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Bounded Sobriety and k-Bounded Sobriety of Q-Cotopological Spaces

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Abstract.
In this paper, we extend bounded sobriety and k-bounded sobriety to the setting of Q-cotopological spaces, where Q is a commutative and integral quantale. The main results are: (1) The category **BSob$Q$-CTop** of all bounded sober Q-cotopological spaces is a full reflective subcategory of the category **SQ-CTop** of all stratified Q-cotopological spaces; (2) We present the relationships among Hausdorff, T_1, sobriety, bounded sobriety and k-bounded sobriety in the setting of Q-cotopological spaces; (3) For a linearly ordered quantale Q, a topological space X is bounded (resp., k-bounded) sober if and only if the corresponding Q-cotopological space $\omega_Q(X)$ is bounded (resp., k-bounded) sober, where ω_Q : **Top** $\longrightarrow$ **SQ-CTop** is the well-known Lowen functor in fuzzy topology.

1. Introduction

In the classical setting, sobriety of topological spaces can be described in terms of open sets as well as closed sets. A topological space X is sober if each irreducible closed subset of X is the closure of exactly one point in X. Moreover, sobriety of topological spaces can be described via the Papert-Papert-Isbell adjunction $\mathcal{O} \dashv pt$ (see [7, 14]) between the category **Top** of topological spaces and the category **Loc** of locales (see [8]). More precisely, a topological space X is sober if $\eta_X : X \longrightarrow pt(\mathcal{O}(X))$ is a bijection, where η_X denotes the unit of the above adjunction.

Extending the theory of sober spaces to the fuzzy setting is an interesting topic in fuzzy topology. In the fuzzy setting, since the table of truth values is not usually a Boolean algebra, there is no natural way to switch between open sets and closed sets. So, it makes a difference whether we postulate sobriety of fuzzy topological spaces in terms of open sets or in terms of closed sets. On one hand, most of the existing works on sobriety of fuzzy topological spaces extend the frame approach in the classical setting. The main method is to establish a fuzzy counterpart of the Papert-Papert-Isbell adjunction, please see Rodabaugh (see [20]), Zhang and Liu (see [29]), Kotzé (see [9, 10]), Srivastava and Khastgir (see [22]), Pultr and Rodabaugh (see [15–18]), Gutiérrez García, Höhle and de Prada Vicente (see [5]), and Yao (see [24, 25]), etc. On the other hand, Kotzé (see [9, 10]) studied the irreducible-closed-set approach to sobriety of fuzzy topological space, when the table of truth values is a frame with an order reversing involution. Recently, for a commutative

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and integral quantale Q, making use of the fuzzy order between closed sets, Zhang (see [28]) established a theory of sobriety for Q-cotopological spaces based on irreducible closed sets.

In the classical setting, studying generalizations of sobriety is also an interesting topic. Zhao and Fan (see [31]) introduced a weak notion of sobriety, called bounded sobriety. They proved that the category **BSob** of all bounded sober spaces is a full reflective subcategory of the category $\mathbf{Top}_0$ of all T_0 spaces. Furthermore, motivated by the definition of the Scott topology on posets, Zhao and Ho (see [32]) introduced a method of deriving a new topology from a given one by using irreducible sets, in a similar way as one derives the Scott topology on a poset from the Alexandroff topology on the poset. This derived topology leads to a weak notion of bounded sobriety, called k-bounded sobriety. Following the idea of Zhang, the main aim of this paper is to extend bounded sobriety and k-bounded sobriety to the setting of Q-cotopological spaces based on irreducible closed sets.

2. Preliminaries

Throughout the paper, we refer to [1] for category theory, to [21] for quantale theory.

Definition 2.1. ([21]) A *commutative and integral quantale* is a triple $(Q, \&, \leq)$ such that $(Q, \leq)$ is a complete lattice with a bottom element 0 and a top element 1, $(Q, \&, 1)$ is a commutative monoid and $p\&(\bigvee_{j\in J} q_j) = \bigvee_{j\in J}(p\&q_j)$ for all $p \in Q$ and $\{q_j\}_{j\in J} \subseteq Q$.

Since $p\&\_$ preserves arbitrary sups, it has a right adjoint, which we shall denote by $p \rightarrow \_$. Thus $p\&q \leq r \Longleftrightarrow q \leq p \rightarrow r$ for all $p, q, r \in Q$.

From now on, unless otherwise stated, Q always denotes a commutative and integral quantale. In fact, a commutative and integral quantale is just a complete residuated lattice (see [2]). Let X be a set. Q^X denotes the set of all Q-subsets of X, that is, the set of all maps from X to Q. Clearly, Q^X is a complete lattice under the pointwise order.

Proposition 2.2. ([21]) *Let Q be a quantale. Then the following statements hold:*
(1) $1 \rightarrow p = p$;
(2) $p \leq q \Longleftrightarrow 1 = p \rightarrow q$;
(3) $p \rightarrow (q \rightarrow r) = (p\&q) \rightarrow r$;
(4) $p\&(p \rightarrow q) \leq q$;
(5) $(\bigvee_{j\in J} p_j) \rightarrow q = \bigwedge_{j\in J}(p_j \rightarrow q)$;
(6) $p \rightarrow (\bigwedge_{j\in J} q_j) = \bigwedge_{j\in J}(p \rightarrow q_j)$.

Definition 2.3. ([2, 23]) Let X be a set. A map $R : X \times X \longrightarrow Q$ is called a *Q-preorder* on X if for all $x, y, z \in X$,
(1) $R(x, x) = 1$ (reflexivity);
(2) $R(x, y)\&R(y, z) \leq R(x, z)$ (transitivity).

