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The Characterization and the Product of Quasi-Ehresmann Transversals

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Abstract. Wang (Filomat 29(5), 985-1005, 2015) introduced and investigated quasi-Ehresmann transversals of semi-abundant semigroups satisfy conditions (CR) and (CL) as the generalizations of orthodox transversals of regular semigroups in the semi-abundant case. In this paper, we give two characterizations for a generalized quasi-Ehresmann transversal to be a quasi-Ehresmann transversal. These results further demonstrate that quasi-Ehresmann transversals are the "real" generalizations of orthodox transversals in the semi-abundant case. Moreover, we obtain the main result that the product of any two quasi-ideal quasi-Ehresmann transversals of a semi-abundant semigroup S which satisfy the certain conditions is a quasi-ideal quasi-Ehresmann transversal of S.

1. Introduction

The concept of inverse transversals of regular semigroups was introduced by Blyth-McFadden [1]. Since then, inverse transversals have attracted much attention and a series of important results have been obtained and generalized (see [1-5,11,13-21,23-26]). If S is a regular semigroup, then *an inverse transversal* of S is an inverse subsemigroup S^o which meets $V(a)$ precisely once for each $a \in S$ (that is, $|V(a) \cap S^o| = 1$), where $V(a) = \{x \in S|\, axa = a \text{ and } xax = x\}$ denotes the set of inverses of a. Since orthodox semigroups can be considered as generalizations of inverse semigroups, Chen [2] generalized inverse transversals to orthodox transversals in the class of regular semigroups and gave a construction theorem for regular semigroups with quasi-ideal orthodox transversals. Chen-Guo [4] obtained some important properties associated with orthodox transversals in the general case. Most recently, Kong, Meng, Zhao [13,15,16,17,21] investigated orthodox transversals and obtained some interesting results. Especially, Kong-Meng [17] acquired the characterization for a generalized orthodox transversal to be an orthodox transversal and present a concrete description of the maximum idempotent separating congruence on regular semigroups with orthodox transversals. If the concept of transversals could be introduced in the E-inversive semigroups, then the congruences [6, 7] on them will be characterized more neatly.

The concept of adequate transversals was introduced for abundant semigroups by El-Qallali [5] as an analogue of inverse transversals, and followed by Chen, Guo, Shum, Kong and Wang etc. [3,11,14,18,19]. In [19], the authors shown that the product of any two quasi-ideal adequate transversals of an abundant

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semigroup S which satisfy the regularity condition is a quasi-ideal adequate transversal of S.

Semi-abundant semigroups satisfy conditions (CR) and (CL) were introduced by Fountain-Gomes-Gould [8] as generalized regular semigroups and studied by many authors[8,9,10,22,26]. Gomes-Gould [9] studied some classes of semi-abundant semigroups satisfy conditions (CR) and (CL) by fundamental approaches and Lawson [22] considered some kinds of semi-abundant semigroups satisfy conditions (CR) and (CL) by category approaches, and Gould [10] gave a survey of investigations of special semi-abundant semigroups satisfy conditions (CR) and (CL), namely restriction semigroups and Ehresmann semigroups. Wang [26] introduced the concept of quasi-Ehresmann transversals of semi-abundant semigroups satisfy conditions (CR) and (CL), as a generalization of the concept of orthodox transversals of regular semigroups, and gave some properties associated with quasi- Ehresmann transversals.

In this paper, we continue along the line of [8,17,19,26] by studying quasi-Ehresmann transversals of semi-abundant semigroups which satisfy conditions (CR) and (CL). In this paper, we give two characterizations for a generalized quasi-Ehresmann transversal to be a quasi-Ehresmann transversal which further demonstrate that quasi-Ehresmann transversals are the "real" generalizations of orthodox transversals in the semi-abundant case. The main purpose of this paper is to show that the product of any two quasi-ideal quasi-Ehresmann transversals of a semi-abundant semigroup S satisfies conditions (CR) and (CL) and satisfies the regularity condition is a quasi-ideal quasi-Ehresmann transversal of S. The corresponding results associate with orthodox transversals and adequate transversals are generalized and enriched.

2. Preliminaries

Let S and S^o be semigroups. Throughout this paper, if no confusion, the set of idempotents of S and S^o are denoted by E and E^o, respectively. For short, the set $V(a) \cap S^o$ is denoted by $V_{S^o}(a)$. If E generates a regular semiband, that is, $\langle E \rangle$ is a regular subsemigroup of S, then S is said to *satisfy the regularity condition*. S^o is called a *quasi-ideal* of S, if $S^oSS^o \subseteq S^o$. We list some basic results as follows which are frequently used in this paper.

Definition 2.1[2] Let S be a regular semigroup with an orthodox subsemigroup of S^o. Then S^o is said to be an *orthodox transversal* of S, if the following two conditions are satisfied:

(1) $(\forall\, a \in S) \quad V_{S^o}(a) \neq \emptyset$;

(2) For any $a, b \in S$, if $\{a, b\} \cap S^o \neq \emptyset$, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$.

Lemma 2.1[17] Let S be a regular semigroup and S^o a subsemigroup of S with $V_{S^o}(a) \neq \emptyset$ for each $a \in S$. Then S^o is an orthodox transversal of S if and only if

$$(\forall a, b \in S) \quad [V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)].$$

The so-called Miller-Clifford theorem will be frequently used in this paper.

Lemma 2.2[12] (1) Let e and f be $\mathcal{D}$-equivalent idempotents of a semigroup S. Then each element a of $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$, such that $aa' = e$ and $a'a = f$;

(2) Let $a, b \in S$. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.

Let S be a semigroup and $a, b \in S$. By $a\mathcal{R}^*b$ we mean that $xa = ya$ if and only if $xb = yb$ for all $x, y \in S^1$. The relation $\mathcal{L}^*$ can be defined dually. $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence on S. A semigroup S is called *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of S contains at least one idempotent. An abundant semigroup S is called *quasi-adequate* if its idempotents form a band. A band B is called a *rectangular band* if it satisfies the identity $abc = ac$ for all $a, b, c \in B$. An *adequate semigroup* is an abundant semigroup in which the idempotents commute.

Let S be an abundant semigroup and U an abundant subsemigroup of S. U is called a $*$*-subsemigroup* of S, if for any $a \in U$, there exist idempotents $e \in L^*_a(S) \cap U$ and $f \in R^*_a(S) \cap U$.

