Filomat 33:7 (2019), 2073–2088
https://doi.org/10.2298/FIL1907073K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ground State Solutions of p-Laplacian Singular Kirchhoff Problem Involving a Riemann-Liouville Fractional Derivative

Mouna Kratou[a,b]

[a] *Departement of mathematics, College of sciences at Dammam, Imam Abdulrahman Bin Faisal University, 31441 Dammam, Kingdom of Saudi Arabia*
[b] *Basic and Applied Scientific Research Center, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, 31441, Dammam, Saudi Arabia.*

Abstract. The purpose of this paper is to study the existence and multiplicity of solutions to the following Kirchhoff equation with singular nonlinearity and Riemann-Liouville Fractional Derivative:

$$(\mathrm{P}_\lambda)\begin{cases} \left(a+b\int_0^T |{}_0D_t^\alpha(u(t))|^p dt\right)^{p-1} {}_tD_T^\alpha\left(\Phi_p({}_0D_t^\alpha u(t))\right) \\ = \frac{\lambda g(t)}{u^\gamma(t)} + f(t,u(t)),\ t\in(0,T); \\ \\ u(0)=u(T)=0, \end{cases}$$

where $a \geq 1,\ b,\ \lambda > 0,\ p > 1$ are constants, $\frac{1}{p} < \alpha \leq 1,\ 0 < \gamma < 1,\ g \in C([0,1])$ and $f \in C^1([0,T]\times\mathbb{R},\mathbb{R})$. Under appropriate assumptions on the function f, we employ variational methods to show the existence and multiplicity of positive solutions of the above problem with respect to the parameter λ.

1. Introduction

The theory of fractional calculus may be used to the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes has had a growing development in fields as physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology (see e.g. [4], [18], [19]). In fact, the subject of fractional differential equations has been gaining more importance and attention in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one may refer to monographs [24, 28] and papers [1, 2, 25–27], and references cited therein. In particular, in the qualitative theory of fractional differential equations, the existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions has attracted great attention.

In the sequel of the above-mentioned works with the new approach to the theory of fractional differential equations, here, in this paper, we investigate the existence of multiple solutions of the Kirchhoff fractional

2010 *Mathematics Subject Classification*. Primary 34A08.; Secondary 34B10, 47H10
Keywords. Riemann Liouville fractional derivative, Kirchhoff-type equations, fibering maps, Nehari manifold, nonlinear singular fractional differential equation, existence of solutions.
Received: 02 July 2018; Accepted: 30 October 2018
Communicated by Maria Alessandra Ragusa
Email address: mmkratou@iau.edu.sa (Mouna Kratou)

problem involving Riemann-Liouville fractional derivative and singular nonlinearity:

$$(\mathrm{P}_\lambda)\begin{cases} \left(a + b\int_0^T |{}_0D_t^\alpha(u(t))|^p dt\right)^{p-1} {}_tD_T^\alpha\left(\Phi_p({}_0D_t^\alpha u(t))\right) \\ = \frac{\lambda g(t)}{u^\gamma(t)} + f(t,u(t)),\ t \in (0,T); \\ \\ u(0) = u(T) = 0, \end{cases}$$

where $a \geq 1,\ b\ \lambda > 0, p > 1$ are constants. $\frac{1}{p} < \alpha \leq 1, 0 < \gamma < 1 < p < r$ and $g \in C([0,T])$, and $\Phi_p : \mathbb{R} \to \mathbb{R}$ is the p-laplacian defined by

$$\Phi_p(s) = |s|^{p-2}s\ (s \neq 0),\ \Phi_p(0) = 0.$$

While $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree $r-1$ that is $f(x,tu) = t^{r-1}f(x,u)$ holds for all $(x,u) \in [0,T] \times \mathbb{R}$. Put $F(x,s) := \int_0^s f(x,t)dt$ and satisfying suitable growth conditions. Precisely, we assume the following:

(H$_1$) $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is homogeneous of degree r that is

$$F(x,tu) = t^r F(x,u)\ (t > 0)\ \text{ for all }\ x \in [0,T],\ u \in \mathbb{R}.$$

(H$_2$) $F^{\pm}(x,u) = max(\pm F(x,u), 0) \neq 0$ for all $u \neq 0$.

Note that, from **(H$_1$)**, f leads to the so-called Euler identity

$$uf(x,u) = rF(x,u)$$

and

$$|F(x,u)| \leq K|u|^r \quad \text{for some constant } K > 0. \tag{1.1}$$

Problem (P_λ) is related to the stationary version of the Kirchhoff equation presented by Kirchhoff in 1883 [23] given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in the above equation have the following meanings: L is the length of the string, h is the area of the cross-section, E is the young modulus of the material, ρ is the mass density and ρ_0 is the initial tension. A feature of problem (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx$, which depends on the average $\frac{1}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx$. Nonlocal effect also finds its applications in biological systems. Moreover, problem (P_λ) has a solid theoretical significance and a sharp physical background. For instance, this problem describes the surface tension of the height of a thin liquid film on a solid surface in lubrication approximation (see [6, 13]).

A parabolic version of problem (1.2) can be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed to be dependent on the energy of the entire system with u being its population density. Alternatively, the movement of particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria), which gives rise to nonlocal parabolic equations. We refer to [30] for details.

Recently, the study of the fractional elliptic equations with regular nolinearities and without a Kirchhoff coefficient has been attracted lot of interest by researchers in nonlinear analysis. The fractional boundary value problem using variational methods has been studied in [3, 5, 7, 16, 17, 20, 21, 31, 32] with references therein. Also, existence and multiplicity results for the Kirchhoff equations with regular nolinearities are shown an always increasing interest. We refer the interested readers to [10–12, 29].

Motivated by above results, in the present paper, we show the existence and multiplicity of nontrivial, non-negative solutions of the Kirchhoff fractional problem involving Riemann-Liouville fractional Derivative and singular nonlinearity (P_λ).

We give below the precise statements of results that we will prove.

Theorem 1.1. *Assume that the hypothesis* ($\mathbf{H}_1$)*-*($\mathbf{H}_3$) *are satisfied. Then, there exists Λ_0 such that for all $\lambda \in (0, \Lambda_0)$, problem possesses at least two nontrivial positive solutions.*

This paper is organized as follows: In Section 2, some definitions and properties on the fractional calculus are presented. The section 3 is devoted to proof some lemmas in preparation for the proof of our main result. While, existence of two solutions (Theorem 1.1) will be presented in sections 4 and 5.

2. Preliminaries

In this section, we give some background theory on the concept of fractional calculus, in particular the Riemann-Liouville operators and results which will used throughout this paper. Let us start by introduce the definition of the fractional integral in the sense of Riemann-Liouville.

Definition 2.1. *(See [1]) Let $[a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$ and u be a real-valued function defined almost everywhere (a.e.) on (a, b). The Riemann-Liouville left-sided and right-sided fractional integrals of a function u*

$$_{a+}I_t^\alpha u(t) = {}_aI_t^\alpha u(t) = \left({}_{a+}I_t^\alpha u\right)(t) = \left({}_aI_t^\alpha u\right)(t)$$

and

$$_tI_{b-}^\alpha u(t) = {}_tI_b^\alpha u(t) = \left({}_tI_{b-}^\alpha u\right)(t) = \left({}_tI_b^\alpha u\right)(t)$$

of order $\alpha \in \mathbb{R}^+$ are defined by

$$_aI_t^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)\, \mathrm{d}s \quad (t \in (a, b]) \tag{2.1}$$

and

$$_tI_b^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s)\, \mathrm{d}s \quad (t \in [a, b)) \tag{2.2}$$

respectively. Here Γ is the familiar Gamma function.

Let $[a, b]$ $(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $L^p(a, b)$ $(1 \le p \le \infty)$ the set of those Lebesgue complex-valued measurable functions u on $[a, b]$ for which $\|u\|_p < \infty$, where

$$\|u\|_p = \left(\int_a^b |u(t)|^p\, \mathrm{d}t\right)^{1/p} \quad (1 \le p < \infty)$$

and

$$\|u\|_\infty = \operatorname*{ess\,sup}_{a \le x \le b} |u(x)|.$$

If $u \in L^1(a,b)$, then ${}_aI_t^\alpha u$ and ${}_tI_b^\alpha u$ are defined a.e. on (a,b).