A Q-preorder R on a set X is called a *Q-order* on X if for all $x, y \in X$, $R(x, y) = R(y, x) = 1$ implies $x = y$ (antisymmetry). The pair (X, R) is called a *Q-preordered set* (resp., *Q-ordered set)* if R is a Q-preorder (resp., Q-order) on X. For convenience, we often write simply X for a Q-preordered set (X, R) and $X(x, y)$ for $R(x, y)$ if no confusion would arise.

Let X be a set. The map $sub_X : Q^X \times Q^X \longrightarrow Q$ is defined by $sub_X(A, B) = \bigwedge_{x\in X}(A(x) \rightarrow B(x))$. Then sub_X is a Q-preorder on Q^X. In particular, if X is a singleton set, then the Q-preordered set (Q^X, sub_X) reduces to the Q-preordered set (Q, e_Q), where $e_Q(p, q) = p \rightarrow q$. For all $p \in Q, A \in Q^X$, we write $p\&A,\ p \rightarrow A \in Q^X$ for the fuzzy sets given by $(p\&A)(x) = p\&A(x)$ and $(p \rightarrow A)(x) = p \rightarrow A(x)$, respectively.

A *fuzzy upper (resp., lower) set* in a Q-preordered set X is a map $\varphi : X \longrightarrow Q$ such that $X(x, y)\&\varphi(x) \leq \varphi(y)$ (resp., $X(x, y)\&\varphi(y) \leq \varphi(x)$) for all $x, y \in X$. A map $f : X \longrightarrow Y$ between Q-preordered sets is *order-preserving* if $X(x, y) \leq Y(f(x), f(y))$ for all $x, y \in X$. A pair (f, g) of order-preserving maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$

is called a *Q-adjunction* (see [23, 26]) between Q-preordered sets X and Y, if $Y(f(x), y) = X(x, g(y))$ for all $x \in X$, $y \in Y$. In this case, f is called the *left adjoint* of g and dually g the *right adjoint* of f.

Let X, Y be sets and $f : X \longrightarrow Y$ a map. Then the *Zadeh forward power set operator* $f^{\rightarrow} : Q^X \longrightarrow Q^Y$ and the *Zadeh backward power set operator* $f^{\leftarrow} : Q^Y \longrightarrow Q^X$ are defined, respectively, by

$$f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x), \quad f^{\leftarrow}(B) = B \circ f$$

for all $A \in Q^X, y \in Y$ and $B \in Q^Y$. It can be easily seen that $(f^{\rightarrow}, f^{\leftarrow})$ is a Q-adjunction between (Q^X, sub_X) and (Q^Y, sub_Y).

Definition 2.4. ([28]) A *Q-topology* on a set X is a subset δ of Q^X such that

(O1) $p_X \in \delta$ for all $p \in Q$;

(O2) $A \wedge B \in \delta$ for all $A, B \in \delta$;

(O3) $\bigvee_{j\in J} A_j \in \delta$ for all $\{A_j\}_{j\in J} \subseteq \delta$.

The pair (X, δ) is called a *Q-topological space*. Note that a Q-topological space in this paper is also called a weakly stratified Q-topological space in [6]. A Q-topology δ on X is *stratified* (see [6]) if $p\&A \in \delta$ for all $p \in Q, A \in \delta$.

It is observed in [6, 27] that a Q-topology δ on a set X is stratified if and only if its corresponding interior operator $\mathcal{K}$ satisfying $sub_X(A, B) \leq sub_X(\mathcal{K}(A), \mathcal{K}(B))$ for all $A, B \in Q^X$, where $\mathcal{K} : Q^X \longrightarrow Q^X$ is defined by $\mathcal{K}(A) = \bigvee\{B \in \delta | B \leq A\}$.

Definition 2.5. ([28]) A *Q-cotopology* on a set X is a subset τ of Q^X such that

(C1) $p_X \in \tau$ for all $p \in Q$;

(C2) $A \vee B \in \tau$ for all $A, B \in \tau$;

(C3) $\bigwedge_{j\in J} A_j \in \tau$ for all $\{A_j\}_{j\in J} \subseteq \tau$.

The pair (X, τ) is called a *Q-cotopological space*, elements in τ are called *closed sets* of (X, τ). A Q-cotopology τ is *stratified* if

(C4) $p \rightarrow A \in \tau$ for all $p \in Q, A \in \tau$.

We often write X for a Q-cotopological space (X, τ). A map $f : X \longrightarrow Y$ between Q-cotopological spaces is *continuous* if $f^{\leftarrow}(A) = A \circ f$ is closed in X whenever A is closed in Y. Let **SQ-CTop** denote the category of stratified Q-cotopological spaces with continuous maps.

Given a Q-cotopological space (X, τ), its *closure operator* $^- : Q^X \longrightarrow Q^X$ is defined by

$$\overline{A} = \bigwedge\{B \in \tau | A \leq B\}$$

for all $A \in Q^X$. One can check that the following conditions hold:

(cl1) $\overline{p_X} = p_X$ for all $p \in Q$;

(cl2) $A \leq \overline{A}$ for all $A \in Q^X$;

(cl3) $\overline{A \vee B} = \overline{A} \vee \overline{B}$ for all $A, B \in Q^X$;

(cl4) $\overline{\overline{A}} = \overline{A}$ for all $A \in Q^X$.

Proposition 2.6. ([28]) *Let (X, τ) be a Q-cotopological space. The following statements are equivalent:*

(1) *X is stratified;*

(2) *$p\&\overline{A} \leq \overline{p\&A}$ for all $p \in Q$ and $A \in Q^X$;*

(3) *The closure operator $^- : (Q^X, sub_X) \longrightarrow (Q^X, sub_X)$ is order-preserving.*

It follows immediately from Proposition 2.6(3) that if X is a stratified Q-cotopological space and B is a closed set in X, then $sub_X(A,B) = sub_X(\overline{A},B)$ for all $A \in Q^X$.