Definition 2.2[5] Let S be an abundant semigroup and S^o a $*$-adequate subsemigroup of S. S^o is called an *adequate transversal* of S, if for each $x \in S$ there exist idempotents $e, f \in S$ and a unique element $\overline{x} \in S^o$ such that $x = e\overline{x}f$, where $e\mathcal{L}\overline{x}^+$ and $f\mathcal{R}\overline{x}^*$.

Let S be a semigroup and $a, b \in S$. That $a\widetilde{\mathcal{R}}b$ means that $ea = a$ if and only if $eb = b$ for all $e \in E$. The relation $\widetilde{\mathcal{L}}$ can be defined dually. Denote $\widetilde{\mathcal{H}} = \widetilde{\mathcal{L}} \cap \widetilde{\mathcal{R}}$. In general, $\widetilde{\mathcal{L}}$ is not a right congruence and $\widetilde{\mathcal{R}}$ is not a left congruence. Obviously, $L \subseteq \widetilde{\mathcal{L}}$ and $R \subseteq \widetilde{\mathcal{R}}$. If $a, b \in RegS$, the set of regular elements of S, then $a\widetilde{\mathcal{R}}b$ ($a\widetilde{\mathcal{L}}b$) if and only if $a\mathcal{R}b$ ($a\mathcal{L}b$). On the relation $\widetilde{\mathcal{R}}$ on a semigroup S, we have the following useful result.

Lemma 2.3 Let S be a semigroup and $a \in S, e \in E$. Then the following statements are equivalent:
(1) $e\widetilde{\mathcal{R}}a$;
(2) $ea = a$ and for all $f \in E$, $fa = a$ implies $fe = e$.

Now, we state the following fundamental concept of our paper. Semi-abundant semigroups satisfy conditions (CR) and (CL) were introduced by Fountain-Gomes-Gould[8].

Definition 2.3 A semigroup S is called *semi-abundant* if each $\widetilde{\mathcal{L}}$-class and each $\widetilde{\mathcal{R}}$ -class of S contains idempotents. In particular, if $\widetilde{\mathcal{L}}$ is a right congruence and $\widetilde{\mathcal{R}}$ is a left congruence on a semi-abundant semigroup S, then we say that *S satisfies conditions (CR) and (CL)*.

A semi-abundant semigroup S satisfies conditions (CR) and (CL) is quasi-Ehresmann if its idempotents form a subsemigroup of S. Certainly, regular semigroups are semi-abundant semigroups satisfy conditions (CR) and (CL), and orthodox semigroups are quasi-Ehresmann semigroups. It is easy to see a semi-abundant semigroup S satisfies conditions (CR) and (CL) is quasi-Ehresmann if and only if $RegS$ is an orthodox subsemigroup of S. Let S be a semi-abundant semigroup satisfies conditions (CR) and (CL). For $\widetilde{\mathcal{K}} \in \{\widetilde{\mathcal{L}}, \widetilde{\mathcal{R}}\}$ and $a \in S$, the $\widetilde{\mathcal{K}}$-class of S containing a is denoted by $\widetilde{\mathcal{K}}_a$.

A semi-abundant subsemigroup U of a semi-abundant semigroup S satisfies conditions (CR) and (CL)is called a $\sim$-subsemigroup of S if

$$\widetilde{\mathcal{L}}(U) = \widetilde{\mathcal{L}}(S) \cap (U \times U),\ \widetilde{\mathcal{R}}(U) = \widetilde{\mathcal{R}}(S) \cap (U \times U),$$

and this equivalent to that there exist idempotents $e, f \in U$ such that $e\widetilde{\mathcal{L}}x$ and $f\widetilde{\mathcal{R}}x$ in S for all $x \in U$. Now, let S be a semi-abundant semigroup satisfies conditions (CR) and (CL) and S^o a quasi-Ehresmann $\sim$-subsemigroup of S. For any $x \in S$, denote

$$\Omega_{S^o}(x) = \{(e, \overline{x}, f) \in E \times S^o \times E : x = e\overline{x}f, e\widetilde{\mathcal{L}}\overline{x}^+; f\widetilde{\mathcal{R}}\overline{x}^* \text{ for some } \overline{x}^+, \overline{x}^* \in E^o\},$$

and $\Gamma_x = \{\overline{x} : (e, \overline{x}, f) \in \Omega_{S^o}(x)\}, I(x) = \{e : (e, \overline{x}, f) \in \Omega_{S^o}(x)\}$, $\Lambda(x) = \{f : (e, \overline{x}, f) \in \Omega_{S^o}(x)\}$, $I = \bigcup_{x \in S} I(x)$, $\Lambda = \bigcup_{x \in S} \Lambda(x)$.

Lemma 2.4[26] Let S be a semi-abundant semigroup satisfies conditions (CR) and (CL) and S^o a quasi-Ehresmann $\sim$-subsemigroup of S. Then $I = \{e \in E : (\exists e^* \in E^o)\ e\widetilde{\mathcal{L}}e^*\}$ *and* $\Lambda = \{f \in E : (\exists f^+ \in E^o)\ f\widetilde{\mathcal{R}}f^+\}$.

Definition 2.4[26] Let S be a semi-abundant semigroup satisfies conditions (CR) and (CL) and S^o a quasi-Ehresmann $\sim$-subsemigroup of S. Then S^o is called a *quasi-Ehresmann transversal* of S if the following three conditions hold:
(1) $\Gamma_x \neq \emptyset$ for all $x \in S$;
(2) $is \in I$ and $si \in RegS$ implies $si \in E$ for all $i \in I$ and $s \in E^o$;
(3) $s\lambda \in \Lambda$ and $\lambda s \in RegS$ implies $\lambda s \in E$ for all $\lambda \in \Lambda$ and $s \in E^o$.

3. Two characterizations of quasi-Ehresmann transversals

Let S be a semi-abundant semigroup satisfies conditions (CR) and (CL) with the set of idempotents E and S^o a quasi-Ehresmann $\sim$-subsemigroup of S with the set of idempotents E^o. S^o is called a *generalized quasi-Ehresmann transversal* of S if $\Gamma_x \neq \emptyset$ for all $x \in S$.

In the following, we shall give two characterizations for a generalized quasi-Ehresmann transversal to be a quasi-Ehresmann transversal which further demonstrate that quasi-Ehresmann transversals are the "real" generalizations of orthodox transversals in the semi-abundant case.