Let $[a,b]$ $(-\infty \le a < b \le \infty)$ and $m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. We denote by $C^m[a,b]$ a space of functions u which are m times continuously differentiable on $[a,b]$ with the norm

$$\|u\|_{C^m} = \sum_{k=0}^{m} \|u^{(k)}\|_C = \sum_{k=0}^{m} \max_{t\in[a,b]} \left|u^{(k)}(t)\right| \quad (m \in \mathbb{N}_0)\,. \tag{2.3}$$

In particular, for $m = 0$, $C^0[a,b] \equiv C[a,b]$ is the space of continuous functions u on $[a,b]$ with the norm

$$\|u\|_C = \max_{t\in[a,b]} |u(t)|. \tag{2.4}$$

When $[a,b]$ is a finite interval and $0 \le \gamma < 1$, we introduce the weighted space $C_\gamma[a,b]$ of functions u given on $(a,b]$, such that the function $(t-a)^\gamma\, u(t) \in C[a,b]$, and

$$\|u\|_{C_\gamma} = \|(t-a)^\gamma\, u(t)\|_C, \quad C_0[a,b] = C[a,b]. \tag{2.5}$$

Definition 2.2. *(See [1]) The Riemann-Liouville left-sided and right-sided fractional derivatives of a function u*

$${}_{a+}D_t^\alpha u(t) = {}_aD_t^\alpha u(t) = \left({}_{a+}D_t^\alpha u\right)(t) = \left({}_aD_t^\alpha u\right)(t)$$

and

$${}_tD_{b-}^\alpha u(t) = {}_tD_b^\alpha u(t) = \left({}_tD_{b-}^\alpha u\right)(t) = \left({}_tD_b^\alpha u\right)(t)$$

of order $\alpha \in \mathbb{R}^+ \cup \{0\}$ are defined by

$$\begin{aligned}{}_aD_t^\alpha u(t) :&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left({}_aI_t^{n-\alpha}\, u(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-s)^{n-\alpha-1}\, u(s)\,\mathrm{d}s \quad (n = [\alpha]+1;\ t > a)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}{}_tD_b^\alpha u(t) :&= \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left({}_tI_b^{n-\alpha}\, u(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (s-t)^{n-\alpha-1}\, u(s)\,\mathrm{d}s \quad (n = [\alpha]+1;\ t < b),\end{aligned} \tag{2.7}$$

respectively, where $[\alpha]$ means the integral part of α.

Remark 2.3. *From [24, pp. 2-3], if u is absolutely continuous on $[a,b]$, then the fractional derivatives ${}_aD_t^\alpha u$ and ${}_tD_b^\alpha u$ exist almost everywhere on $[a,b]$ and can be represented in the forms*

$${}_aD_t^\alpha u(t) = {}_aI_t^{1-\alpha} u'(t) + \frac{u(a)}{(t-a)^\alpha \Gamma(1-\alpha)} \tag{2.8}$$

and

$${}_tD_b^\alpha u(t) = -{}_tI_b^{1-\alpha} u'(t) + \frac{u(b)}{(b-t)^\alpha \Gamma(1-\alpha)}. \tag{2.9}$$

In particular, we have

$${}_aD_t^\alpha u(t) = {}_aI_t^{1-\alpha} u'(t) \quad \textit{and} \quad {}_tD_b^\alpha u(t) = -{}_tI_b^{1-\alpha} u'(t) \quad (u(a) = 0 = u(b)). \tag{2.10}$$

The left-sided and right-sided Caputo fractional derivatives ${}_a^C D_t^\alpha u(t)$ *and* ${}_t^C D_b^\alpha u(t)$ *of order* $\alpha \in \mathbb{R}^+ \cup \{0\}$ *with, here,* $0 < \alpha < 1$ *are defined by*

$$ {}_a^C D_t^\alpha u(t) = {}_a D_t^\alpha [u(t) - u(a)] \tag{2.11}$$

and

$$ {}_t^C D_b^\alpha u(t) = {}_t D_b^\alpha [u(t) - u(b)], \tag{2.12}$$

respectively.

We find from (2.8)–(2.12) *that, if* u *is absolutely continuous on* $[a, b]$, $u(a) = 0 = u(b)$, *and* $0 < \alpha < 1$, *then the Riemann-Liouville fractional integrals and the Caputo fractional derivatives coincide:*

$$ {}_a^C D_t^\alpha u(t) = {}_a I_t^{1-\alpha} u'(t) \quad \textit{and} \quad {}_t^C D_b^\alpha u(t) = -\, {}_t I_b^{1-\alpha} u'(t). \tag{2.13}$$

The semigroup property of the fractional integration operators ${}_a I_t^\alpha$ and ${}_t I_b^\alpha$ are given by the following remark.

Remark 2.4. *([24, Lemma 2.3]) If* $\alpha, \beta \in \mathbb{R}^+$, *then the equations*

$$ \left({}_a I_t^\alpha {}_a I_t^\beta u\right)(t) = \left({}_a I_t^{\alpha+\beta} u\right)(t) \quad \textit{and} \quad \left({}_t I_b^\alpha {}_t I_b^\beta u\right)(t) = \left({}_t I_b^{\alpha+\beta} u\right)(t) \tag{2.14}$$

are satisfied at almost every point $t \in [a, b]$ *for* $f(t) \in L^p(a, b)$ $(1 \le p \le \infty)$. *If* $\alpha + \beta > 1$, *then the relations in* (2.14) *hold at any point of* $[a, b]$.

The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration.

Remark 2.5. *([24, Lemma 2.4]) If* $\alpha \in \mathbb{R}^+$ *and* $f(t) \in L^p(a, b)$ $(1 \le p \le \infty)$, *then the following equalities*

$$ \left({}_a D_t^\alpha {}_a I_t^\alpha u\right)(t) = f(t) \quad \textit{and} \quad \left({}_t D_b^\alpha {}_t I_b^\alpha u\right)(t) = f(t) \tag{2.15}$$

hold almost everywhere on $[a, b]$.

Remark 2.6. *([24, Lemma 2.4]) The fractional integration operators* ${}_a I_t^\alpha$ *and* ${}_t I_b^\alpha$ *with* $\alpha \in \mathbb{R}^+$ *are bounded in* $L^p(a, b)$ $(1 \le p \le \infty)$*:*

$$ \|{}_a I_t^\alpha u\|_p \le \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \|u\|_p \quad \textit{and} \quad \|{}_t I_b^\alpha u\|_p \le \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \|u\|_p. \tag{2.16}$$

In the same way, we give another classical result on the boundedness of the left fractional integral from $L^p(a, b)$ to $C_a(a, b)$ which completes Remark 2.6 in the case $0 < \frac{1}{p} < \alpha < 1$.

Remark 2.7. *([9, Property 4]) Let* $0 < \frac{1}{p} < \alpha < 1$ *and* $q = \frac{p}{p-1}$. *Then, for any* $u \in L^p(a, b)$, ${}_a I_t^\alpha u$ *is Hölder continuous on* $(a, b]$ *with exponent* $\alpha - \frac{1}{p} > 0$, *that is, there exists a constant* $M \in \mathbb{R}^+$ *such that*

$$ \left| {}_a I_{t_2}^\alpha u\,(t_2) - {}_a I_{t_1}^\alpha u\,(t_1) \right| \le M\,(t_2 - t_1)^{\alpha - 1/p}$$

for any $a < t_1 < t_2 \le b$. *Moreover,* $\lim_{t \to a} {}_a I_t^\alpha u(t) = 0$. *Consequently,* ${}_a I_t^\alpha u$ *can be continuously extended by* 0 *at* $t = a$. *Finally, for any* $u \in L^p(a, b)$, ${}_a I_t^\alpha u \in C_a(a, b)$, *and*

$$ \|{}_a I_t^\alpha u\|_\infty \le \frac{(b-a)^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)\left((\alpha-1)q+1\right)^{\frac{1}{q}}} \|u\|_p. \tag{2.17}$$

The following formula which is often called fractional integration by parts will also be required.