Let (X,τ) be a Q-cotopological space. Define $\Omega(\tau) : X \times X \longrightarrow Q$ by

$$\Omega(\tau)(x,y) = \bigwedge_{A\in\tau}(A(y) \to A(x)).$$

Then $\Omega(\tau)$ is a Q-preorder on X, called the *specialization Q-preorder* of (X,τ). Clearly, each closed set in (X,τ) is a fuzzy lower set in the Q-preordered set $(X,\Omega(\tau))$.

We often write $\Omega(X)$ for the Q-preordered set obtained by equipping X with its specialization Q-preorder.

Proposition 2.7. ([19]) *Let X be a stratified Q-cotopological space. Then $\Omega(X)(x,y) = \overline{1_y}(x)$ for all $x,y \in X$.*

3. Bounded Sober Q-Cotopological Spaces

Let X be a topological space. A closed set F in X is *irreducible* if it is non-empty and for all closed sets A,B in X, $F \subseteq A \cup B$ implies either $F \subseteq A$ or $F \subseteq B$. The *specialization preorder* $\leq$ on X is defined by $x \leq y$ if and only if $x \in cl(\{y\})$, where $cl(\{y\})$ denotes the closure of $\{y\}$. If X is T_0, then $\leq$ is a partial order, called the *specialization order*. A closed set F in X is *bounded* if there exists $x \in X$ such that $F \subseteq cl(\{x\})$. A topological space is *bounded sober* if each bounded irreducible closed set in it is the closure of exactly one point. Bounded sobriety is an interesting property in non-Hausdorff topology and domain theory. Now we hope to extend the theory of bounded sober spaces to the setting of Q-cotopological spaces.

Definition 3.1. ([28]) Let X be a Q-cotopological space. A closed set F in X is *irreducible* if $\bigvee_{x\in X} F(x) = 1$ and $sub_X(F, A \vee B) = sub_X(F,A) \vee sub_X(F,B)$ for all closed sets A,B in X.

Definition 3.2. Let X be a Q-cotopological space. An irreducible closed set F in X is *bounded* if there exists $x \in X$ such that $F \leq \overline{1_x}$.

Example 3.3. Let X be a stratified Q-cotopological space. Then $\overline{1_x}$ is a bounded irreducible closed set for each $x \in X$.

Definition 3.4. A stratified Q-cotopological space X is called *bounded sober* if every bounded irreducible closed set in X is the closure of 1_x for a unique $x \in X$.

A stratified Q-cotopological space X is called *sober* if every irreducible closed set in X is the closure of 1_a for a unique $a \in X$. A Q-cotopological space X is said to be T_0 if $x \neq y$ implies $\overline{1_x} \neq \overline{1_y}$ for all $x,y \in X$. A Q-cotopological space X is called T_1 if $\overline{1_x} = 1_x$ for each $x \in X$.

Remark 3.5. (1) Every bounded sober Q-cotopological space is T_0.

(2) Every sober Q-cotopological space is clearly bounded sober, but the converse may not be true, please see Example 3.14.

(3) A sober Q-cotopological space may not be T_1. For example, let $Q = ([0,1], \&)$ with & being the Lukasiewicz t-norm. The Alexandroff Q-cotopology τ_{AL} on the Q-preordered set $([0,1], d_R)$ is the Q-cotopology consisting of all its fuzzy lower sets, where $d_R(x,y) = y \to x$ for all $x,y \in [0,1]$. Then $([0,1], \tau_{AL})$ is sober (see Proposition 4.3 in [28]), but not T_1. In fact, for each $x \in X$, since $\overline{1_x} = x \to id$ (see (F2) of Section 4 in [28]), we have that $\overline{1_x}(y) = x \to y$ for all $y \in [0,1]$. When $x < y$, $\overline{1_x}(y) = 1$, but $1_x(y) = 0$. Then $\overline{1_x} \neq 1_x$, and thus $([0,1], \tau_{AL})$ is not T_1.

Proposition 3.6. *Let X be a stratified T_1 Q-cotopological space. Then it is bounded sober.*

Proof. Suppose that F is a bounded irreducible closed set in X. Then there exists $x \in X$ such that $F \leq \overline{1_x} = 1_x$. For all $y \in X$, if $y \neq x$, then $F(y) \leq 1_x(y) = 0$. Since F is irreducible, $\bigvee_{y\in X} F(y) = F(x) = 1$. Thus, $F = 1_x = \overline{1_x}$. □

A Q-cotopological space X is *Hausdorff* if the diagonal $\triangle : X \times X \longrightarrow Q$, given by

$$\triangle(x,y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

is a closed set in the product space $X \times X$. If Q is a linearly ordered quantale, then every stratified Hausdorff Q-cotopological space is sober (see Proposition 3.11 in [28]) and T_1.

Let **BSob$Q$-CTop** denote the category of bounded sober Q-cotopological spaces with continuous maps. Given a stratified Q-cotopological space X, let $B(X)$ denote the set of all bounded irreducible closed sets in X. For each closed set F in X, define a map $K_F : B(X) \longrightarrow Q$ by

$$K_F(A) = sub_X(A,F).$$

Lemma 3.7. *Let (X,τ) be a stratified Q-cotopological space. Then the following statements hold:*
(1) $K_{p_X}(A) = p$ *for all* $p \in Q$ *and* $A \in B(X)$;
(2) $K_{F_1} \vee K_{F_2} = K_{F_1 \vee F_2}$ *for all* $F_1, F_2 \in \tau$;
(3) $\bigwedge_{j\in J} K_{F_j} = K_{\bigwedge_{j\in J} F_j}$ *for all* $\{F_j\}_{j\in J} \subseteq \tau$;
(4) $K_{p\to F} = p \to K_F$ *for all* $p \in Q$ *and all* $F \in \tau$;
(5) $sub_X(F_1,F_2) = sub_{B(X)}(K_{F_1},K_{F_2})$ *for all* $F_1, F_2 \in \tau$.

