direction we first provide an appropiate bound for $\beta_p[x,y]$ by using Douglas' lemma [3]. Next, we prove that inequality $\alpha_p[A,B] \leq \beta_p[A,B]$ holds for normal and double commute operators. By some examples we show that the mentioned conditions are essential.

2. Main results

We begin with the following lemma which plays basic role in the sequel.

Lemma 2.1. *(**Douglas' lemma** [3, Theorem 1]) If A and B are bounded operators on a Hilbert space $\mathcal{H}$ such that $A^*A \leq \lambda^2 B^*B$ for some $\lambda \geq 0$, then there exists a unique operator $C \in B(\mathcal{H})$ so that $A = CB$, $ker(A^*) = ker(C^*)$, $im(C^*) \subseteq \overline{im(B)}$ and $\|C\|^2 = \inf\{\mu : A^*A \leq \mu B^*B\}$.*

Let $A, B \in B(\mathcal{H})$, and

$$A = U|A| \quad \text{and} \quad B = V|B| \tag{8}$$

be polar decompositions of A and B, respectively. One may obtain this from Douglas' lemma by considering $A^*A = |A||A|$ and $B^*B = |B||B|$. Moreover, if $A^*A \leq \lambda^2 B^*B$ and $B^*B \leq \mu^2 A^*A$ for some $\lambda, \mu \geq 0$, then there exist unique operators $C, D \in B(\mathcal{H})$ such that

$$A = C|B| \quad \text{and} \quad B = D|A|. \tag{9}$$

The following theorem is our first main result. Note that invertibility of $|A|$ and $|B|$ is not needed.

Theorem 2.2. *Let $A, B \in B(\mathcal{H})$ be as in (8) and (9), $p > 1$ and let $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$|C|B|^p - D|A|^p|^2 \leq |(C-V)|B|^p|^2 + |B|(|A|^{p-1} - |B|^{p-1})^2|B| - (T + T^*) \tag{10}$$

$$\leq r|(C-V)|B|^p|^2 + s|B|(|A|^{p-1} - |B|^{p-1})^2|B| \tag{11}$$

where $T = (|A|^{p-1} - |B|^{p-1})^2|B|V^*(C-V)|B|^p$.

Proof. Let I be the identity operator on $\mathcal{H}$. Since $V^*V \leq I$, we observe that

$$\left|V|B|(|A|^{p-1} - |B|^{p-1})\right|^2 \leq |B|(|A|^{p-1} - |B|^{p-1})^2|B|. \tag{12}$$

Hence

$$\begin{aligned} |C|B|^p - D|A|^p|^2 &= \left|C|B|^p - V|B||A|^{p-1}\right|^2 = \left|C|B|^p - V|B|^p + V|B|^p - V|B||A|^{p-1}\right|^2 \\ &= \left|(C-V)|B|^p - V|B|(|A|^{p-1} - |B|^{p-1})\right|^2 \\ &= |(C-V)|B|^p|^2 + \left|V|B|(|A|^{p-1} - |B|^{p-1})\right|^2 - (T + T^*) \\ &\leq |(C-V)|B|^p|^2 + |B|(|A|^{p-1} - |B|^{p-1})^2|B| - (T + T^*), \end{aligned}$$

which is inequality (10). To prove (11), we first note that $(r-1)(s-1) = 1$. This together with (12) implies that

$$\begin{aligned} & r\,|(C-V)|B|^p|^2 + s|B|(|A|^{p-1} - |B|^{p-1})^2|B| \\ &\quad - \left(|(C-V)|B|^p|^2 + |B|(|A|^{p-1} - |B|^{p-1})^2|B| - (T+T^*)\right) \\ &= (r-1)\,|(C-V)|B|^p|^2 + (s-1)|B|(|A|^{p-1} - |B|^{p-1})^2|B| + T + T^* \\ &\geq (r-1)|(C-V)|B|^p|^2 + (s-1)|V|B|(|A|^{p-1} - |B|^{p-1})|^2 + T + T^* \\ &= \left|\sqrt{r-1}(C-V)|B|^p + \sqrt{s-1}V|B|(|A|^{p-1} - |B|^{p-1})\right|^2 \\ &\geq 0, \end{aligned}$$

which completes the proof. □

Remark 2.3. *By the proof above, we see that the equality in (10) holds if and only if* $(|A|^{p-1} - |B|^{p-1})|B|V^*V|B| = |A|^{p-1} - |B|^{p-1}$*, and the equality in (11) holds if and only if* $(|A|^{p-1} - |B|^{p-1})|B|V^*V|B| = |A|^{p-1} - |B|^{p-1}$ *and* $r(C-V)|B|^p = sV|B|(|A|^{p-1} - |B|^{p-1})$.