Remark 2.8. *([9, Property 3]) Let $0 < \alpha < 1$ and p, q are such that*

$$p \geq 1,\ q \geq 1 \ \text{ and } \ \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$$

(and $p \neq 1 \neq q$ in the case $1/p + 1/q = 1 + \alpha$). Then, for all $u \in L^p(a, b)$ and all $v \in L^q(a, b)$, we have

$$\int_a^b {}_aI_t^\alpha u(t) \cdot v(t)\,\mathrm{d}t = \int_a^b u(t) \cdot {}_tI_b^\alpha v(t)\,\mathrm{d}t, \tag{2.18}$$

and

$$\int_a^b u(t)\, {}_a^CD_t^\alpha v(t)\,\mathrm{d}t = v(t)_tI_b^{1-\alpha}u(t)\Big|_{t=a}^{t=b} + \int_a^b v(t)\, {}_aD_t^\alpha u(t)\,\mathrm{d}t. \tag{2.19}$$

Moreover, if $v(a) = v(b) = 0$, then we have

$$\int_a^b u(t)\, {}_aD_t^\alpha v(t)\,\mathrm{d}t = \int_a^b v(t)\, {}_a^CD_t^\alpha u(t)\,\mathrm{d}t. \tag{2.20}$$

3. Nehari manifold and fibering map analysis

To show the existence of solutions to the problem (P_λ), we will use critical point theory (see, *e.g.*, [22]). We begin by introduce some notations and results which will be used. The set of all functions $u \in C^\infty([0, 1], \mathbb{R})$ with $u(0) = u(1) = 0$ is denoted by $C_0^\infty([0, 1], \mathbb{R})$. For $\alpha \in \mathbb{R}^+$, we define the fractional derivative space $E_0^{\alpha,p}$ as the closure of $C_0^\infty([0, 1], \mathbb{R})$ with the norm

$$\|u\|_{\alpha,p} = \left(\|u\|_p^p + \|{}_0^CD_t^\alpha u\|_p^p\right)^{\frac{1}{p}}. \tag{3.1}$$

We summarize some properties for the space $E_0^{\alpha,p}$ in the following remark.

Remark 3.1. *(see [22, Remark 3.1])*

(i) The space $E_0^{\alpha,p}$ is the space of functions $u \in L^p[0, 1]$ having an α-order Caputo fractional derivative ${}_0^CD_t^\alpha u \in L^p[0, 1]$ and $u(0) = u(1) = 0$.

(ii) For any $u \in E_0^{\alpha,p}$ $(0 < \alpha < 1)$, since $u(0) = 0$, we have (see (2.11))

$${}_0^CD_t^\alpha u(t) = {}_0D_t^\alpha u(t) \quad (t \in [0, 1]). \tag{3.2}$$

(iii) The space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Lemma 3.2. *[22, Proposition 3.2] Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Then, for all $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_p \leq \frac{T^\alpha}{\Gamma(\alpha + 1)}\|{}_0D_t^\alpha u\|_p. \tag{3.3}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{\widetilde{p}} = 1$, we have

$$\|u\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)\left((\alpha - 1)\widetilde{p} + 1\right)^{\frac{1}{\widetilde{p}}}}\|{}_0D_t^\alpha u\|_p. \tag{3.4}$$

Incorporating (3.2) and (3.3) into the norm (3.1), we can consider the space $E_0^{\alpha,p}$ with respect to the form

$$\|u\|_{\alpha,p} = \|{}_0D_t^{\alpha}u\|_p \tag{3.5}$$

in the following analysis.

Lemma 3.3. *(see [22, Proposition 3.3]) Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$*. Assume that* $\alpha > \frac{1}{p}$ *and the sequence* $\{u_n\}$ *converges weakly to* u *in* $E_0^{\alpha,p}$*, i.e.,* $\{u_n\} \rightharpoonup u$*. Then,* $\{u_n\} \to u$ *in* $C([0,1])$*, that is,* $\|u_n - u\|_\infty \to 0$ *as* $n \to \infty$*.*

Associated to the problem (P_λ) we define the functional $E_\lambda : E_0^{\alpha,p} \to \mathbb{R}$ given by

$$E_\lambda(u) = \frac{1}{bp^2}\left(a + b\|u\|_{\alpha,p}^p\right)^p - \frac{\lambda}{1-\gamma}\int_0^T g(t)|u|^{1-\gamma}\,\mathrm{d}t - \frac{1}{r}\int_0^T F(t,u)\,\mathrm{d}t - \frac{a^p}{bp^2}.$$

One can easily verify that the energy functional $E_\lambda(u)$ is not bounded below on the space $E_0^{\alpha,p}$. But we will show that $E_\lambda(u)$ is bounded below on this Nehari manifold and we will extract solutions by minimizing the functional on suitable subsets. In order to investigate the problem (P_λ), we define the constraint set

$$\mathcal{N}_\lambda := \left\{u \in E_0^{\alpha,p} \setminus \{0\} : t(u)u = 0\right\},$$

where $t(u)$ is the zero of the map $\Phi_u : (0,\infty) \to \mathbb{R}$ defined as

$$\Phi_u(t) = E_\lambda(tu).$$

Now as we know that the Nehari manifold is closely related to the behaviour of the functions $\Phi_u : s \mapsto E_\lambda(su)$ for $s > 0$ defined by

$$\Phi_u(s) = \frac{1}{bp^2}\left(a + bs^p\|u\|_{\alpha,p}^p\right)^p - \frac{\lambda s^{1-\gamma}}{1-\gamma}\int_0^T g(t)|u|^{1-\gamma}\,\mathrm{d}t - \frac{s^r}{r}\int_0^T F(t,u)\,\mathrm{d}t - \frac{a^p}{bp^2}.$$

Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [14]. For $u \in E_0^{\alpha,p}$, we have

$$\begin{aligned}
\Phi_u'(s) &= \left(a + bs^p\|u\|_{\alpha,p}^p\right)^{p-1} s^{p-1}\|u\|_{\alpha,p}^p - \frac{\lambda}{s^\gamma}\int_0^T g(t)|u(t)|^{1-\gamma}dt - s^{r-1}\int_0^T F(t,u(t))dt,\\
\Phi_u''(s) &= (p-1)s^{p-2}\|u\|_{\alpha,p}^p\left(a + bs^p\|u\|_{\alpha,p}^p\right)^{p-1} + bp(p-1)s^{2p-2}\|u\|_{\alpha,p}^{2p}\left(a + bs^p\|u\|_{\alpha,p}^p\right)^{p-2}\\
&+ \lambda\frac{\gamma}{s^{\gamma+1}}\int_0^T g(t)|u(t)|^{1-\gamma}dt - (r-1)s^{r-2}\int_0^T F(t,u(t))dt.
\end{aligned}$$

Note that $\mathcal{N}_\lambda$ contains every nonzero solution of (P_λ), and $u \in \mathcal{N}_\lambda$ if and only if

$$\left(a + b\|u\|_{\alpha,p}^p\right)^{p-1}\|u\|_{\alpha,p}^p - \lambda\int_0^T g(t)|u|^{1-\gamma}\,\mathrm{d}t - \int_0^T F(t,u)\,\mathrm{d}t = 0. \tag{3.6}$$

Lemma 3.4. *Let* $u \in E_0^{\alpha,p}$*, then* $tu \in \mathcal{N}_\lambda$ *if and only if* $\Phi_u'(t) = 0$*.*

Proof. Let $su \in \mathcal{N}_\lambda$. This means that

$$\begin{aligned}
0 &= \left(a + b\|su\|_{\alpha,p}^p\right)^{p-1}\|su\|_{\alpha,p}^p - \lambda\int_0^T g(t)|su|^{1-\gamma}\,\mathrm{d}t - \int_0^T F(t,su)\,\mathrm{d}t\\
&= \Phi_u'(s).
\end{aligned}$$