Theorem 3.1 *Let S be a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal S^o. Then S^o is a quasi-Ehresmann transversal of S if and only if*

$$(\forall a, b \in RegS), \ \ [V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)].$$

Proof. (Sufficiency) Let $f \in E^o, e \in I$ with $e\mathcal{L}e^* \in E^o$. By means of S^o is quasi-Ehresmann and $ef\widetilde{\mathcal{L}}e^*f \in E^o$ we have

$$fe^* \cdot ef \cdot fe^* = f \cdot e^*e \cdot ff \cdot e^* = fe^* \cdot fe^* = fe^*$$

$$ef \cdot fe^* \cdot ef = e \cdot ff \cdot e^*e \cdot f = ef \cdot e^*f = ef.$$

Thus $fe^* \in V_{S^o}(fe^*) \cap V_{S^o}(ef)$, by the condition, we obtain $V_{S^o}(fe^*) = V_{S^o}(ef)$. From S^o is quasi-Ehresmann, we deduce that E^o is a band and so is the semilattice Y of rectangular bands $E_\alpha (\alpha \in Y)$. Since e^*f and fe^* are in the same rectangular band, and so are inverses of each other. Hence $e^*f \in V_{S^o}(ef)$ and so

$$ef = (ef)(e^*f)(ef) = (ef)^2$$

That is ef is idempotent and we have in fact proved $IE^o \subseteq E$.

If fe is regular, take $x \in V_{S^o}(fe)$ and $x^o \in V_{S^o}(x)$. Then exf is idempotent and $exf \in V(fe)$ with $exf\widetilde{\mathcal{L}}e^*xf \in S^o$. Let $(e^*xf)^* \in E^o$ with $(e^*xf)^*\widetilde{\mathcal{L}}e^*xf$ since $\widetilde{\mathcal{L}}$ is a left congruence. Then $exf\mathcal{L}(e^*xf)^* \in E^o$ and so $(e^*xf)^* \in V_{S^o}(exf) \cap V_{S^o}((e^*xf)^*)$. From the assumption and S^o is quasi-Ehresmann, we have $V_{S^o}(exf) = V_{S^o}((e^*xf)^*)$ and hence $V_{S^o}(exf) \subseteq E^o$. Meanwhile we deduce that the regular elements of S^o form an orthodox subsemigroup of S^o, and so $fx^oe^* \in V_{S^o}(e^*xf)$ since $e^*, f \in E^o$. Hence

$$fx^oe^* \cdot exf \cdot fx^oe^* = fx^oe^* \cdot e^*xf \cdot fx^oe^* = fx^oe^*$$

and

$$exf \cdot fx^oe^* \cdot exf = e \cdot e^*xf \cdot fx^oe^* \cdot e^*xf = e \cdot e^*xf = exf$$

since $e^*\mathcal{L}e$ with e, e^* are idempotent. So, $fx^oe^* \in V_{S^o}(exf)$. Similarly, one can prove that $e^*xf \in V_{S^o}(fe) \cap V_{S^o}(fx^oe^*)$. Thus $fx^oe^* \in E^o$ and $V_{S^o}(fe) = V_{S^o}(fx^oe^*) \subseteq E^o$ and consequently $x \in E^o$. Therefore $e^*xf \in E^o$ and

$$fe = fe \cdot e^*xf \cdot fe = fe \cdot exf \cdot e^*xf \cdot fe = fex \cdot fe^* \cdot xfe.$$

Premultiplying and postmultiplying by x, we obtain

$$x = xfex = xfex \cdot fe^* \cdot xfex = xfe^*x.$$

Thus $fe^*x\mathcal{L}x$ with $fe^*x \in E^o$, from which we deduce that $fe^*xf = fe^*x \cdot xf\mathcal{L}xf$. By means of $fe^*xf, xf \in E^o$, we have $fe^*xf \in V_{S^o}(xf)$. It is obvious that $xf \in V(fe)$ and $xf \in E^o$ implies that $V_{S^o}(fe) = V_{S^o}(xf)$ and so $fe^*xf \in V_{S^o}(fe)$. Therefore $fe = fe \cdot fe^*xf \cdot fe = f(ef)(ef)e^*xffe = fe \cdot fefe^*xffe = fefe$ since ef is idempotent, and so fe is idempotent. Up to now, we have in fact proved if fe is regular, then it is idempotent. Dually, we can proved that $E^o\Lambda \subseteq E$ and if for all $\lambda \in \Lambda, f \in E^o$, if λf is regular, then it is idempotent.

(Necessity) By [26, Theorem 3.6 (4)], the condition is necessary. □

Theorem 3.2 *Let S be a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal S^o. Then S^o is a quasi-Ehresmann transversal if and only if for any regular elements $a \in S$, $b \in S^o$, if ba is regular, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$; and if ab is regular, then $V_{S^o}(b)V_{S^o}(a) \subseteq V_{S^o}(ab)$.*

Proof. (Necessity) For any regular elements $a \in S$, $b \in S^o$, take $a^o \in V_{S^o}(a)$, $b^o \in V_{S^o}(b)$, if S^o is a quasi-Ehresmann transversal, then by the definition, $aa^ob^ob \in IE^o \subseteq E$. If ba is regular, take $(ba)^o \in V_{S^o}(ba)$, then

$$(b^obaa^o)(a(ba)^ob)(b^obaa^o) = b^o(baa^oa)(ba)^o(bb^oba)a^o = b^o(ba)(ba)^o(ba)a^o = b^o(ba)a^o = b^obaa^o.$$

Thus b^obaa^o is regular and so $b^obaa^o \in E^oI \subseteq E$. Therefore

$$a^ob^o \cdot ba \cdot a^ob^o = a^o(aa^ob^ob)(aa^ob^ob)b^o = a^o \cdot aa^ob^ob \cdot b^o = a^ob^o$$

$$ba \cdot a^ob^o \cdot ba = b(b^obaa^o)(b^obaa^o)a = b \cdot b^obaa^o \cdot a = ba$$

and so $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$. Similarly, if ab is regular, then $V_{S^o}(b)V_{S^o}(a) \subseteq V_{S^o}(ab)$.

