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# Some Numerical Radius Inequalities for Products of Hilbert Space Operators

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**Abstract.** We prove several numerical radius inequalities for products of two Hilbert space operators. Some of our inequalities improve well-known ones. More precisely, we prove that, if $A, B \in \mathbb{B}(\mathscr{H})$ such that $A$ is self-adjoint with $\lambda_1 = \min \lambda_i \in \sigma(A)$ (the spectrum of $A$) and $\lambda_2 = \max \lambda_i \in \sigma(A)$. Then

$$\omega(AB) \leq \|A\|\omega(B) + \left(\|A\| - \frac{|\lambda_1 + \lambda_2|}{2}\right) D_B$$

where $D_B = \inf_{\lambda \in \mathbb{C}} \|B - \lambda I\|$. In particular, if $A > 0$ and $\sigma(A) \subseteq [k\|A\|, \|A\|]$, then

$$\omega(AB) \leq (2-k)\|A\|\,\omega(B).$$

## 1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ be the $C^*$-algebra of all bounded linear operators on Hilbert space $\mathscr{H}$. For $A \in \mathbb{B}(\mathscr{H})$, let $w(A)$ and $\|A\|$, denote the numerical radius and the usual operator norm of $A$, respectively. It is well-known that for every $A \in \mathbb{B}(\mathscr{H})$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1}$$

The first inequality becomes an equality if $A^2 = 0$ . The second inequality becomes an equality if $A$ is normal. Many authors have obtained several refinements and reverse for the inequalities in (1) see e.g., [3, 9, 10]. It has been shown in [6], that if $A \in \mathbb{B}(\mathscr{H})$, then

$$\|A\|^2 \leq D^2{}_A + \omega^2(A). \tag{2}$$

It is well-known to all that the submultiplicative property is not hold for the numerical radius, see [4]. If $A$ and $B$ are bounded linear operators in $\mathbb{B}(\mathscr{H})$, then

$$w(AB) \leq 4w(A)w(B). \tag{3}$$

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In the case $AB = BA$, then

$$\omega(AB) \leq 2\omega(A)\omega(B).$$

If $A$ is an isometry and $AB = BA$, or a unitary operator that commutes with another operator $B$, then

$$w(AB) \leq w(B).$$

Concerning the inequality (3), it is shown in [1] that if $A, B \in \mathbb{B}(\mathscr{H})$, then

$$\omega(AB) \leq \omega(A)(D_B + \|B\|). \tag{4}$$

Also, if $A > 0$, then

$$\omega(AB) \leq \frac{3}{2}\|A\|\,\omega(B). \tag{5}$$

If $A$ and $B$ are operators in $\mathbb{B}(\mathscr{H})$, we write the direct sum $A \oplus B$ for the $2 \times 2$ operator matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, regarded as an operator on $\mathscr{H} \oplus \mathscr{H}$. Thus

$$\omega(A \oplus B) = \max\{\omega(A), \omega(B)\} \tag{6}$$

and

$$\|A \oplus B\| = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\| = \max\{\|A\|, \|B\|\}. \tag{7}$$

Some numerical radius inequalities for certain $2 \times 2$ operator matrices is obtained in [8]. More precisely,

$$\sqrt[2n]{\max(\omega((AB)^n), \omega((BA)^n)} \leq \omega\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{\|A\| + \|B\|}{2} \tag{8}$$

for $n = 1, 2, \ldots$.

For other results and historical comments on the numerical radius see [7].
In Section 2, we introduce some new refinements of numerical radius inequalities for products of two operators and establish reverse for inequality (2). Also, we obtain upper and lower bounds for the numerical radius of the off-diagonal parts of $2 \times 2$ operator matrices.

## 2. Main Results

We use the following lemma due to Dragomir [5] to establish reverse for inequality (2).