This give the proof of the Lemma 3.4. □

From the lemma 3.4, we have that the elements in $\mathcal{N}_\lambda$ correspond to stationary points of the maps $\Phi_u(t)$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(1) = 0$. Hence, it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection $\Phi_u(s)$ defined as follows:

$$\mathcal{N}_\lambda^+ = \{u \in \mathcal{N}_\lambda : \Phi''_u(1) > 0\} = \left\{su \in E_0^{\alpha,p} : \Phi'_u(s) = 0, \Phi''_u(s) > 0\right\},$$
$$\mathcal{N}_\lambda^- = \{u \in \mathcal{N}_\lambda : \Phi''_u(1) < 0\} = \left\{su \in E_0^{\alpha,p} : \Phi'_u(s) = 0, \Phi''_u(s) < 0\right\},$$
$$\mathcal{N}_\lambda^0 = \{u \in \mathcal{N}_\lambda : \Phi''_u(1) = 0\} = \left\{su \in E_0^{\alpha,p} : \Phi'_u(s) = 0, \Phi''_u(s) = 0\right\}.$$

Our first result is the following

Lemma 3.5. *E_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$. Then, using (1.1) and (3.4), we obtain

$$\int_0^T F(t,u(t))dt \le K\int_0^T |u|^r dt \le \frac{KT^{1+r(\alpha-\frac{1}{p})}}{\beta^r}\|u\|^r, \tag{3.7}$$

and

$$\int_0^T g(t)|u|^{1-\gamma}dt \le \|g\|_\infty \int_0^T |u|^{1-\gamma}dt \le \|g\|_\infty \frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}}{\beta^{1-\gamma}}\|u\|^{1-\gamma}. \tag{3.8}$$

Consequently, from (3.7) and (3.8), we obtain

$$\begin{aligned}
E_\lambda(u) &= \frac{1}{bp^2}\left(a + b\|u\|_{\alpha,p}^p\right)^p - \frac{\lambda}{1-\gamma}\int_0^T g(t)|u|^{1-\gamma}\,\mathrm{d}t - \frac{1}{r}\int_0^T F(t,u)\,\mathrm{d}t - \frac{a^p}{bp^2}\\
&= \left(a + b\|u\|_{\alpha,p}\right)^p\left(\frac{1}{bp^2}(a + b\|u\|_{\alpha,p}^p) - \frac{1}{r}\|u\|_{\alpha,p}^p\right) - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{r}\right)\int_0^T g(t)|u|^{1-\gamma}\,\mathrm{d}t - \frac{a^p}{bp^2}\\
&\ge \left(a + b\|u\|_{\alpha,p}\right)^p\left(\frac{a}{bp^2} + (\frac{1}{p^2} - \frac{1}{r})\|u\|_{\alpha,p}^p\right) - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{r}\right)\frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}\|g\|_\infty}{(\Gamma(\alpha))^{1-\gamma}\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1-\gamma}{\widetilde{p}}}}\|u\|_{\alpha,p}^{1-\gamma} - \frac{a^p}{bp^2}.
\end{aligned}$$

Since $0 < 1-\gamma < p^2 < r$, $E_\lambda(u)$ is coercive and bounded below on $\mathcal{N}_\lambda$. The proof of the Lemma 3.5 is now completed. $\square$

Lemma 3.6. *Given $u \in \mathcal{N}_\lambda^-$ (respectively $\mathcal{N}_\lambda^+$) with $u \ge 0$, for all $v \in E_0^{\alpha,p}$ with $v \ge 0$, there exist $\varepsilon > 0$ and a continuous function ω such that for all $k \in \mathbb{R}$ with $|k| < \varepsilon$ we have*

$$\omega(0) = 1 \quad \text{and} \quad \omega(u+kv) \in \mathcal{N}_\lambda^- \ (\text{respectively } \ \mathcal{N}_\lambda^+).$$

Proof. We introduce the function $\psi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ define by:

$$\psi(t,k) = t^{p+\gamma-1}\left(a + b\|u+kv\|_{\alpha,p}^p\right)^{p-1}\|u+kv\|_{\alpha,p}^p - \lambda\int_0^T g(s)(u+kv)^{1-\gamma}ds - t^{r+\gamma-1}\int_0^T F(s,u+kv)ds.$$

Hence,

$$\psi_t(t,k) = (p+\gamma-1)t^{p+\gamma-2}\left(a + b\|u+kv\|_{\alpha,p}^p\right)^{p-1}\|u+kv\|_{\alpha,p}^p - (r+\gamma-1)t^{r+\gamma-2}\int_0^T F(s,u+kv)ds,$$

is continuous on $\mathbb{R}\times\mathbb{R}$. Since $u \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$, we have $\psi(1,0) = 0$, and

$$\psi_t(1,0) = (p+\gamma-1)\left(a + b\|u\|_{\alpha,p}^p\right)^{p-1}\|u\|_{\alpha,p}^p - (r+\gamma-1)\int_0^T F(t,u)dt < 0.$$

Therefore, applying the implicit function theorem to the function ψ at the point $(1,0)$. So, we obtain the existence of a parameter $\delta > 0$ and a positive continuous function ω satisfying

$$\omega(0) = 1,\ \omega(k)(u + kv) \in \mathcal{N}_\lambda,\ \forall\, k \in \mathbb{R},\ |k| < \delta.$$

Hence, taking $\varepsilon > 0$ possibly smaller enough, we get

$$\omega(k)(u + kv) \in \mathcal{N}_\lambda^-,\ \forall\, k \in \mathbb{R},\ |k| < \varepsilon.$$

The case $u \in \mathcal{N}_\lambda^+$ may be obtained in the same way. This completes the proof of the Lemma 3.6. □

Lemma 3.7. *There exists Λ_0, such that for $0 < \lambda < \Lambda_0$ there exist s^- and s^+ such that*

$$\Phi_u(s_0) = \Phi_u(s_1)$$

and

$$\Phi_u'(s_0) < 0 < \Phi_u'(s_1);$$

that is, $s_0 u \in \mathcal{N}_\lambda^-$ and $s_1 u \in \mathcal{N}_\lambda^+$.

Proof. Fix $u \in E_0^{\alpha,p}$. Then it follows that

$$\begin{aligned}
\Phi_u'(s) &= \left(a + bs^p\|u\|_{\alpha,p}^p\right)^{p-1} s^{p-1}\|u\|_{\alpha,p}^p - \frac{\lambda}{s^\gamma}\int_0^T g(t)|u(t)|^{1-\gamma}dt - s^{r-1}\int_0^T F(t,u(t))dt \\
&\geq s^{p-1}\|u\|_{\alpha,p}^p - \frac{\lambda}{s^\gamma}\int_0^T g(t)|u(t)|^{1-\gamma}dt - s^{r-1}\int_0^T F(t,u(t))dt \\
&= s^{p-1}\left(\|u\|_{\alpha,p}^p - \psi(s)\right)
\end{aligned}$$

where $\psi(s) = \lambda s^{1-p-\gamma}\int_0^T g(t)|u(t)|^{1-\gamma}dt + s^{r-p}\int_0^T F(t,u(t))dt$. Since, the first derivative of the function ψ is given by

$$\psi'(s) = \lambda(1-p-\gamma)s^{-p-\gamma}\int_0^T g(t)|u(t)|^{1-\gamma}dt + (r-p)s^{r-p-1}\int_0^T F(t,u(t))dt,$$

it is simple to verify that $\psi(s)$, attains it's maximum at

$$s_{\max} = \left(\frac{\lambda(1-p-\gamma)\int_0^T g(t)|u(t)|^{1-\gamma}dt}{(r-p)\int_0^T F(t,u(t))dt}\right)^{\frac{1}{r+\gamma-1}}.$$