Proof. (1) $K_{p_X}(A) = sub_X(A,p_X) = \bigwedge_{x\in X}(A(x) \to p_X(x)) = 1 \to p = p$.
(2) $K_{F_1\vee F_2}(A) = sub_X(A, F_1 \vee F_2) = sub_X(A,F_1) \vee sub_X(A,F_2) = K_{F_1}(A) \vee K_{F_2}(A)$.
(3) $K_{\bigwedge_{j\in J} F_j}(A) = sub_X(A, \bigwedge_{j\in J} F_j) = \bigwedge_{j\in J} sub_X(A,F_j) = \bigwedge_{j\in J} K_{F_j}$.
(4) $K_{p\to F}(A) = sub_X(A, p \to F) = \bigwedge_{x\in X}(A(x) \to (p \to F(x))) = p \to K_F(A)$.
(5) On one hand, since $sub_X(A,F_1) \& sub_X(F_1,F_2) \leq sub_X(A,F_2)$,

$$sub_X(F_1,F_2) \leq \bigwedge_{A\in B(X)} (sub_X(A,F_1) \to sub_X(A,F_2)) = sub_{B(X)}(K_{F_1},K_{F_2}).$$

On the other hand,

$$\begin{aligned} sub_{B(X)}(K_{F_1},K_{F_2}) &= \bigwedge_{A\in B(X)} (K_{F_1}(A) \to K_{F_2}(A)) \\ &= \bigwedge_{A\in B(X)} (sub_X(A,F_1) \to sub_X(A,F_2)) \\ &\leq \bigwedge_{x\in X} (sub_X(\overline{1_x},F_1) \to sub_X(\overline{1_x},F_2)) \\ &= \bigwedge_{x\in X} (sub_X(1_x,F_1) \to sub_X(1_x,F_2)) \\ &= \bigwedge_{x\in X} (F_1(x) \to F_2(x)) \\ &= sub_X(F_1,F_2). \end{aligned}$$

This completes the proof. □

By the above lemma, $\{K_F \mid F$ is a closed set in $X\}$ is a stratified Q-cotopology on $B(X)$. We write $B(X)$ for the resulting Q-cotopological space.

Theorem 3.8. *Let X be a stratified Q-cotopological space. Then $B(X)$ is bounded sober.*

Proof. Suppose that K_F is a bounded irreducible closed set in $B(X)$. Since K_F is the closure of 1_F in $B(X)$, it suffices to prove that $F \in B(X)$. We prove this conclusion in three steps.
(1) Since K_F is a bounded irreducible closed set in $B(X)$, then

$$\bigvee_{A\in B(X)} K_F(A) = \bigvee_{A\in B(X)} sub_X(A,F) = 1.$$

For each $A \in B(X), x \in X$,

$$A(x)\&sub_X(A,F) = A(x)\&\bigwedge_{z\in X}(A(z) \to F(z)) \leq F(x),$$

it follows that

$$\bigvee_{x\in X} F(x) \geq \bigvee_{x\in X}\bigvee_{A\in B(X)} A(x)\&sub_X(A,F) = \bigvee_{A\in B(X)} sub_X(A,F) = 1.$$

(2) For all closed sets B, C in X, since K_F is a bounded irreducible closed set in $B(X)$, then

$$\begin{aligned} sub_X(F, B \vee C) &= sub_{B(X)}(K_F, K_{B\vee C}) \\ &= sub_{B(X)}(K_F, K_B \vee K_C) \\ &= sub_{B(X)}(K_F, K_B) \vee sub_{B(X)}(K_F, K_C) \\ &= sub_X(F,B) \vee sub_X(F,C). \end{aligned}$$

(3) Since K_F is a bounded irreducible closed set in $B(X)$, there exists $A \in B(X)$ such that $K_F \leq \overline{1_A}$. It follows from Proposition 2.7 that $K_A = \overline{1_A}$. By Lemma 3.7(5), we conclude that

$$sub_X(F,A) = sub_{B(X)}(K_F, K_A) = sub_{B(X)}(K_F, \overline{1_A}) = 1.$$

So $F \leq A$. Since A is bounded, it follows that F is bounded. Thus $F \in B(X)$ and $K_F = \overline{1_F}$. □

Proposition 3.9. *Let X be a stratified Q-cotopological space. Define*

$$\kappa : X \longrightarrow B(X)$$

by $\kappa(x) = \overline{1_x}$. *Then*

(1) $\kappa : X \longrightarrow B(X)$ *is continuous;*

(2) *X is bounded sober if and only if* κ *is a homeomorphism.*

Proof. (1) Let F be a closed set in X. For all $x \in X$,

$$\kappa^{\leftarrow}(K_F)(x) = K_F(\overline{1_x}) = sub_X(\overline{1_x}, F) = sub_X(1_x, F) = F(x).$$

Then $\kappa^{\leftarrow}(K_F) = F$, and thus κ is continuous.

(2) Sufficiency. By Theorem 3.8, it is clear.

Necessity. Since X is bounded sober, for all $A \in B(X)$, there exists a unique $x \in X$ such that $A = \overline{1_x} = \kappa(x)$. Then κ is a bijection. Let F be a closed set in X. For all $x \in X$, we have that

$$\kappa^{\rightarrow}(F)(\overline{1_x}) = \bigvee_{\kappa(y)=\overline{1_x}} F(y) = F(x) = K_F(\overline{1_x}).$$

This shows that κ is a continuous closed bijection, hence a homeomorphism. □

Lemma 3.10. *Let X, Y be stratified Q-cotopological spaces,* $f : X \longrightarrow Y$ *be a continuous map. Then* $\overline{f^{\rightarrow}(A)} \in B(Y)$ *for all* $A \in B(X)$.