(Sufficiency) For any regular elements $t_1, t_2 \in S^o$, if $V(t_1) \cap V(t_2) \neq \emptyset$, take $t \in V(t_1) \cap V(t_2)$ and $t_1^o \in V_{S^o}(t_1)$. From $t_2t\mathcal{L}t_1t\mathcal{R}t_1t_1^o$, by Lemma 2.2, $t_2\mathcal{R}t_2tt_1t_1^o\mathcal{L}t_1^o$ and $(t_2tt_1t_1^o)^2 = t_2(tt_1t_1^ot_2tt_1)t_1^o = t_2tt_1t_1^o$ since by the assumption $t_1^ot_2 \in V_{S^o}(tt_1)$. Similarly, $t_2\mathcal{L}t_1^ot_1tt_2\mathcal{R}t_1^o$ with $t_1^ot_1tt_2 \in E$. Thus

$$t_1^ot_2t_1^o = t_1^ot_2tt_1tt_2t_1^o = t_1^o(t_2tt_1t_1^o)t_1tt_2t_1^o = (t_1^ot_1tt_2)t_1^o = t_1^o$$

and

$$t_2t_1^ot_2 = t_2(tt_2t_1^ot_1)t_1^o(t_1t_1^ot_2t)t_2 = t_2tt_1t_1^ot_1tt_2 = t_2tt_1tt_2 = t_2tt_2 = t_2.$$

Hence $t_1^o \in V_{S^o}(t_2)$, that is, $V_{S^o}(t_1) \cap V_{S^o}(t_2) \neq \emptyset$. Therefore $V_{S^o}(t_1) = V_{S^o}(t_2)$ since the regular elements of S^o form an orthodox subsemigroup of S.

For any $e \in S$, if $V_{S^o}(e) \cap E^o \neq \emptyset$, take $f \in V_{S^o}(e) \cap E^o$. Then for any $e^o \in V_{S^o}(e)$, we have $e \in V(f) \cap V(e^o)$ and so by the above result, $V_{S^o}(f) = V_{S^o}(e^o)$. Consequently, e^o is an inverse of f in S^o and $e^o \in E^o$ since S^o is quasi-Ehresmann. That is, if $V_{S^o}(e) \cap E^o \neq \emptyset$, then $V_{S^o}(e) \subseteq E^o$.

Let $e, f \in I$ with $e\mathcal{L}f$. Take $h \in E^o$ such that $h\mathcal{L}e\mathcal{L}f$, then $h \in V_{S^o}(e) \cap V_{S^o}(f)$. For any $g \in V_{S^o}(e)$, by the above result we have $g \in E^o$. It is easy to see that $ghg \in V_{S^o}(gfg)$ and $ghg \in V_{S^o}(geg) = V_{S^o}(g)$. Then gfg and g have a common inverse ghg. Consequently $ghg \cdot gfg \cdot ghg = ghg$ and thus $gfg = g$. Since $ge\mathcal{L}e\mathcal{L}f$, by Lemma 2.2, $fg\mathcal{R}f$ and so $fgf = f$. Thus $g \in V_{S^o}(f)$ and so $V_{S^o}(e) \subseteq V_{S^o}(f)$. Similarly, we have the reverse inclusion and hence $V_{S^o}(e) = V_{S^o}(f)$. Dually, if $e, f \in \Lambda$ with $e\mathcal{R}f$, then $V_{S^o}(e) = V_{S^o}(f)$.

It is easy to see that if $a \in RegS$, then for any $a^o \in V_{S^o}(a)$, we have $V_{S^o}(a) = V_{S^o}(a^oa)a^oV_{S^o}(aa^o)$.

For $a, b \in RegS$, if $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset$, take $c^o \in V_{S^o}(a) \cap V_{S^o}(b)$. Then $V_{S^o}(a) = V_{S^o}(c^oa)c^oV_{S^o}(ac^o)$ and $V_{S^o}(b) = V_{S^o}(c^ob)c^oV_{S^o}(bc^o)$. It follows from $ac^o, bc^o \in I$ and $ac^o\mathcal{L}bc^o$ that $V_{S^o}(ac^o) = V_{S^o}(bc^o)$. Similarly, $V_{S^o}(c^oa) = V_{S^o}(c^ob)$. Therefore $V_{S^o}(a) = V_{S^o}(b)$ and so by Theorem 3.1 S^o is a quasi-Ehresmann transversal. □

Obviously, a regular semigroup with an orthodox transversal is a semi-abundant semigroup satisfies conditions (CR) and (CL) with a generalized quasi-Ehresmann transversal. Comparing Lemma 2.1 with Theorem 3.1, and Definition 2.1 with Theorem 3.2, it is illustrated by these two points of view that the transversal is a quasi-Ehresmann transversal. Thus, quasi-Ehresmann transversals are the generalization of orthodox transversals in the semi-abundant case.

By means of the properties of adequate transversal[3,*Theorem*3.3], one can easily observe that an abundant semigroup with an adequate transversal is a semi-abundant semigroup satisfies conditions (CR) and (CL) with a quasi-Ehresmann transversal.

In the following, we will investigate when a quasi-Ehresmann transversal is an orthodox transversal and when a quasi-Ehresmann transversal is an adequate transversal, respectively. We have the following results.

Theorem 3.3 *Let S^o be a quasi-Ehresmann transversal of the semi-abundant semigroup S satisfies conditions (CR) and (CL). Then*

(i) S^o is an orthodox transversal of S if and only if S is a regular semigroup.

(ii) if S and S^o are abundant, then S^o is an adequate transversal of S if and only if S^o is an adequate semigroup.

Proof. (i) (Sufficiency) If S is regular, every element in S is regular, and so $V_{S^o}(a) \neq \emptyset$ for each $a \in S$. It follows from Theorem 3.1 that for any $a, b \in S$, $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset$ implies that $V_{S^o}(a) = V_{S^o}(b)$. Thus, S^o is an orthodox transversal of S by Lemma 2.1.