Moreover,

$$\begin{aligned}
\psi(s_{\max}) &= \left(\frac{r-p}{p+\gamma-1}+1\right)s_{\max}^{r-p}\int_0^T F(t,u(t))dt \\
&= \left(\frac{r-p}{p+\gamma-1}+1\right)\left(\frac{\lambda(1-p-\gamma)\int_0^T g(t)|u(t)|^{1-\gamma}dt}{(r-p)\int_0^T F(t,u(t))dt}\right)^{\frac{r-p}{r+\gamma-1}}\int_0^T F(t,u(t))dt \\
&= \left(\frac{r+\gamma-1}{p+\gamma-1}\right)\left(\frac{\lambda(1-p-\gamma)}{r-p}\right)^{\frac{r-p}{r+\gamma-1}}\left(\int_0^T g(t)|u(t)|^{1-\gamma}dt\right)^{\frac{r-p}{r+\gamma-1}}\left(\int_0^T F(t,u(t))dt\right)^{\frac{p+\gamma-1}{r+\gamma-1}} \\
&\leq \left(\frac{r+\gamma-1}{p+\gamma-1}\right)\left(\frac{\lambda(1-p-\gamma)}{r-p}\right)^{\frac{r-p}{r+\gamma-1}}\left(\int_0^T g(t)|u(t)|^{1-\gamma}dt\right)^{\frac{r-p}{r+\gamma-1}}\left(\int_0^T F(t,u(t))dt\right)^{\frac{p+\gamma-1}{r+\gamma-1}}.
\end{aligned}$$

Consequently, from (3.7) and (3.8), one has

$$\psi(s_{\max}) \leq \left(\frac{r+\gamma-1}{p+\gamma-1}\right)\left(\frac{\lambda(1-p-\gamma)}{r-p}\right)^{\frac{r-p}{r+\gamma-1}}\left(\|g\|_\infty \frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}}{\beta^{1-\gamma}}\right)^{\frac{(r-p)(\gamma-1)}{p(r+\gamma-1)}}\left(\frac{KT^{1+r(\alpha-\frac{1}{p})}}{\beta^r}\right)^{\frac{r(1-p-\gamma)}{p(r+\gamma-1)}}\|u\|^p_{\alpha,p}.$$

Hence, $\|u\|^p_{\alpha,p} > \psi(s_{\max})$ and so $\Phi'_u(s) > 0$ for all

$$0 < \lambda < \frac{r-p}{1-p-\gamma}\left(\frac{p+\gamma-1}{r+\gamma-1}\right)^{\frac{r+\gamma-1}{r-p}}\left(\frac{\|g\|_\infty \beta^{1-\gamma}}{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}}\right)^{\frac{1-\gamma}{p(r-p)}}\left(\frac{\beta^r}{KT^{1+r(\alpha-\frac{1}{p})}}\right)^{\frac{r(p+\gamma-1)}{p(r-p)}}$$

On the other hand, we have

$$\begin{aligned}
\Phi'_u(s) &= \left(a+bs^p\|u\|^p_{\alpha,p}\right)^{p-1}s^{p-1}\|u\|^p_{\alpha,p} - \frac{\lambda}{s^\gamma}\int_0^T g(t)|u(t)|^{1-\gamma}dt - s^{r-1}\int_0^T F(t,u(t))dt\\
&\leq \left(a+bs^p\|u\|^p_{\alpha,p}\right)^{p} - \frac{\lambda}{s^\gamma}\int_0^T g(t)|u(t)|^{1-\gamma}dt - s^{r-1}\int_0^T F(t,u(t))dt
\end{aligned}$$

since $1-\gamma < p < r$, there exist $0 < s_0 < s_{\max} < s_1$ such that $\Phi'_u(s_0) < 0$, $\Phi'_u(s_1) < 0$. Note that $\mathcal{N}^0_\lambda = \emptyset$, we deduce that there exist s_1, s_0 such that $\Phi'_u(s_1) = \Phi'_u(s_0) = 0$ and $\Phi''_u(s_1) > 0 > \Phi''_u(s_0) = 0$. Thus, Φ_u has a local minimum at $s = s_0$ and a local maximum at $s = s_1$, that is Φ_u is decreasing in $(0, s_0)$ and increasing in (s_0, s_1). Hence, $s_1 u \in \mathcal{N}^+_\lambda$ and $s_0 u \in \mathcal{N}^-_\lambda$. The proof of Lemma 3.7 is now completed. □

Now, we prove the following crucial Lemma:

Lemma 3.8. *Suppose $\lambda \in (0, \Lambda_0)$, $\mathcal{N}^0_\lambda = \emptyset$.*

Proof. We proceed by contradiction to prove that $\mathcal{N}^0_\lambda = \emptyset$ for all $\lambda \in (0, \Lambda_0)$. Let us suppose that there exists $u_0 \in \mathcal{N}^0_\lambda$. Then, it follows that

$$\begin{aligned}
\Phi''_u(s) &= (p-r)\|u\|^p_{\alpha,p}\left(a+b\|u\|^p_{\alpha,p}\right)^{p-1} + bp(p-1)\|u\|^{2p}_{\alpha,p}\left(a+b\|u\|^p_{\alpha,p}\right)^{p-2}\\
&+ \lambda(r+\gamma-1)\int_0^T g(t)|u(t)|^{1-\gamma}dt\\
&= (p+\gamma-1)\|u\|^p_{\alpha,p}\left(a+b\|u\|^p_{\alpha,p}\right)^{p-1} + bp(p-1)\|u\|^{2p}_{\alpha,p}\left(a+b\|u\|^p_{\alpha,p}\right)^{p-2}\\
&- (r+\gamma-1)\int_0^T F(t,u)dt. \qquad (3.9)
\end{aligned}$$

Furthermore, if $u \in \mathcal{N}^0_\lambda$, then

$$\begin{aligned}
(p+\gamma-1)\frac{a^{p-1}}{p}\|u\|^p_{\alpha,p} &\leq (p+\gamma-1)\|u\|^p_{\alpha,p}\left(a+b\|u\|^p_{\alpha,p}\right)^{p-1} + bp(p-1)\|u\|^{2p}_{\alpha,p}\left(a+b\|u\|^p_{\alpha,p}\right)^{p-2}\\
&= (r+\gamma-1)\int_0^T F(t,u)dt\\
&\leq (r+\gamma-1)\frac{KT^{1+r(\alpha-\frac{1}{p})}}{\beta^r}\|u\|^r_{\alpha,p}
\end{aligned}$$

and

$$\begin{aligned}
(p-r)\frac{a^{p-1}\Gamma(\alpha+1)}{pT^{\alpha}}\|u\|_{\alpha,p}^{p} &\leq (p-r)\|u\|_{\alpha,p}^{p}\left(a+b\|u\|_{\alpha,p}^{p}\right)^{p-1} \\
&\leq (p-r)\|u\|_{\alpha,p}^{p}\left(a+b\|u\|_{\alpha,p}^{p}\right)^{p-1} - bp(p-1)\|u\|_{\alpha,p}^{2p}\left(a+b\|u\|_{\alpha,p}^{p}\right)^{p-2} \\
&\leq \lambda(r+\gamma-1)\int_0^T g(t)|u(t)|^{1-\gamma}dt \\
&\leq \lambda(r+\gamma-1)\|g\|_{\infty}\frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}}{\beta^{1-\gamma}}\|u\|_{\alpha,p}^{1-\gamma}.
\end{aligned}$$

Consequently,

$$\left(\frac{a^{p-1}(p+\gamma-1)\beta^{r}}{p(r+\gamma-1)KT^{1+r(\alpha-\frac{1}{p})}}\right)^{\frac{1}{r-p}} \leq \|u\|_{\alpha,p} \leq \left(\frac{\lambda p(r+\gamma-1)\|g\|_{\infty}T^{1+\alpha(1-\gamma)(\alpha-\frac{1}{p})}}{a^{p-1}(p-r)\beta^{1-\gamma}\Gamma(\alpha+1)}\right)^{\frac{1}{p+\gamma-1}}. \tag{3.10}$$

Therefore,

$$\lambda \geq \Lambda_0 := \left(\frac{a^{p-1}(p+\gamma-1)\beta^{r}}{p(r+\gamma-1)KT^{1+r(\alpha-\frac{1}{p})}}\right)^{\frac{p+\gamma-1}{r-p}} \frac{a^{p-1}(p-r)\beta^{1-\gamma}\Gamma(\alpha+1)}{p(r+\gamma-1)\|g\|_{\infty}T^{1+\alpha+(1-\gamma)(\alpha-\frac{1}{p})}}.$$

Hence $\mathcal{N}_{\lambda}^{0} = \emptyset$ for all $\lambda \in (0, \Lambda_0)$. $\square$

By Lemmas 3.5 and 3.7, we can write $\mathcal{N}_{\lambda} = \mathcal{N}_{\lambda}^{+} \cup \mathcal{N}_{\lambda}^{-}$ and define

$$c_{\lambda}^{+} = \inf_{u\in\mathcal{N}_{\lambda}^{+}} E_{\lambda}(u) \quad \text{and} \quad c_{\lambda}^{-} = \inf_{u\in\mathcal{N}_{\lambda}^{-}} E_{\lambda}(u).$$

4. Existence of minimizer on $\mathcal{N}_{\lambda}^{+}$

In this section, we will show that the minimum of E_{λ} is achieved in $\mathcal{N}_{\lambda}^{+}$. Also, we show that this minimizer is also the first solution of (P_{λ}).