Proof. Let F_1, F_2 be closed sets in Y.

$$\begin{aligned} sub_Y(\overline{f^{\rightarrow}(A)}, F_1 \vee F_2) &= sub_Y(f^{\rightarrow}(A), F_1 \vee F_2) \\ &= sub_X(A, f^{\leftarrow}(F_1 \vee F_2)) \\ &= sub_X(A, f^{\leftarrow}(F_1)) \vee sub_X(A, f^{\leftarrow}(F_2)) \\ &= sub_Y(\overline{f^{\rightarrow}(A)}, F_1) \vee sub_Y(\overline{f^{\rightarrow}(A)}, F_2). \end{aligned}$$

Since A is bounded, there exists $x \in X$ such that $A \leq \overline{1_x}$. As f is continuous, it preserves the specialization Q-preorder. For all $y \in Y$, by Proposition 2.7, we conclude that

$$\begin{array}{rcl} f^{\rightarrow}(A)(y) & = & \bigvee\limits_{f(z)=y} A(z) \\ & \leq & \bigvee\limits_{f(z)=y} \overline{1_x}(z) \\ & = & \bigvee\limits_{f(z)=y} \Omega(X)(z,x) \\ & \leq & \bigvee\limits_{f(z)=y} \Omega(Y)(f(z),f(x)) \\ & = & \Omega(Y)(y,f(x)) \\ & = & \overline{1_{f(x)}}(y). \end{array}$$

This means $f^{\rightarrow}(A) \leq \overline{1_{f(x)}}$. Then $\overline{f^{\rightarrow}(A)} \leq \overline{1_{f(x)}}$, and thus $\overline{f^{\rightarrow}(A)} \in B(Y)$. □

Theorem 3.11. **BSob$Q$-CTop** *is a full reflective subcategory of* **SQ-CTop**.

Proof. Let X be a stratified Q-cotopological space. We shall prove that $\kappa : X \longrightarrow B(X)$ is universal. Suppose that Y is a bounded sober Q-cotopological space and $f : X \longrightarrow Y$ is a continuous map. We need to prove that there exists a unique continuous map $f^* : B(X) \longrightarrow Y$, such that $f = f^* \circ \kappa$.

(1) Existence. For all $A \in B(X)$, by Lemma 3.10, $\overline{f^{\rightarrow}(A)} \in B(Y)$. Since Y is bounded sober, there is a unique $y \in Y$ such that $\overline{f^{\rightarrow}(A)} = \overline{1_y}$. Define $f^* : B(X) \longrightarrow Y$ by

$$f^*(A) = y.$$

Clearly, f^* is well defined. For each closed set B in Y, since f is continuous, $f^{\leftarrow}(B)$ is a closed set in X. We conclude that

$$\begin{array}{rcl} K_{f^{\leftarrow}(B)}(A) & = & sub_X(A, f^{\leftarrow}(B)) \\ & = & sub_Y(f^{\rightarrow}(A), B) \\ & = & sub_Y(\overline{f^{\rightarrow}(A)}, B) \\ & = & sub_Y(\overline{1_{f^*(A)}}, B) \\ & = & B(f^*(A)) \\ & = & (f^*)^{\leftarrow}(B)(A). \end{array}$$

Then $(f^*)^{\leftarrow}(B) = K_{f^{\leftarrow}(B)}$, and thus f^* is continuous. For all $x \in X$, since $1_{f(x)} \leq f^{\rightarrow}(\overline{1_x}) \leq \overline{f^{\rightarrow}(1_x)} = \overline{1_{f(x)}}$, it follows that $\overline{1_{f(x)}} = \overline{f^{\rightarrow}(\overline{1_x})}$. Then $f = f^* \circ \kappa$.

(2) Uniqueness. If $g : B(X) \longrightarrow Y$ is a continuous map such that $f = g \circ \kappa$. For all $A \in B(X)$, it suffices to prove that $\overline{f^{\rightarrow}(A)} = \overline{1_{g(A)}}$. On one hand, for all $x \in X$,

$$\begin{array}{rcl} A(x) & = & sub_X(\overline{1_x}, A) \\ & = & \Omega(B(X))(\kappa(x), A) \\ & \leq & \Omega(Y)(g(\kappa(x)), g(A)) \\ & = & \Omega(Y)(f(x), g(A)) \\ & = & \overline{1_{g(A)}}(f(x)), \end{array}$$

Then $f^{\rightarrow}(A) \leq \overline{1_{g(A)}}$, and thus $\overline{f^{\rightarrow}(A)} \leq \overline{1_{g(A)}}$. On the other hand, we first show that $\overline{\kappa^{\rightarrow}(A)} = K_A$. Since $\kappa^{\rightarrow}(A) \leq K_A$, we have that $\overline{\kappa^{\rightarrow}(A)} \leq K_A$. Conversely, since $\overline{\kappa^{\rightarrow}(A)}$ is a closed set in $B(X)$, there is a closed set F in X such that $\overline{\kappa^{\rightarrow}(A)} = K_F$. For each $x \in X$,

$$A(x) \leq \kappa^{\rightarrow}(A)(\overline{1_x}) \leq K_F(\overline{1_x}) = F(x).$$

Then $K_A \leq K_F = \overline{\kappa^{\rightarrow}(A)}$. It follows that

$$\overline{f^{\rightarrow}(A)}(g(A)) = \overline{g^{\rightarrow} \circ \kappa^{\rightarrow}(A)}(g(A)) \geq g^{\rightarrow}(K_A)(g(A)) \geq K_A(A) = 1.$$

Then $\overline{f^{\rightarrow}(A)} \geq \overline{1_{g(A)}}$, and thus $\overline{f^{\rightarrow}(A)} = \overline{1_{g(A)}}$. □

In [28], Zhang proved that the category **Sob***Q***-CTop** of all sober Q-cotopological spaces is a full reflective subcategory of **S***Q***-CTop**. Obviously, **Sob***Q***-CTop** is a full subcategory of **BSob***Q***-CTop**. By Theorem 3.11, we can easily obtain that **Sob***Q***-CTop** is a full reflective subcategory of **BSob***Q***-CTop**.