(Necessity) If S^o is an orthodox transversal, every element x^o in S^o is regular. For any $a \in S, a = e\overline{a}f$ with $e, f \in E, e\mathcal{L}\overline{a}^{+} \in E^o, f\mathcal{R}\overline{a}^{*} \in E^o$. Since $\overline{a}$ is regular, $\overline{a}^{*}\mathcal{L}\overline{a}\mathcal{R}\overline{a}^{+}$ implies that, $\overline{a}$ has a unique inverse $x \in R_{\overline{a}^{*}} \cap L_{\overline{a}^{+}}$, such that $\overline{a}x = \overline{a}^{+}, x\overline{a} = \overline{a}^{*}$. Consequently, $axa = e\overline{a}f \cdot x\overline{a}x \cdot e\overline{a}f = e\overline{a} \cdot x\overline{a}x \cdot \overline{a}f = e(\overline{a}x\overline{a}x\overline{a})f = e\overline{a}f = a$ since $f\mathcal{R}\overline{a}^{*} = x\overline{a}e\mathcal{L}\overline{a}^{+} = \overline{a}x$. That is, a is regular and therefore S is a regular semigroup.

(ii) The necessary condition is obvious.

(Sufficiency) Let $a \in S, a = e\overline{a}f$ with $e, f \in E, e\mathcal{L}\overline{a}^{+} \in E^o, f\mathcal{R}\overline{a}^{*} \in E^o, a = i\overline{b}j$ with $i, j \in E, i\mathcal{L}\overline{b}^{+} \in E^o, j\mathcal{R}\overline{b}^{*} \in E^o$. It follows from $e\widetilde{\mathcal{R}}a\widetilde{\mathcal{R}}i\mathcal{L}\overline{b}^{+}$ that $\overline{b}^{+}e \cdot i \cdot \overline{b}^{+}e = \overline{b}^{+}i\overline{b}^{+}e = \overline{b}^{+}e$, so $\overline{b}^{+}e$ is regular and by Theorem 3.1, $\overline{b}^{+}e \in E$. Since $\overline{b}^{+}\mathcal{R}\overline{b}^{+}e\mathcal{L}e\mathcal{L}\overline{a}^{+}$, if S^o is adequate by Lemma 2.2, $\overline{a}^{+}\mathcal{R}\overline{a}^{+}\overline{b}^{+}\mathcal{L}\overline{b}^{+}$ with $\overline{a}^{+}\overline{b}^{+} \in E$, and so $\overline{a}^{+}\mathcal{L}\overline{b}^{+}\overline{a}^{+}\mathcal{R}\overline{b}^{+}$. By S^o is adequate, the idempotents in S^o commute and so $\overline{a}^{+}\overline{b}^{+} = \overline{b}^{+}\overline{a}^{+}$. Hence $\overline{a}^{+}, \overline{b}^{+}$ are in the same $\mathcal{H}-$class and so $\overline{a}^{+} = \overline{b}^{+}$. Similarly, $\overline{a}^{*} = \overline{b}^{*}$. Therefore $\overline{a} = \overline{a}^{+}a\overline{a}^{*} = \overline{b}^{+}a\overline{b}^{*} = \overline{b}$ and consequently, S^o is in fact the adequate transversal of S. □

Therefore, by Theorem 3.3 we can say that quasi-Ehresmann transversals are the "real" common generalization of orthodox transversals and adequate transversals in the semi-abundant case.

4. The main theorem

In 1986, Saito[25] had proved that the product of any two quasi-ideal inverse transversals of a regular semigroup S is a quasi-ideal inverse transversal of S. In 2011, we[19] had obtained that the product of any two quasi-ideal adequate transversals of an abundant semigroup S which satisfy the regularity condition is a quasi-ideal adequate transversal of S. In this section, we acquire that the product of any two quasi-ideal quasi-Ehresmann transversals of a semi-abundant semigroup S satisfies conditions (CR) and (CL) and satisfies the regularity condition is a quasi-ideal quasi-Ehresmann transversal of S. Furthermore, all of the quasi-ideal quasi-Ehresmann transversals of S form a rectangular band.

Let H and J be subsets of a semigroup S and write HJ for $\{hj : h \in H, j \in J\}$. Clearly $(\forall H, J, K \subseteq S)\ (HJ)K = H(JK)$ and we denote it by HJK.

Lemma 4.1 Let S^o be a quasi-ideal quasi-Ehresmann transversal of the semi-abundant semigroup S satisfies conditions (CR) and (CL) and H a subset of S. Then

(1) $HSS^o = HS^o$ and $S^oSH = S^oH$;

(2) HS^o and S^oH are both subsemigroups and quasi-ideals of S;

(3) for any $x \in RegS$, if $|V(x) \cap H| \geq 1$, then $|V(x) \cap HS^o| \geq 1$ and $|V(x) \cap S^oH| \geq 1$.

Proof. (1) Let $h \in H, x \in S$ and $s \in S^o$. Then $h = e_h\overline{h}f_h$ with $f_h\mathcal{R}\overline{h}^{*} \in E^o$ and so $hxs = h\overline{h}^{*}f_hxs \in HS^oSS^o \subseteq HS^o$. It is obvious that $hs = hf_hs \in HSS^o$ and thus $HSS^o = HS^o$. Similarly, $S^oSH = S^oH$.

(2) It is easy to see $HS^o \cdot HS^o \subseteq H \cdot S^oSS^o \subseteq HS^o$, thus HS^o is a subsemigroup of S. Similarly, $HS^o \cdot S \cdot HS^o \subseteq H \cdot S^oSS^o \subseteq HS^o$ and so HS^o is a quasi-ideal of S. There is a dual result for S^oH.

(3) For any regular element $x \in S$, take $x' \in V(x) \cap H$, then for any $x^o \in V_{S^o}(x)$, $x'xx^o \in V(x) \cap HSS^o = V(x) \cap HS^o$, that is $|V(x) \cap HS^o| \geq 1$. Similarly, $|V(x) \cap S^oH| \geq 1$. □

Lemma 4.2 Let S^o, $S^\diamond$ be quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup S satisfies conditions (CR) and (CL). For every $a \in RegS$, we have $V_{S^\diamond S^o}(a) = V_{S^\diamond}(a) \cdot a \cdot V_{S^o}(a)$.