**Lemma 2.1.** *For any $a, b \in \mathscr{H}$ and $b \neq 0$, we have*

$$\inf_{\lambda \in \mathbb{C}} \|a - \lambda b\|^2 = \frac{\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2}{\|b\|^2}. \tag{9}$$

**Theorem 2.2.** *If $A \in \mathbb{B}(\mathscr{H})$, then*

$$D^2{}_A + m^2(A) \leq \|A\|^2,$$

*where $m(A) = \inf\{\, |\langle Tx, x\rangle| : \|x\| = 1 \,\}$.*

*Proof.* Suppose that $x \in \mathscr{H}$ with $\|x\| = 1$. Choose $a = Ax, b = x$ in (9) to give

$$\begin{aligned}\inf_{\lambda\in\mathbb{C}} \|Ax - \lambda x\|^2 &= \|Ax\|^2 - |\langle Ax, x\rangle|^2 \\ &\leq \|A\|^2 - |\langle Ax, x\rangle|^2 \\ &\leq \|A\|^2 - m^2(A)\end{aligned}$$

and so

$$\inf_{\lambda\in\mathbb{C}} \|Ax - \lambda x\|^2 \leq \|A\|^2 - m^2(A).$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ gives

$$D^2{}_A \leq \|A\|^2 - m^2(A).$$

This completes the proof. □

The following result may be stated as well.

**Theorem 2.3.** *If $A, B \in \mathbb{B}(\mathscr{H})$, then*

$$\omega(AB) \leq \omega(A)\omega(B) + D_A D_B. \tag{10}$$

*Proof.* On account of [2, Theorem 7], we have

$$|\langle x, y\rangle - \langle x, z\rangle\langle z, y\rangle| \leq \left(\frac{1}{4}|\alpha - \beta|^2 + \left\|x - \frac{\alpha+\beta}{2}z\right\|^2\right)^{\frac{1}{2}}\left(\frac{1}{4}|\lambda - \mu|^2 + \left\|y - \frac{\lambda+\mu}{2}z\right\|^2\right)^{\frac{1}{2}},$$

for all $x, y, z \in \mathscr{H}$ with $\|z\| = 1$, and for every $\alpha, \beta, \lambda, \mu \in \mathbb{C}$. If replace $\lambda = \mu$ and $\alpha = \beta$, then

$$|\langle x, y\rangle - \langle x, z\rangle\langle z, y\rangle| \leq \|x - \alpha z\|\, \|y - \lambda z\|. \tag{11}$$

Choose $z = u$, $y = B^*u$ and $x = Au$ in (11) to give

$$|\langle BAu, u\rangle - \langle Au, u\rangle\langle Bu, u\rangle| \leq \|Au - \alpha u\|\, \|B^*u - \lambda u\|.$$

Taking the supremum over $u \in H$ with $\|u\| = 1$ gives

$$\omega(BA) - \omega(A)\omega(B) \leq \|A - \alpha I\|\, \|B^* - \lambda I\|,$$

which is exactly the desired result. □

Let $R_A$ denote the radius of the smallest disk in the complex plane containing $\sigma(A)$. It is known (see, e.g., [11]) that $D_A = R_A$ for any normal operator $A$. The following corollaries are immediate consequences of Theorem 2.3.

**Corollary 2.4.** *Let $A, B \in \mathbb{B}(\mathscr{H})$ such that $A$ is normal. Then*

$$\omega(AB) \leq \omega(B)(\|A\| + 2R_A).$$

**Corollary 2.5.** *Let $A, B \in \mathbb{B}(\mathscr{H})$. If $D_A \leq k\|A\|$, then*

$$\omega(AB) \leq \omega(A)(2kD_B + \omega(B)) \tag{12}$$

**Remark 2.6.** *In some cases the inequality* (10) *strengthen* (4)*. For example, if $A \in \mathbb{B}(\mathscr{H})$ is an invertble operator and $\|A\|\, \|A^{-1}\| \leq \frac{2\sqrt{3}}{3}$, then*

$$\omega(AB) \leq \omega(A)(D_B + \omega(B)). \tag{13}$$

*Also, for $k \leq \frac{3}{4}$ the inequality* (12) *strengthen* (3).