Lemma 4.1. *If $0 < \lambda < \Lambda_0$, then for all $u \in \mathcal{N}_{\lambda}^{+}$, $c_{\lambda}^{+} < 0$.*

Proof. Let $u \in \mathcal{N}_{\lambda}^{+}$, then we have $\phi_{u_0}''(1) > 0$ which gives

$$\begin{aligned}
&(p-1)\|u\|_{\alpha,p}^{p}\left(a+b\|u\|_{\alpha,p}^{p}\right)^{p-1} + bp(p-1)\|u\|_{\alpha,p}^{2p}\left(a+b\|u\|_{\alpha,p}^{p}\right)^{p-2} + \lambda\gamma\int_0^T g(t)|u(t)|^{1-\gamma}dt \\
&-(r-1)\int_0^T F(t,u(t))dt > 0.
\end{aligned} \tag{4.1}$$

Now multiplying equation (3.6) by $(r-1)$ and subtracting from equation (4.1), we get

$$\lambda(r+\gamma-1)\int_0^T g(t)|u|^{1-\gamma}\,\mathrm{d}t > \|u\|_{\alpha,p}^{p}\left(a+b\|u\|_{\alpha,p}^{p}\right)^{p-2}\left((r-p)a+b\|u\|_{\alpha,p}^{p}(r-p^{2})\right). \tag{4.2}$$

Now using equation (3.6) and (4.2), we get

$$
\begin{aligned}
E_\lambda(u) &= \frac{1}{bp^2}\left(a+b\|u\|^p_{\alpha,p}\right)^p - \frac{\lambda}{1-\gamma}\int_0^T g(t)|u|^{1-\gamma}\,\mathrm{d}t - \frac{1}{r}\int_0^T F(t,u)\,\mathrm{d}t - \frac{a^p}{bp^2}\\
&= \left(a+b\|u\|^p_{\alpha,p}\right)^{p-1}\left(\frac{a}{bp^2}+(\frac{1}{p^2}-\frac{1}{r})\|u\|^p_{\alpha,p}\right) - \lambda\left(\frac{1}{1-\gamma}-\frac{1}{r}\right)\int_0^T g(t)|u|^{1-\gamma}\,\mathrm{d}t - \frac{a^p}{bp^2}\\
&\leq \left(a+b\|u\|^p_{\alpha,p}\right)^{p-1}\left(\frac{a}{bp^2}+(\frac{1}{p^2}-\frac{1}{r})\|u\|^p_{\alpha,p}\right) - \frac{\left((r-p)a+b\|u\|^p_{\alpha,p}(r-p^2)\right)}{r(1-\gamma)}\|u\|^p_{\alpha,p}\left(a+b\|u\|^p_{\alpha,p}\right)^{p-2}\\
&\leq -b\left(\frac{(r-p^2)(p^2-(1-\gamma))}{p^2r(1-\gamma)}\right)\left(a+b\|u\|^p_{\alpha,p}\right)^{p-2}\|u\|^{2p}_{\alpha,p} < 0
\end{aligned}
$$

since $0 < a < 1$, and $b > 0$. Thus,

$$
c^+_\lambda = \inf_{u\in\mathcal{N}^+_\lambda} E_\lambda(u) < 0 \qquad \text{for all } \lambda \in (0,\Lambda_0). \tag{4.3}
$$

□

Theorem 4.2. *If $0 < \lambda < \Lambda_0$, then there exists $u_0 \in \mathcal{N}^+_\lambda$ satisfying $E_\lambda(u_0) = \inf_{u\in\mathcal{N}^+_\lambda} E_\lambda(u)$.*

Proof. Since E_λ is bounded below on $\mathcal{N}_\lambda$ and so on $\mathcal{N}^+_\lambda$. Then, using the Ekeland variational principle [15], there exist a minimizing sequence $\{u_n\} \subset \mathcal{N}^+_\lambda$ such that

$$
E_\lambda(u_n) \to \inf_{u\in\mathcal{N}^+_\lambda} E_\lambda(u) \ \text{ as } n\to\infty.
$$

Since E_λ is coercive, $\{u_n\}$ is bounded in $E^{\alpha,p}_0$. Then there exists a subsequence, still denoted by u_n and $u_0 \in E^{\alpha,p}_0$ such that, as $n \to \infty$,

$$
\begin{gathered}
u_n \rightharpoonup u_0, \text{ weakly in } E^{\alpha,p}_0\\
u_n \to u_0, \text{ strongly in } L^q(\Omega) \text{ for all } 1 \leq q < p^*,\\
u_n \to u_0, \text{ a.e. in } \Omega.
\end{gathered}
$$

Now, using Lemma 3.3, we get that, as $n \to \infty$,

$$
\begin{aligned}
\int_0^T g(t)u_n^{1-\gamma}\mathrm{d}t &\leq \int_0^T g(t)u_0^{1-\gamma}\mathrm{d}t + \int_0^T g(t)\mid u_n-u_0\mid^{1-\gamma}\mathrm{d}t\\
&\leq \int_0^T g(t)u_0^{1-\gamma}\mathrm{d}x + Tg(t)\parallel u_n-u_0\parallel^{1-\gamma}_\infty\\
&= \int_0^T g(t)u_0^{1-\gamma}\mathrm{d}t + o(1).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\int_0^T g(t)u_0^{1-\gamma}\mathrm{d}t &\leq \int_0^T u_n^{1-\gamma}\mathrm{d}t + \int_0^T g(t)\mid u_n-u_0\mid^{1-\gamma}\mathrm{d}t\\
&\leq \int_0^T g(t)u_n^{1-\gamma}\mathrm{d}t + Tg(t)\parallel u_n-u_0\parallel^{1-\gamma}_\infty\\
&= \int_0^T g(t)u_n^{1-\gamma}\mathrm{d}t + o(1).
\end{aligned}
$$

Thus,

$$\int_0^T g(t)u_n^{1-\gamma}\mathrm{d}t = \int_0^T g(t)u_0^{1-\gamma}\mathrm{d}x + o(1). \tag{4.4}$$

On the other hand, by [8] there exists $l \in L^r(\mathbb{R}^N)$ such that

$$|u_n(x)| \leq l(x), \quad \text{as } n \to \infty$$

for any $1 \leq r < p^*$. Therefore by Dominated convergence Theorem we have that

$$\lim_{n\to\infty} \int_0^T F(t, u_n)dt = \int_0^T F(t, u_0)dt. \tag{4.5}$$

Moreover, by Lemma (3.7), there exists s_1 such that $s_1 u_0 \in \mathcal{N}^+_{\lambda,\mu}$. Now, we shall prove $u_n \to u_0$ strongly in $E_0^{\alpha,p}$. Suppose otherwise, then either

$$\|u_0\|_{E_0^{\alpha,p}} \leq \liminf_{n\to\infty} \|u_n\|_{E_0^{\alpha,p}}.$$

Thus, since $u_n \in \mathcal{N}^+_\lambda$, one has

$$\begin{aligned}
\lim_{n\to\infty} \Phi'_{u_n}(s_1) &= \lim_{n\to\infty} \left(\left(a + bs^p \|u_n\|^p_{\alpha,p}\right)^{p-1} s_1^{p-1} \|u_n\|^p_{\alpha,p} - \frac{\lambda}{s_1^\gamma} \int_0^T g(t)|u_n(t)|^{1-\gamma}dt - s_1^{r-1} \int_0^T F(t, u_n(t))dt \right) \\
&> \left(a + bs_1^p \|u\|^p_{\alpha,p}\right)^{p-1} s_1^{p-1} \|u\|^p_{\alpha,p} - \frac{\lambda}{s_1^\gamma} \int_0^T g(t)|u(t)|^{1-\gamma}dt - s_1^{r-1} \int_0^T F(t, u(t))dt = \Phi'_{u_0}(s_1) = 0
\end{aligned}$$