Proof. Let $a^\diamond \in V_{S^\diamond}(a), a^o \in V_{S^o}(a)$. Then $a^\diamond aa^o \in S^\diamond SS^o = S^\diamond S^o$ and $a^\diamond aa^o \in V(a)$, and so $V_{S^\diamond}(a) \cdot a \cdot V_{S^o}(a) \subseteq$

$V_{S^\diamond S^o}(a)$. For every $x^\diamond y^o \in V_{S^\diamond S^o}(a)$, we have

$$a = ax^\diamond y^o a,\ \ x^\diamond y^o = x^\diamond y^o \cdot a \cdot x^\diamond y^o.$$

Hence

$$x^\diamond y^o = x^\diamond y^o \cdot aa^\diamond aa^o a \cdot x^\diamond y^o = x^\diamond y^o aa^\diamond \cdot a \cdot a^o ax^\diamond y^o.$$

and

$$x^\diamond y^o aa^\diamond \in S^\diamond SS^\diamond \subseteq S^\diamond,\ \ a^o ax^\diamond y^o \in S^o SS^o \subseteq S^o,$$

On the other hand,

$$a \cdot x^\diamond y^o aa^\diamond \cdot a = a \cdot x^\diamond y^o \cdot a = a,$$

$$x^\diamond y^o aa^\diamond \cdot a \cdot x^\diamond y^o aa^\diamond = x^\diamond y^o ax^\diamond y^o aa^\diamond = x^\diamond y^o aa^\diamond.$$

Thus $x^\diamond y^o aa^\diamond \in V_{S^\diamond}(a)$ and dually, $a^o ax^\diamond y^o \in V_{S^o}(a)$. Therefore $V_{S^\diamond S^o}(a) \subseteq V_{S^\diamond}(a) \cdot a \cdot V_{S^o}(a)$. □

Lemma 4.3 Let S^o be a quasi-ideal quasi-Ehresmann transversal of the semi-abundant semigroup S satisfies conditions (CR) and (CL). For any $x, y \in S$, there exist $\overline{x} \in \Gamma_x, \overline{y} \in \Gamma_y$ such that $x = e_x\overline{x}f_x, e_x\mathcal{L}\overline{x}^+, f_x\mathcal{R}\overline{x}^*$ for some $\overline{x}^+, \overline{x}^* \in E^o$ and $y = e_y\overline{y}f_y, e_y\mathcal{L}\overline{y}^+, f_y\mathcal{R}\overline{y}^*$ for $\overline{y}^+, \overline{y}^* \in E^o$. Then

(1) $\overline{x}f_xe_y\overline{y} \in \Gamma_{xy}$;

(2) $e_x(\overline{x}f_xe_y)^+ \in I_{xy}$;

(3) $(f_xe_y\overline{y})^* f_y \in \Lambda_{xy}$.

Proof. Certainly

$$xy = e_x\overline{x}f_xe_y\overline{y}f_y = e_x(\overline{x}f_xe_y)^+(\overline{x}f_xe_y\overline{y})(f_xe_y\overline{y})^* f_y,$$

where $e_x(\overline{x}f_xe_y)^+ \in IE^o \subseteq E$, $(f_xe_y\overline{y})^* f_y \in E^o\Lambda \subseteq E$ and $\overline{x}f_xe_y\overline{y} \in S^o$ since S^o is a quasi-ideal. Since $\widetilde{\mathcal{R}}, \mathcal{R}$ are left congruences and $\widetilde{\mathcal{L}}, \mathcal{L}$ are right congruences, we have

$$e_x(\overline{x}f_xe_y)^+ \,\mathcal{L}\, \overline{x}^+(\overline{x}f_xe_y)^+ \,\widetilde{\mathcal{R}}\, \overline{x}^+(\overline{x}f_xe_y) = \overline{x}f_xe_y\overline{y}^+ \,\widetilde{\mathcal{R}}\, \overline{x}f_xe_y\overline{y},$$

$$(f_xe_y\overline{y})^* f_y \,\mathcal{R}\, (f_xe_y\overline{y})^*\overline{y}^* \,\widetilde{\mathcal{L}}\, (f_xe_y\overline{y})\overline{y}^* = \overline{x}^* f_xe_y\overline{y} \,\widetilde{\mathcal{L}}\, \overline{x}f_xe_y\overline{y}.$$

Therefore the above properties valid. □

In what follows S^o and $S^\diamond$ will denote a pair of quasi-Ehresmann transversal of the semi-abundant semigroup S satisfies conditions (CR) and (CL) and E_{S^o} and $E_{S^\diamond}$ will denote the idempotents of them respectively to avoid confusion. For the sake of simplicity, in $S^\diamond$, we still denote the typical idempotent that $\widetilde{\mathcal{L}}$-related and $\widetilde{\mathcal{R}}$-related to $a \in S^\diamond$ by a^* and a^+ respectively. For any $x \in S$, we write $x = e_x\overline{x}f_x$ and $x = i_x\widetilde{x}\lambda_x$ as the decompositions of x in S^o and $S^\diamond$ respectively. Then $\widetilde{x} \in S^\diamond$ has the same meaning as in Definition 2.4. More precisely, $i_x, \lambda_x \in E$ and $\widetilde{x}^*, \widetilde{x}^+ \in E_{S^\diamond}$ with $\widetilde{x}^*\widetilde{\mathcal{L}}\widetilde{x}\widetilde{\mathcal{R}}\widetilde{x}^+$ and $i_x\mathcal{L}\widetilde{x}^+$, $\lambda_x\mathcal{R}\widetilde{x}^*$, and so $i_x\widetilde{\mathcal{R}}x\widetilde{\mathcal{L}}\lambda_x$.

Let $S^\diamond$ and S^o be quasi-Ehresmann transversals of the semi-abundant semigroup S satisfies conditions (CR) and (CL). Denote

$$I(S^\diamond, S^o) = \{aa^o :\ a \in RegS \cap S^\diamond, a^o \in V_{S^o}(a)\},$$

$$\Lambda(S^o, S^\diamond) = \{a^\diamond a :\ a \in RegS \cap S^o, a^\diamond \in V_{S^\diamond}(a)\}.$$

Theorem 4.4 *Let $S^\diamond$ and S^o be a pair of quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup S satisfies conditions (CR) and (CL) and satisfies the regularity condition. Then*

$$I(S^\diamond, S^o) = \Lambda(S^o, S^\diamond) = I_o \cap \Lambda_\diamond.$$

Proof. For any $aa^o \in I(S^\diamond, S^o)$, where $a \in RegS \cap S^\diamond, a^o \in V_{S^o}(a)$, certainly, $a \in V_{S^\diamond}(a^o)$ and so $aa^o = a^{o\diamond}a^o \in \Lambda(S^o, S^\diamond)$. Thus $I(S^\diamond, S^o) \subseteq \Lambda(S^o, S^\diamond)$ and dually $\Lambda(S^o, S^\diamond) \subseteq I(S^\diamond, S^o)$. Consequently, $I(S^\diamond, S^o) = \Lambda(S^o, S^\diamond)$ and we denote it by W. From the above definitions, it is clear that $W \subseteq I_o \cap \Lambda_\diamond$.