From now on we shall use the notations

$$\alpha_p[A,B] = \left|A|A|^{p-1} - B|B|^{p-1}\right| \quad \text{and} \quad \beta_p[A,B] = \left|A|B|^{p-1} - B|A|^{p-1}\right|,$$

where A and B are operators in $B(\mathcal{H})$ with invertible absolute values.

Corollary 2.4. *Let* $A, B \in B(\mathcal{H})$ *be operators where* $|A|$ *and* $|B|$ *are invertible, and let* $r, s > 1$ *with* $\frac{1}{r} + \frac{1}{s} = 1$. *Then*

$$\beta_p^2[A,B] \leqslant r|B|^{p-1}\,|A-B|^2\,|B|^{p-1} + s|B|\left(|A|^{p-1} - |B|^{p-1}\right)^2|B|. \tag{13}$$

The equality in (13) holds if and only if $r(A-B)|B|^{p-1} = sB(|A|^{p-1} - |B|^{p-1})$.

Proof. Since $|A|$ and $|B|$ are invertible, it is easy to verify that $|A| \geq mI$ and $|B| \geq nI$ for some $m, n > 0$. Thus

$$A^*A \leq \lambda^2 B^*B, \quad B^*B \leq \mu^2 A^*A, \tag{14}$$

for some $\lambda, \mu > 0$ with $\mu^2\lambda^2 \geq 1$. By Douglas' lemma there exist unique operators $C, D \in B(\mathcal{H})$ such that $A = C|B|$ and $B = D|A|$. Then $C = A|B|^{-1}$ and $D = B|A|^{-1}$. We also have $V = B|B|^{-1}$. These together with Theorem 2.2 imply that

$$\begin{aligned} \beta_p^2[A,B] &= \left|A|B|^{p-1} - B|A|^{p-1}\right|^2 = |C|B|^p - D|A|^p|^2 \\ &\leq r|(C-V)|B|^p|^2 + s|B|(|A|^{p-1} - |B|^{p-1})^2|B| \\ &= r\left|(A-B)|B|^{p-1}\right|^2 + s|B|(|A|^{p-1} - |B|^{p-1})^2|B| \\ &= r|B|^{p-1}\,|A-B|^2\,|B|^{p-1} + s|B|(|A|^{p-1} - |B|^{p-1})^2|B|, \end{aligned}$$

which is the desired inequality. Considering Remark 2.3, the equality in (13) holds if and only if $r(C-V)|B|^p = sV|B|(|A|^{p-1} - |B|^{p-1})$. Substituting $C = A|B|^{-1}$ and $V = B|B|^{-1}$ we have $r(A - B)|B|^{p-1} = sV|B|(|A|^{p-1} - |B|^{p-1})$ which is equivalent with $r(A - B)|B|^{p-1} = sB(|A|^{p-1} - |B|^{p-1})$. □

Remark 2.5. *Interchanging the operators A and B in (13) we also have*

$$\beta_p^2[A, B] \leqslant r|A|^{p-1}\,|A - B|^2\,|A|^{p-1} + s|A|\left(|B|^{p-1} - |A|^{p-1}\right)^2|A|. \tag{15}$$

The equality in (13) holds if and only if $r(A - B)|A|^{p-1} = sA(|A|^{p-1} - |B|^{p-1})$.

Next, we provide sufficient and essential conditions for inequality $\alpha_p[A, B] \leq \beta_p[A, B]$.

Theorem 2.6. *Let A and B be normal operators such that $AB = BA$, and $|A|$ and $|B|$ are invertible. If $p < 1$, then*

$$\alpha_p[A, B] \leq \beta_p[A, B]. \tag{16}$$

For $p > 1$ the inequality reverses. The equality holds if and only if $|A| = |B|$.