An element a in a lattice L is a *coprime* if for all $b, c \in L$, $a \leq b \vee c$ implies that either $a \leq b$ or $a \leq c$. A complete lattice L is said to be have enough coprimes if each element in L can be written as the join of a set of coprimes. Clearly, every linearly ordered quantale has enough coprimes. We say that a quantale Q has enough coprimes if the complete lattice Q has enough coprimes.

Let Q be a quantale and X be a topological space. We say that a map $\lambda : X \longrightarrow Q$ is *upper semicontinuous* if for all $p \in Q$, $\lambda_{[p]} = \{x \in X \mid \lambda(x) \geq p\}$ is a closed set in X. If Q is a quantale with enough coprimes, then for each topological space X, the set of upper semicontinuous maps $X \longrightarrow Q$ forms a stratified Q-cotopology on X(see [28]). We write $\omega_Q(X)$ for the resulting stratified Q-cotopological space.

Let X be a topological space. For each closed set $A \in X$, $1_A : X \longrightarrow Q$ is obviously upper semicontinuous. Then every closed set in X is also a closed set in $\omega_Q(X)$. Moreover, for each $A \subseteq X$, $\overline{1_A} = 1_{cl\{A\}}$, where $cl\{A\}$ is the closure of A in X. The following example illustrates that a stratified T_1 Q-cotopological space may not be sober.

Example 3.12. Let Q be a linearly ordered quantale and X be an infinite set. If $(X, \mathcal{T})$ is a finite complement space, then $(X, \mathcal{T})$ is a T_1 space, but not a sober space. Thus the stratified Q-cotopological space $\omega_Q(X)$ is T_1, but not sober.

The correspondence $X \longmapsto \omega_Q(X)$ defines an *embedding functor* ω_Q : **Top** $\longrightarrow$ **S***Q***-CTop**. This functor is one of the well-known Lowen functors in fuzzy topology (see [13]). The following proposition presents that for a linearly ordered quantale Q, the notion of bounded sobriety for Q-cotopological spaces is a good extension in the sense of Lowen.

Proposition 3.13. *Let Q be a linearly ordered quantale. Then a topological space X is bounded sober if and only if the stratified Q-cotopological space $\omega_Q(X)$ is bounded sober.*

Proof. Sufficiency. Let K be a bounded irreducible closed set in X. Then 1_K is a bounded irreducible closed set in $\omega_Q(X)$. In fact, by the proof of Proposition 3.14 in [28], 1_K is an irreducible closed set in $\omega_Q(X)$. It suffices to prove that 1_K is bounded. Since K is bounded, there exists $x \in X$ such that $K \subseteq cl(\{x\})$. Thus $1_K \leq 1_{cl(\{x\})} = \overline{1_x}$. Since $\omega_Q(X)$ is bounded sober, there is a unique $z \in X$ such that $1_K = \overline{1_z} = 1_{cl(\{z\})}$. Hence $K = cl(\{z\})$.
Necessity. Let λ be a bounded irreducible closed set in $\omega_Q(X)$. By the proof of Proposition 3.14 in [28], we have that $\lambda = 1_K$ for some irreducible closed set K in X. Since λ is bounded, there is $x \in X$ such that $\lambda \leq \overline{1_x} = 1_{cl(\{x\})}$. Then $K \subseteq cl(\{x\})$, and thus K is a bounded irreducible closed set in X. Since X is bounded sober, there is a unique $t \in X$ such that $K = cl(\{t\})$. Hence $\lambda = 1_K = 1_{cl(\{t\})} = \overline{1_t}$. □

Example 3.14. Let Q be a linearly ordered quantale and $\mathbb{N}$ be the set of natural numbers. The *Alexandroff topology* $\Upsilon(\mathbb{N})$ on $\mathbb{N}$ is the topology consisting of all its upper subsets. Then $(\mathbb{N}, \Upsilon(\mathbb{N}))$ is a bounded sober space (see Proposition 2 in [31]). By Proposition 3.13, the stratified Q-cotopological space $\omega_Q(\mathbb{N})$ is bounded sober. We can check that a map $f : \mathbb{N} \longrightarrow Q$ is upper semicontinuous if and only if it is antitone. Define a map $g : \mathbb{N} \longrightarrow Q$ by $g(x) = 1$. Then g is an irreducible closed set in $\omega_Q(\mathbb{N})$, but it is not the closure of 1_a for any $a \in \mathbb{N}$.

4. k-Bounded Sober Q-Cotopological Spaces

A topological space X is called *k-bounded sober* if for every irreducible closed set F whose supremum exists, there is a unique point $x \in X$ such that $F = cl(\{x\})$. It is weaker than bounded sobriety. k-bounded sobriety plays an important role in non-Hausdorff topology and domain theory, for instance, for each poset P, upper topology on P is k-bounded sober. The aim of this section is to generalize k-bounded sobriety to the setting of Q-cotopological spaces.

Definition 4.1. ([11]) Let X be a Q-preordered set and $A \in Q^X$. A *supremum* of A in X is an element $\sup A$ in X such that $X(\sup A, x) = \bigwedge_{y\in X} (A(y) \to X(y, x))$ for all $x \in X$.

Definition 4.2. A stratified Q-cotopological space X is called *k-bounded sober*, if every irreducible closed set whose supremum exists in X under the specialization Q-preorder is the closure of 1_x for a unique $x \in X$.

Remark 4.3. (1) Every k-bounded sober Q-cotopological space is T_0.