Therefore, $\Phi'_{u_n}(s_1) > 0$ for n large enough. Since $u_n \in \mathcal{N}^+_\lambda$, we have $s_{max}(u_n) > 1$. Moreover $\Phi'_{u_n}(1) = 0$ and $\Phi_{u_n}(1)$ is increasing for $s \in (0, s_{max}(u_n))$. This implies that $\Phi_{u_n}(s) < 0$ for all $s \in (0, 1]$ and n sufficiently large. We obtain $1 < s_1 < s_{max}(u_0)$. But $s_1 u_0 \in \mathcal{N}^+_\lambda$ and

$$E_\lambda(s_1 u_0) = \inf_{1<s<s_{max}(u_0)} E_\lambda(su_0).$$

Which implies that

$$E_\lambda(s_1 u_0) < E_\lambda(u_0) = \lim_{n\to\infty} E_\lambda(u_n) = c^+_\lambda$$

which gives a contradiction. Thus, $u_n \to u_0$ strongly in $E_0^{\alpha,p}$, and $E_\lambda(u_0) = \inf_{u\in\mathcal{N}^+_\lambda} E_\lambda(u)$. The proof of the Theorem 4.2 is now completed. □

5. Existence of minimizer on $\mathcal{N}^-_\lambda$

In this section, we shall show the existence of second solution by proving the existence of minimizer of E_λ on $\mathcal{N}^-_\lambda$.

Lemma 5.1. *If $0 < \lambda < \Lambda_0$, then for all $u \in \mathcal{N}^+_\lambda$, $c^-_\lambda > 0$.*

Proof. Let $v \in \mathcal{N}^-_\lambda$, then we have $\phi''_v(1) < 0$ which gives,

$$\begin{aligned}
(p+\gamma-1)\frac{a^{p-1}}{p}\|v\|^p_{\alpha,p} &\leq (p+\gamma-1)\|u\|^p_{\alpha,p}\left(a + b\|v\|^p_{\alpha,p}\right)^{p-1} + bp(p-1)\|v\|^{2p}_{\alpha,p}\left(a + b\|v\|^p_{\alpha,p}\right)^{p-2} \\
&\leq (r+\gamma-1)\frac{KT^{1+r(\alpha-\frac{1}{p})}}{\beta^r}\|v\|^r_{\alpha,p},
\end{aligned}$$

this implies

$$\|v\|_{\alpha,p} \geq \left(\frac{(p+\gamma-1)a^{p-1}\beta^{\gamma}}{p(r+\gamma-1)KT^{1+r(\alpha-\frac{1}{p})}}\right)^{\frac{1}{r-p}}.$$

Subsequently,

$$\begin{aligned}
E_\lambda(v) &\geq \left(a+b\|v\|_{\alpha,p}\right)^p\left(\frac{a}{bp^2}+(\frac{1}{p^2}-\frac{1}{r})\|v\|^p_{\alpha,p}\right)-\lambda\left(\frac{1}{1-\gamma}-\frac{1}{r}\right)\frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}\|g\|_\infty}{(\Gamma(\alpha))^{1-\gamma}\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1-\gamma}{\widetilde{p}}}}\|v\|^{1-\gamma}_{\alpha,p}\\
&\geq \frac{a^{p-1}}{p}\|v\|^p_{\alpha,p}+(\frac{1}{p^2}-\frac{1}{r})bpa^{p-2}\|v\|^{2p}_{\alpha,p}-\lambda\left(\frac{1}{1-\gamma}-\frac{1}{r}\right)\frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}\|g\|_\infty}{(\Gamma(\alpha))^{1-\gamma}\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1-\gamma}{\widetilde{p}}}}\|v\|^{1-\gamma}_{\alpha,p}\\
&= \|v\|^{1-\gamma}_{\alpha,p}\left(\frac{a^{p-1}}{p}\|v\|^{p+\gamma-1}_{\alpha,p}+(\frac{1}{p^2}-\frac{1}{r})bpa^{p-2}\|v\|^{2p+\gamma-1}_{\alpha,p}-\lambda\left(\frac{1}{1-\gamma}-\frac{1}{r}\right)\frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}\|g\|_\infty}{(\Gamma(\alpha))^{1-\gamma}\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1-\gamma}{\widetilde{p}}}}\right)\\
&\geq \left(\frac{(p+\gamma-1)a^{p-1}\beta^{\gamma}}{p(r+\gamma-1)KT^{1+r(\alpha-\frac{1}{p})}}\right)^{\frac{1-\gamma}{r-p}}\left(\frac{a^{p-1}}{p}\left(\frac{(p+\gamma-1)a^{p-1}\beta^{\gamma}}{p(r+\gamma-1)KT^{1+r(\alpha-\frac{1}{p})}}\right)^{\frac{p+\gamma-1}{r-p}}\right.\\
&\quad\left.+(\frac{1}{p^2}-\frac{1}{r})bpa^{p-2}\left(\frac{(p+\gamma-1)a^{p-1}\beta^{\gamma}}{p(r+\gamma-1)KT^{1+r(\alpha-\frac{1}{p})}}\right)^{\frac{2p+\gamma-1}{r-p}}-\lambda\left(\frac{1}{1-\gamma}-\frac{1}{r}\right)\frac{T^{1+(1-\gamma)(\alpha-\frac{1}{p})}\|g\|_\infty}{(\Gamma(\alpha))^{1-\gamma}\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1-\gamma}{\widetilde{p}}}}\right).
\end{aligned}$$

Thus, if $0<\lambda<\Lambda_0$, then $E_\lambda(v)>k_0$ for all $v\in\mathcal{N}^-_{\lambda,\mu}$ for some $k_0=k_0(\gamma,\beta,p,a,T,\lambda,r,\tilde{p},\Gamma)>0$. Therefore $c^-_\lambda>k_0$ follows from the definition c^-_λ. This completes the proof of the Lemma 5.1. □

Theorem 5.2. *If $0<\lambda<\Lambda_0$, then there exists $v_0\in\mathcal{N}^-_\lambda$ satisfying $E_\lambda(v_0)=\inf_{v\in\mathcal{N}^-_\lambda}E_\lambda(v)$.*

Proof. Since E_λ is bounded below on $\mathcal{N}_{\lambda,\mu}$ and so on $\mathcal{N}^-_\lambda$. Then, there exists $\{v_n\}\subset\mathcal{N}^-_\lambda$ be a sequence such that

$$E_\lambda(v_n)\to\inf_{v\in\mathcal{N}^-_\lambda}E_\lambda(v)\ \text{ as } n\to\infty.$$