Conversely, suppose that $x \in I_o \cap \Lambda_\diamond$. Since $x \in \Lambda_\diamond$, we have $x = x^\diamond x$ for some $x^\diamond \in V_{S^\diamond}(x)$ with $x^\diamond \in E_{S^\diamond}$ and so $x^\diamond = xx^\diamond$. Similarly, $x \in I_o$ implied that $x = xx^o$ for some $c^o \in V_{S^o}(x)$ with $c^o \in E_{S^o}$ and so $x^o = x^o x$. Let $x^{\diamond o} \in V_{S^o}(x^\diamond)$. From $c^o \mathcal{L}\, x \,\mathcal{R}\, x^\diamond \,\mathcal{R}\, x^\diamond x^{\diamond o}$, by Lemma 2.2, we deduce that $x^o \,\mathcal{R}\, x^o x^\diamond x^{\diamond o} \,\mathcal{L}\, x^\diamond x^{\diamond o} \,\mathcal{L}\, x^{\diamond o}$ with $x^o x^\diamond x^{\diamond o} \in E^o I_o \subseteq E^o$ since S^o is a quasi-ideal and S satisfies the regularity condition. Thus $x^o \,\mathcal{L}\, x^{\diamond o}x^o \,\mathcal{R}\, x^{\diamond o}$. Certainly, $x^o \,\mathcal{R}\, x^o x^\diamond \,\mathcal{L}\, x^\diamond$ and so by Lemma 2.2, $x^{\diamond o} \,\mathcal{R}\, x^{\diamond o}x^o x^\diamond \,\mathcal{L}\, x^o x^\diamond$ and $x^{\diamond o}x^o x^\diamond \,\widetilde{\mathcal{H}}\, x^{\diamond o}x^\diamond \in I_o \cap \Lambda_\diamond$. Consequently, $x^\diamond x^{\diamond o} x^o \,\mathcal{H} x$ and so $x^\diamond x^{\diamond o}x^o = x$ since $x \in E$ and $x^\diamond x^{\diamond o} \cdot x^o \in I_o E^o \subseteq E$. Also $(x^{\diamond o}x^o x^\diamond)^2 = x^{\diamond o}x^o(x^\diamond x^{\diamond o}x^o)x^\diamond = x^{\diamond o}x^o x x^\diamond = x^{\diamond o}x^o x^\diamond$ and $x^{\diamond o}x^o x^\diamond \in E$. Therefore

$$x^\diamond \cdot x^{\diamond o}x^o \cdot x^\diamond = xx^\diamond = x^\diamond \quad \text{and} \quad x^{\diamond o}x^o \cdot x^\diamond \cdot x^{\diamond o}x^o = x^{\diamond o}x^o x = x^{\diamond o}x^o$$

and so $x^{\diamond o}x^o \in V_{S^o}(x^\diamond)$. Hence $x = x^\diamond \cdot x^{\diamond o}x^o \in I(S^\diamond, S^o) = W$. □

Theorem 4.5 *Let $S^\diamond$ and S^o be quasi-ideal quasi-Ehresmann transversals of the semi-abundant semigroup S satisfies conditions (CR) and (CL) and satisfies the regularity condition. Then $S^\diamond S^o$ is a quasi-ideal quasi-Ehresmann transversal of S.*

Proof. It is evident that $S^\diamond S^o$ is a subsemigroup and a quasi-ideal of S. For any $x \in S^\diamond S^o$, there exist $s^\diamond \in S^\diamond, t^o \in S^o$ such that $x = s^\diamond t^o$. It follows from S^o is a quasi-ideal of S and Lemma 4.3 that $e_{s^\diamond}(\overline{s^\diamond} f_{s^\diamond} e_{t^o})^+ \in I_{s^\diamond t^o} = I_x$ and we denote it by e_x. It is obvious that $i_{s^\diamond} \in E_{S^\diamond}$ since $s^\diamond \in S^\diamond$ and so from $e_{s^\diamond}\widetilde{\mathcal{R}} i_{s^\diamond} \in E_{S^\diamond}$ we deduce that $e_{s^\diamond} \in I_o \cap \Lambda_\diamond$. Thus by Theorem 4.4 there exists $a \in Reg(S^\diamond)$ such that $e_{s^\diamond} = aa^o$ and so

$$e_x = e_{s^\diamond}(\overline{s^\diamond} f_{s^\diamond} e_{t^o})^+ = aa^o(\overline{s^\diamond} f_{s^\diamond} e_{t^o})^+ \in S^\diamond S^o.$$

Similarly, $\lambda_x \in S^\diamond S^o$. Thus $e_x, \lambda_x \in E_{S^\diamond S^o}$, and so from $e_x \widetilde{\mathcal{R}} x \widetilde{\mathcal{L}} \lambda_x$ we deduce that $S^\diamond S^o$ is semi-abundant. It is a routine matter to show that $e_x \widetilde{\mathcal{R}}(S) x \widetilde{\mathcal{L}}(S) \lambda_x$, thus $S^\diamond S^o$ is a $\sim$-semi-abundant subsemigroup of S.

Let e be an idempotent of $S^\diamond S^o$. Then $e = as$ for some $a \in S^\diamond$, $s \in S^o$. Since $(asa)(sas)(asa) = asa, (sas)(asa)(sas) = sas$ and $sas \in S^o$, we have $sas \in V_{S^o}(asa)$, so that $e = asasas = asa(asa)^o$. Since $asa \in S^\diamond$, each idempotent of $S^\diamond S^o$ is of the form bb^o for some regular element $b \in S^\diamond$. Let e and f be idempotents of $S^\diamond S^o$. Then $e = bb^o$ and $f = cc^o$ for some regular elements $b, c \in S^\diamond$ with $b^o \in V_{S^o}(b)$ and $c^o \in V_{S^o}(c)$. For any $l \in E^o$, by the regularity condition, lcc^o is regular and so $lcc^o \in E$ since S^o is a quasi-Ehresmann transversal of the semi-abundant semigroup S satisfies conditions (CR) and (CL). Thus $lcc^o \in E \cap S^o = E^o$ since S^o is also a quasi-ideal of S. Therefore $ef = bb^o cc^o = bb^o(b^{o*}cc^o) \in I_o E^o \subseteq E$ and $S^\diamond S^o$ is a quasi-Ehresmann semigroup.