Proof. It follows from (14) that $|A| \leq |B|$ or $|B| \leq |A|$. On the other hand α_p and β_p are symmetric. So, without loss of generality we may assume that $|A| \leq |B|$. By the Löwner-Heinz inequality, it is sufficient to prove that $\alpha_p^2[A, B] \leq \beta_p^2[A, B]$. First, we note that

$$\begin{aligned}
\beta_p^2[A, B] - \alpha_p^2[A, B] &= \left|A|B|^{p-1} - B|A|^{p-1}\right|^2 - \left|A|A|^{p-1} - B|B|^{p-1}\right|^2 \\
&= \left(|B|^{p-1}A^* - |A|^{p-1}B^*\right)\left(A|B|^{p-1} - B|A|^{p-1}\right) \\
&\quad - \left(|A|^{p-1}A^* - |B|^{p-1}B^*\right)\left(A|A|^{p-1} - B|B|^{p-1}\right) \\
&= |B|^{p-1}|A|^2|B|^{p-1} + |A|^{p-1}|B|^2|A|^{p-1} - |B|^{p-1}A^*B|A|^{p-1} - |A|^{p-1}B^*A|B|^{p-1} \\
&\quad - \left(|A|^{p-1}|A|^2|A|^{p-1} + |B|^{p-1}|B|^2|B|^{p-1} - |A|^{p-1}A^*B|B|^{p-1} - |B|^{p-1}B^*A|A|^{p-1}\right) \\
&= |B|^{p-1}(|A|^2 - |B|^2)|B|^{p-1} - |A|^{p-1}(|A|^2 - |B|^2)|A|^{p-1} \\
&\quad + |B|^{p-1}(B^*A - A^*B)|A|^{p-1} - |A|^{p-1}(B^*A - A^*B)|B|^{p-1}.
\end{aligned} \tag{17}$$

Let $C(A)$ be the C^*-algebra generated by A and I. Since A is normal, then $C(A)$ is a commutative C^*-algebra. Moreover, by the Fuglede-Putnam theorem, A and B are double commuting operators. Double commutativity of A and B implies that A and A^* commute with B (and B^*) and so all elements of $C(A)$ especially $|A|$ and $|A|^{p-1}$ commute with B (and B^*). Until now, we know that the operators $A, A^*, |A|$ and $|A|^{p-1}$ commute with B and B^*. Therefore, they commute with all elements of $C(B)$ especially $|B|$ and $|B|^{p-1}$. Thus

$$\begin{aligned}
&|B|^{p-1}(|A|^2 - |B|^2)|B|^{p-1} - |A|^{p-1}(|A|^2 - |B|^2)|A|^{p-1} \\
&\qquad = \left((|B|^2)^{p-1} - (|A|^2)^{p-1}\right)\left(|A|^2 - |B|^2\right) \geq 0
\end{aligned} \tag{18}$$

and

$$|B|^{p-1}(B^*A - A^*B)|A|^{p-1} = |A|^{p-1}(B^*A - A^*B)|B|^{p-1}.$$

These together with (17) imply that

$$\beta_p^2[A, B] - \alpha_p^2[A, B] = (|B|^{p-1}|A| - |A|^{p-1}|B|)^2 \geq 0.$$

For the case $p > 1$, the factors in (18) have different sign. This completes the proof. □

The following examples show that all the hypotheses of Theorem 2.6 are essential, i.e., if any one omitted, inequality (16) no longer holds.

Example 2.7. *Let*

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

be matrix representations of two operators A and B on a two-dimensional Hilbert space with respect to some fixed orthonormal basis. It is clear that A is not normal but AB = BA. Using the software MAPLE 16 we observe that

$$\left| A|B|^{-1} - B|A|^{-1} \right| = \frac{1}{\sqrt{4810}} \begin{bmatrix} 29 & -11 \\ -11 & 163 \end{bmatrix},$$

$$\left| A|A|^{-1} - B|B|^{-1} \right| = \sqrt{\frac{2}{5}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the matrix $\beta_0[A, B] - \alpha_0[A, B]$ has two eigenvalues with different sign. Hence it is not positive and so inequality (16) does not hold.

Example 2.8. *Let*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

be matrix representations of two operators A and B on a two-dimensional Hilbert space with respect to some fixed orthonormal basis. It is clear that A and B are normal but A does not commute with B. Again, using the software MAPLE 16 we observe that

$$\left| A|B|^{-1} - B|A|^{-1} \right| = \frac{1}{\sqrt{37}} \begin{bmatrix} \frac{23}{2} & -5 \\ -5 & 4 \end{bmatrix},$$

$$\left| A|A|^{-1} - B|B|^{-1} \right| = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and the matrix $\left| A|B|^{-1} - B|A|^{-1} \right| - \left| A|A|^{-1} - B|B|^{-1} \right|$ has two eigenvalues with different sign. Hence it is not positive and so inequality (16) does not hold.

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