Since E_λ is coercive, $\{v_n\}$ is bounded in $E^{\alpha,p}_0$. Then there exists a subsequence, still denoted by v_n and $v_0\in E^{\alpha,p}_0$ such that, as $n\to\infty$,

$$v_n\rightharpoonup v_0 \text{ weakly in} E^{\alpha,p}_0$$

$$v_n\to v_0\ \text{ strongly in } L^q(\Omega)\ \text{ for all } 1\leq q<p^*,$$

$$v_n\to v_0\ \text{ a.e. in } \Omega.$$

Moreover, as in Lemma 4.2, we have

$$\lim_{n\to\infty}\int_0^T g(t)|v_n|^{1-\gamma}dt=\int_0^T g(t)|v_0|^{1-\gamma}dt$$

and

$$\lim_{n\to\infty}\int_0^T F(t,v_n)dt=\int_0^T F(t,v_0)dt.$$

Moreover, by Lemma (3.7), there exists s_0 such that $s_0v_0\in\mathcal{N}^-_\lambda$. Now, we prove $v_n\to v_0$ strongly in $E^{\alpha,p}_0$. Suppose otherwise, then either

$$\|v_0\|_{E^{\alpha,p}_0}\leq\liminf_{n\to\infty}\|v_n\|_{E^{\alpha,p}_0}$$

Thus, since $v_n \in \mathcal{N}_\lambda^-$, $E_\lambda(s_0 v_0) \le E_\lambda(v_0)$, for all $s_0 \ge 0$ we have

$$E_\lambda(s_0 v_0) < \lim_{n\to\infty} E_\lambda(s_0 v_n) \le \lim_{n\to\infty} E_\lambda(v_n) = c_\lambda^-$$

which gives a contradiction. Thus, $v_n \to v_0$ strongly in $E_0^{\alpha,p}$ and $E_\lambda(v_0) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v)$. The proof of the Theorem 5.2 is now completed. □

Proof. [Proof of Theorem 1.1] Now to prove the Theorem 1.1, let us start by proving the existence of non-negative solutions. First, by Theorems 4.2, 5.2, we conclude that there exist $u_0 \in \mathcal{N}_\lambda^+$, $v_0 \in \mathcal{N}_\lambda^-$ satisfying

$$E_\lambda(u_0) = \inf_{u\in\mathcal{N}_\lambda^+} E_\lambda(u)$$

and

$$E_\lambda(v_0) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v).$$

Moreover, since $E_\lambda(u_0) = E_\lambda(|u_0|)$ and $|u_0| \in \mathcal{N}_\lambda^+$ and similarly $E_\lambda(v_0) = E_\lambda(|v_0|)$ and $(|v_0|) \in \mathcal{N}_\lambda^-$, so we may assume $(u_0, v_0) \ge 0$. By Lemma 3.4, we may assume that (u_0, v_0) are nontrivials nonnegatives solutions of problem (P_λ). Finally, it remain to show that the solutions found in Theorems 4.2, 5.2, are distinct. Since $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset$, then, (u_0, v_0) are distinct. The proof of the Theorem 1.1 is now completed. □

6. Conclusion

Theorem 1.1 shows the multiplicity of positive solutions for a class of Kirchhoff fractional problem involving Riemann-Liouville fractional Derivative and singular nonlinearity by using the Nehari manifold approach and some variational techniques. This result generalises Theorem 1.1 and 1.2 in [12] which only concerns the case when the nonlinearity $f(t,x)$ is (p^2-1)-superlinear in x at infinity and the nonlinearity $f(t,x)$ is (p^2-1)-sublinear in x at infinity. The proof used to prove Theorem 1.1 does not modify the quasilinear operator $\Phi_p(s) = |s|^{p-2}s$ $(s \neq 0)$, $\Phi_p(0) = 0$ in (P_λ). Therefore, this approach could be considered for more general quasilinear operators to get multiplicity results. We suggest to extend the methods developed in this paper to the more general framework of Musielak-Orlicz-Sobolev spaces for a collection of stationary problems studied in these function spaces.

Acknowledgments

The author would like to thank the anonymous referees for their carefully reading this paper and their useful comments.

References

[1] R. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions,* Acta Appl. Math., **109** (2010), 973–1033.
[2] R. Agarwal, M. Belmekki, M. Benchohra, *A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative,* Adv. Difference Equ., **9** (2009), 47 pages.
[3] R.P. Agarwal, D. O'Regan, S. Stanek, *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations,* J. Math. Anal. Appl., **371** (2010) 57–68.
[4] A.M. Alghamdi, S. Gala, M.A. Ragusa, *New regularity criteria for the 3D hall-MHD equations*, Annales Polonici Mathematici, **121** (1) (2018),7–20 .
[5] Z. Bai, H. Lu, *Positive solutions for boundary value problem of nonlinear fractional differential equation,* J. Math. Anal. Appl., **311** (2005) 495-505.
[6] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, *The fractional order governing equation of Lévy motion,* Water Resour. Res., **36** (2000) 1413-1423.
[7] M. Benchohra, S. Hamani, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions,* Nonlinear Anal., **71** (2009) 2391-2396.
[8] H. Brezis, *Analyse fonctionelle, in: Théorie et Applications,* Masson, Paris, 1983.
[9] L. Bourdin, *Existence of a weak solution for fractional EulerLagrange equations*, J. Math. Anal. Appl. **399** (2013) 239-251.

[10] P. D'Ancona, S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data,* Invent. Math. **108** (1992) 247–262.
[11] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.A. Soriano, *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation,* Adv. Differential Equations, **6** (2001) 701–730.
[12] T. Chen, W. Liu, H. Jin, *Ground State Solutions of Kirchhoff-type Fractional Dirichlet Problem with p-Laplacian,* to appear's.
[13] J. Cresson, *Inverse problem of fractional calculus of variations for partial differential equations,* Commun. Nonlin. Sci. Numer. Simul., **15** (2010) 987–996.
[14] P. Drabek and S. I. Pohozaev,*Positive solutions for the p-Laplacian: application of the fibering method*, Proc. Royal Soc. Edinburgh Sect A, **127** (1997) 703–726.
[15] I. Ekeland, *On the variational principle,* J. Math. Anal. Appl., **47,** (1974),324–353.
[16] A. Ghanmi, K. Saoudi, *The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator,* Fractional Differential Calculus, **6** (2) (2016), 201–217.
[17] A. Ghanmi, K. Saoudi, *A multiplicity results for a singular problem involving the fractional p-Laplacian operator,* Complex Variables and Elliptic Equation, **61** (2016) 1199-1216.
[18] S. Gala, Z. Guo Z., M.A. Ragusa, *A remark on the regularity criterion of Boussinesq equations with zero heat conductivity*, Applied Mathematics Letters 27, (2014) 70-73.
[19] S. Gala, M.A. Ragusa, Y. Sawano, H. Tanaka, *Uniqueness criterion of weak solutions for the dissipative quasi-geostrophic equations in Orlicz-Morrey spaces*, Applicable Analysis, **93** (2) (2014) 356-368.
[20] A. Ghanmi, M. Kratou, K. Saoudi, *A multiplicity results for a singular problem involving a Riemann-Liouville fractional derivative*, Filomat, 32:2 (2018), 653–669.
[21] W. Jiang, *The existence of solutions to boundary value problems of fractional differential equations at resonance*, Nonlinear Anal., **74** (2011) 1987–1994.
[22] F. Jiao, Y. Zhou, *Existence of solutions for a class of fractional boundary value problems via critical point theory, Comput. Math. Appl.*, **62** (2011), 1181–1199.
[23] G. Kirchhoff, *Mechanik, Teubner*, Leipzig, 1883.
[24] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.
[25] R. A. Khan, M. Rehman, H. Johnny, *Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions,* Fract. Differ. Calc., **1**(1) (2011), 29–43.
[26] V. Lakshmikantham, *Theory of fractional functional differential equations,* Nonlinear Anal., **69** (2008), 3337–3343.
[27] Y. Liu, *Existence of three positive solutions for boundary value problems of singular fractional differential equations,* Fract. Differ. Calc., **2** (2012), no. 1, 55–69.
[28] K. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley and Sons, New York, 1993.
[29] S.I. Pohozaev, *A certain class of quasilinear hyperbolic equations,* Mat. Sb. (NS), **96** (1975) 152-166.
[30] P. Pucci, M. Xiang, B. Zhang, *Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations, **54** (2015), no. 3, 2785-2806.
[31] K. Saoudi, *A critical Fractional elliptic equation with singular nonlinearities*, Fractional Calculus and Applied Analysis **20** (2017) no 6, 1507-1530.
[32] K. Saoudi, J. Choi, P. Agarwal, A. Ghanmi, *The Nehari manifold for a boundary value problem involving Riemann-Liouville fractional derivative*, Adances in difference equations, (2018) 2018:263.
[33] J. F. Xu, Z. Yang, *Multiple positive solutions of a singular fractional boundary value problem,* Appl. Math. E-Notes, **10** (2010), 259-267.