(2) Every bounded sober Q-cotopological space is clearly k-bounded sober, but the converse may not be true, please see Example 4.10.

Let X be a Q-preordered set. The stratified Q-cotopology on X generated by $\{X(-, x) \mid x \in X\}$ is called the *upper Q-cotopology* on X, denoted by τ_ν.

Proposition 4.4. *For each Q-ordered set X, (X, τ_ν) is k-bounded sober.*

Proof. Let F be an irreducible closed set in X with $\sup F$ existing. Then $sub_X(F, A) \leq A(\sup F)$ for all $A \in \tau_\nu$. In fact, it suffices to prove $sub_X(F, A) \leq A(\sup F)$ for the case of $A = X(-, x)$ for all $x \in X$. Obviously,

$$sub_X(F, X(-, x)) = \bigwedge_{y\in X}(F(y) \to X(y, x)) = X(\sup F, x).$$

Since F is an irreducible closed set in X, we have that $F(\sup F) \geq sub_X(F, F) = 1$. Then $F = \overline{1_{\sup F}}$. Assume that there exists $b \in X$ such that $F = \overline{1_b}$. Then $\overline{1_{\sup F}} = \overline{1_b}$. Since the antisymmetry of the Q-order, $\sup F = b$. Thus τ_ν is k-bounded sober. □

Let X be a stratified Q-cotopological space and $KB(X)$ denote the set of all irreducible closed sets in X whose suprema exist. Clearly, $\overline{1_x} \in KB(X)$. For each closed set F in X, define a map $H_F : KB(X) \longrightarrow Q$ by

$$H_F(A) = sub_X(A, F).$$

Lemma 4.5. *Let (X, τ) be a stratified Q-cotopological space. Then the following statements hold:*

(1) $H_{p_X}(A) = p$ *for all $p \in Q$ and $A \in KB(X)$;*

(2) $H_{F_1} \vee H_{F_2} = H_{F_1\vee F_2}$ *for all $F_1, F_2 \in \tau$;*

(3) $\bigwedge_{j\in J} H_{F_j} = H_{\bigwedge_{j\in J} F_j}$ *for all $\{F_j\}_{j\in J} \subseteq \tau$;*

(4) $H_{p\to F} = p \to H_F$ *for all $p \in Q$ and all $F \in \tau$;*

(5) $sub_X(F_1, F_2) = sub_{KB(X)}(H_{F_1}, H_{F_2})$ *for all $F_1, F_2 \in \tau$.*

Proof. Similar to the proof of Lemma 3.7. □

By the above lemma, $\{H_F \mid F$ is a closed set in $X\}$ is a stratified Q-cotopology on $KB(X)$. We write $KB(X)$ for the resulting Q-cotopological space.

Lemma 4.6. *Let (X, τ) be a stratified Q-cotopological space and $A \in KB(X)$. Then $H_A = \overline{1_A}$.*

Proof. For all $B \in KB(X)$, by Proposition 2.7,

$$\begin{aligned} H_A(B) &= sub_X(B,A) \\ &= \bigwedge_{F\in\tau}(sub_X(A,F) \to sub_X(B,F)) \\ &= \bigwedge_{F\in\tau}(H_F(A) \to H_F(B)) \\ &= \Omega(KB(X))(B,A) \\ &= \overline{1_A}(B). \end{aligned}$$

This means $H_A = \overline{1_A}$. □

Theorem 4.7. *Let X be a stratified Q-cotopological space. Then $KB(X)$ is k-bounded sober.*

Proof. Suppose that H_F is an irreducible closed set in $KB(X)$, whose supremum exists. Denote this supremum $\sup H_F$ by some $E \in KB(X)$. By the proof of Theorem 3.8, F is an irreducible closed set in X. Next we shall prove that $\sup E$ is a supremum of F. On one hand, since $\sup H_F = E$, for all $A \in KB(X)$,

$$H_F(A) \le \Omega(KB(X))(A,E) \le H_E(E) \to H_E(A) = H_E(A).$$

By Lemma 4.5(5), $sub_X(F,E) = sub_{KB(X)}(H_F, H_E) = 1$. Then $F(x) \le E(x) \le \Omega(X)(x, \sup E)$ for all $x \in X$, and thus $\Omega(X)(\sup E, y) \le \bigwedge_{x\in X}(F(x) \to \Omega(X)(x,y))$. On the other hand, since $\sup H_F = E$, for all B in $KB(X)$,

$$\bigwedge_{A\in KB(X)} (H_F(A) \to \Omega(KB(X))(A,B)) = \Omega(KB(X))(E,B).$$

This means $\bigwedge_{A\in KB(X)} (sub_X(A,F) \to sub_X(A,B)) = sub_X(E,B)$. Then for all $y \in X$,

$$\begin{aligned} \bigwedge_{x\in X}(F(x) \to \Omega(X)(x,y)) &= \bigwedge_{x\in X}(F(x) \to \overline{1_y}(x)) \\ &= sub_X(F,\overline{1_y}) \\ &\le \bigwedge_{A\in KB(X)} (sub_X(A,F) \to sub_X(A,\overline{1_y})) \\ &= sub_X(E,\overline{1_y}) \\ &= \bigwedge_{x\in X}(E(x) \to \Omega(X)(x,y)) \\ &= \Omega(X)(\sup E, y). \end{aligned}$$

Thus $\bigwedge_{x\in X}(F(x) \to \Omega(X)(x,y)) = \Omega(X)(\sup E, y)$. It follows that $\sup E$ is a supremum of F. Therefore $F \in KB(X)$. By Lemma 4.6, $H_F = \overline{1_F}$. Let $B \in KB(X)$ with $H_F = \overline{1_B}$. Then $sub_X(B,F) = \overline{1_F}(B) = \overline{1_B}(B) = 1$ and $sub_X(F,B) = \overline{1_B}(F) = \overline{1_F}(F) = 1$. Thus $B = F$. □

Proposition 4.8. *Let X be a stratified Q-cotopological space. Define*

$$\iota : X \longrightarrow KB(X)$$

by $\iota(x) = \overline{1_x}$. Then $\iota : X \longrightarrow KB(X)$ is continuous.