In the next result we give a new upper and lower bound for $\omega\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$.

**Corollary 2.7.** *If $A, B \in \mathbb{B}(\mathscr{H})$, then*

$$\max(\omega(A), \omega(B)) - D_{T_1} \leq \omega\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \leq \max(\omega(A), \omega(B)) + D_{T_2},$$

*where* $T_1 = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$ *and* $T_2 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

*Proof.* By (6),

$$\begin{aligned} \max(\omega(A), \omega(B)) &= \omega\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\ &= \omega\left(\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right) \qquad \text{(by Theorem 2.3)} \\ &\leq \omega\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} + D_{T_1} \end{aligned}$$

and so

$$\max(\omega(A), \omega(B)) \leq \omega\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} + D_{T_1}. \tag{14}$$

On the other hand,

$$\begin{aligned} \omega\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} &= \omega\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right) \\ &\leq \omega\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + D_{T_2} \end{aligned}$$

and so

$$\omega\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \leq \max(\omega(A), \omega(B)) + D_{T_2}. \tag{15}$$

The result follows from (14) and (15). □

The following result may be stated as well.

**Theorem 2.8.** *Let $A, B \in \mathbb{B}(\mathscr{H})$. If $\lambda_1 = \min \lambda_i \in \sigma(A)$, $\lambda_2 = \max \lambda_i \in \sigma(A)$ and $A$ be a self-adjoint operator, then*

$$\omega(AB) \leq \|A\|\omega(B) + \left(\|A\| - \frac{|\lambda_1 + \lambda_2|}{2}\right) D_B.$$

*Proof.* Since $A$ is a self-adjoint operator, then $D_A = \frac{\lambda_2 - \lambda_1}{2}$. It follows from the inequality (10) that

$$\begin{aligned} \omega(AB) &\leq \|A\|\,\omega(B) + \frac{\lambda_2 - \lambda_1}{2} D_B \\ &= \|A\|\,\omega(B) + \frac{\lambda_2 - \lambda_1 + |\lambda_2 + \lambda_1|}{2} D_B - \frac{|\lambda_2 + \lambda_1|}{2} D_B \end{aligned}$$

and so

$$\omega(AB) \leq \|A\|\,\omega(B) + \left(\max(|\lambda_1|, |\lambda_2|) - \frac{|\lambda_2 + \lambda_1|}{2}\right) D_B. \tag{16}$$

Since $\|A\| = \max(|\lambda_1|, |\lambda_2|)$, the result follows from (16). □

The following corollaries are immediate consequences of Theorem 2.8.

**Corollary 2.9.** *Let $A, B \in \mathbb{B}(\mathscr{H})$ such that $A > 0$, then*

$$\omega(AB) \leq \|A\| \left( \omega(B) + \frac{1}{2} D_B \right).$$

**Corollary 2.10.** *Let $A, B \in \mathbb{B}(\mathscr{H})$. If $A > 0$ and $\sigma(A) \subseteq [k\|A\|, \|A\|]$, then*

$$\omega(AB) \leq (2 - k)\|A\| \, \omega(B). \tag{17}$$

*Proof.* By Corollary 2.8,

$$\begin{aligned} \omega(AB) &\leq \|A\|\omega(B) + \left( \|A\| - \frac{|\lambda_1 + \lambda_2|}{2} \right) D_B \\ &\leq \|A\| \left( \omega(B) + \frac{1-k}{2} D_B \right). \end{aligned}$$

Therefore,

$$\omega(AB) \leq (2 - k)\|A\|\omega(B).$$

This completes the proof. □

In some cases, for $\frac{1}{2} \leq k \leq 1$, the inequality (17) strengthen (5).

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