Proof. Similar to the proof of Proposition 3.9(1). □

The following proposition presents that for a linearly ordered quantale Q, the notion of k-bounded sobriety for Q-cotopological spaces is a good extension in the sense of Lowen.

Proposition 4.9. *Let Q be a linearly ordered quantale. Then a topological space X is k-bounded sober if and only if the stratified Q-cotopological space $\omega_Q(X)$ is k-bounded sober.*

Proof. Sufficiency. Let K be an irreducible closed set whose supremum exists under the specialization preorder of X. By Proposition 3.13, we have that 1_K is a bounded irreducible closed set in $\omega_Q(X)$. Next, we shall prove that $\sup 1_K = \bigvee K$. For all $y \in X$,

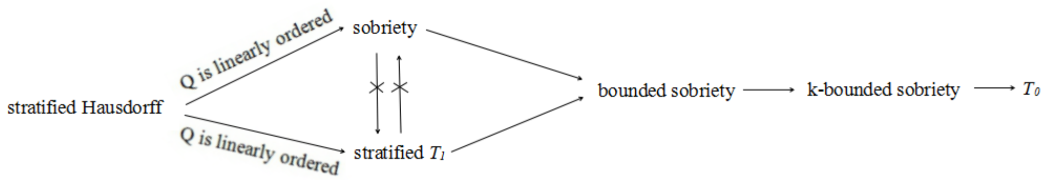
$$\begin{aligned}\bigwedge_{x\in X}(1_K(x) \to \Omega(\omega_Q(X))(x,y)) &= \bigwedge_{x\in K}(1 \to \Omega(\omega_Q(X))(x,y))\\ &= \bigwedge_{x\in K}\Omega(\omega_Q(X))(x,y)\\ &= \Omega(\omega_Q(X))(\bigvee K, y).\end{aligned}$$

Then $\sup 1_K = \bigvee K$, and thus 1_K is an irreducible closed set whose supremum exists in $\omega_Q(X)$. Since $\omega_Q(X)$ is k-bounded sober, there is a unique $x \in X$ such that $1_K = \overline{1_x} = 1_{cl(\{x\})}$. Hence $K = cl(\{x\})$.

Necessity. Let λ be an irreducible closed set in $\omega_Q(X)$, whose supremum exists. By the proof of Proposition 3.13, we have that $\lambda = 1_K$ for some bounded irreducible closed set K in X. Next, we shall prove that $\bigvee K = \sup\lambda$. Since $\lambda(x) \le \Omega(\omega_Q(X))(x, \sup\lambda) = \overline{1_{\sup\lambda}}(x) = 1_{cl(\{\sup\lambda\})}(x)$ for all $x \in X$, $\lambda = 1_K \le \overline{1_{\sup\lambda}} = 1_{cl(\{\sup\lambda\})}$. Then $K \subseteq cl(\{\sup\lambda\})$. Let a be a upper bound of K. Then $K \subseteq cl(\{a\})$, and thus $\lambda = 1_K \le 1_{cl(\{a\})} = \overline{1_a}$. It follows that $\sup\lambda \le a$. Therefore K is an irreducible closed set in X, whose supremum exists. Since X is k-bounded sober, there is a unique $t \in X$ such that $K = cl(\{t\})$. Hence $\lambda = 1_K = 1_{cl(\{t\})} = \overline{1_t}$. □

Example 4.10. Let Q be a linearly ordered quantale and $\mathbb{Q}$ be the poset of all rational numbers with the conventional order. The upper topology $\nu(\mathbb{Q})$ on $\mathbb{Q}$ is the topology generated by sets of the form $X - \downarrow x$ for $x \in \mathbb{Q}$, where $\downarrow x = \{a \in \mathbb{Q} \mid a \le x\}$. Then $(\mathbb{Q}, \nu(\mathbb{Q}))$ is k-bounded sober, but not bounded sober (see Example 4.14 in [32]). By Proposition 3.13 and 4.9, the stratified Q-cotopological space $\omega_Q(\mathbb{Q})$ is k-bounded sober, but not bounded sober.

The following diagram indicates the relationships of Hausdorff, T_1, T_0, sobriety, bounded sobriety, and k-bounded sobriety in the setting of Q-cotopological spaces.



5. Concluding Remarks

In this paper, we prove that **BSob*Q*-CTop** is a full reflective subcategory of **S*Q*-CTop**. Let **KBSob*Q*-CTop** denote the category of k-bounded sober Q-cotopological spaces with continuous maps. A natural question raised here is that whether **KBSob*Q*-CTop** is a full reflective subcategory of **S*Q*-CTop**. In fact, in the classical setting, the above corresponding problem is also unsolved, that is, whether the category **KBSob** of k-bounded sober spaces is a full reflective subcategory of $\mathbf{Top}_0$. In the concluding remarks of [32], Zhao and Ho asked whether $KB(X)$ (the set of all closed irreducible sets of a T_0 space X whose suprema exist) is the canonical k-bounded sobrification of X in the sense of Keimel and Lawson with respect to the map $x \longmapsto cl(\{x\})$. Zhao, Lu and Wang (see [30]) constructed a counterexample to illustrate that $(KB(X), cl)$ is not the canonical k-bounded sobrification of a T_0 space X in the sense of Keimel and Lawson. Although we possess no answer on the above question, we know that $(KB(X), \iota)$ is not universal with respect to I, where I : **KBSob*Q*-CTop**$\longrightarrow$ **S*Q*-CTop** is the inclusion functor.

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