For any $x \in S$, there exist $a, b \in RegS$ such that $e_x = aa^o, \lambda_x = b^\diamond b$, where $a^o \in V_{S^o}(a), b^\diamond \in V_{S^\diamond}(b)$. Thus

$$x = e_x x \lambda_x = aa^o x b^\diamond b = aa^o(a^{o\diamond}a^o x b^\diamond b^{\diamond o})b^\diamond b,$$

where $a^{o\diamond} \in V_{S^\diamond}(a^o)$, $b^{\diamond o} \in V_{S^o}(b^\diamond)$, and consequently,

$$e_x = aa^o \mathcal{L} a^{o\diamond}a^o \in E_{S^\diamond S^o}, \ \lambda_x = b^\diamond b \mathcal{R} b^\diamond b^{\diamond o} \in E_{S^\diamond S^o}.$$

Since $a^{o\diamond}a^o x b^\diamond b^{\diamond o}\lambda_x = a^{o\diamond}a^o x \lambda_x = a^{o\diamond}a^o x$, we have $a^{o\diamond}a^o x b^\diamond b^{\diamond o}\widetilde{\mathcal{R}} a^{o\diamond}a^o x$. From $x\widetilde{\mathcal{R}} e_x$ and $\widetilde{\mathcal{R}}$ is a left congruence we deduce that

$$a^{o\diamond}a^o x \,\widetilde{\mathcal{R}}\, a^{o\diamond}a^o e_x = a^{o\diamond}a^o \in E_{S^\diamond S^o}.$$

Similarly,

$$a^{o\diamond}a^o x b^\diamond b^{\diamond o} \,\widetilde{\mathcal{L}}\, x b^\diamond b^{\diamond o} \,\widetilde{\mathcal{L}}\, b^\diamond b^{\diamond o} \in E_{S^\diamond S^o}.$$

Consequently, $x = e_x(a^{o\diamond}a^o x b^\diamond b^{\diamond o})\lambda_x$ with $e_x, \lambda_x \in E, e_x \mathcal{L}(a^{o\diamond}a^o x b^\diamond b^{\diamond o})^+ = a^{o\diamond}a^o \in E_{S^\diamond S^o}$ and $\lambda_x \mathcal{R}(a^{o\diamond}a^o x b^\diamond b^{\diamond o})^* = b^\diamond b^{\diamond o} \in E_{S^\diamond S^o}$. Therefore, $S^\diamond S^o$ is a generalized quasi-Ehresmann transversal of S.

For regular elements $c \in S$, $d \in S^\diamond S^o$, take $c' \in V_{S^\diamond S^o}(c)$, $d' \in V_{S^\diamond S^o}(d)$, then by Lemma 4.2, there exist $c^\diamond \in V_{S^\diamond}(c)$, $c^o \in V_{S^o}(c)$, $d^\diamond \in V_{S^\diamond}(d)$, $d^o \in V_{S^o}(d)$, such that $c' = c^\diamond cc^o$, $d' = d^\diamond dd^o$. Since $d \in S^\diamond S^o$, $d^o \in V_{S^o}(d)$, we have $d \in V_{S^\diamond S^o}(d^o)$. By Lemma 4.2, there exist $(d^o)^\diamond \in V_{S^\diamond}(d^o)$, $(d^o)^o \in V_{S^o}(d^o)$, such that $d = (d^o)^\diamond d^o(d^o)^o$. So

$$c'cdd' = c^\diamond cc^o cdd^\diamond dd^o = c^\diamond cdd^o = c^\diamond c(d^o)^\diamond d^o(d^o)^o d^o = c^\diamond c(d^o)^\diamond d^o \in \Lambda_\diamond\Lambda_\diamond \subseteq \Lambda_\diamond,$$

and $c'cdd'$ is idempotent. On the other hand,

$$dd'c'c = dd^\diamond dd^o c^\diamond cc^o c = dd^o c^\diamond c = (d^o)^\diamond d^o(d^o)^o d^o c^\diamond c = (d^o)^\diamond d^o c^\diamond c \in \Lambda_\diamond\Lambda_\diamond \subseteq \Lambda_\diamond,$$

and $dd'c'c \in E$. Thus

$$cdd'c'cd = c \cdot c'cdd' \cdot c'cdd' \cdot d = cc'cdd'd = cd,$$

$$d'c'cdd'c' = d' \cdot dd'c'c \cdot dd'c'c \cdot c' = d'dd'cc'c = d'c',$$

and so $V_{S^\diamond S^o}(d)V_{S^\diamond S^o}(c) \subseteq V_{S^\diamond S^o}(cd)$. Similarly, $V_{S^\diamond S^o}(c)V_{S^\diamond S^o}(d) \subseteq V_{S^\diamond S^o}(dc)$.

It follows from Theorem 3.2 that $S^\diamond S^o$ is a quasi-Ehresmann transversal. Since $S^\diamond S^o$ is a quasi-ideal, therefore $S^\diamond S^o$ is a quasi-ideal quasi-Ehresmann transversal of S. □

Theorem 4.6 *Let S be a semi-abundant semigroup satisfying conditions (CR) and (CL) and the the regularity condition. If S has a quasi-ideal quasi-Ehresmann transversal, then all quasi-ideal quasi-Ehresmann transversals of S form a rectangular band.*

Proof. If S^o is a quasi-ideal quasi-Ehresmann transversal of S, then $S^oS^o = S^o$. To see this, for $s^o \in S^o$, $s^o = s^o(s^o)^* \in S^oS^o$, hence $S^o \subseteq S^oS^o$ and the reverse inclusion is obvious. By Theorem 4.5, all quasi-ideal quasi-Ehresmann transversals of S form a semigroup and so form a band.

Let $S^o, S^\diamond, S^\square$ be arbitrary three quasi-ideal quasi-Ehresmann transversals of S. For any $a^o \in S^o, x \in S, b^\diamond \in S^\diamond$, we have

$$a^o xb^\diamond = a^o xe_{b^\diamond}(\overline{b^\diamond})^+ b^\diamond \in S^oSSS^oS^\diamond \subseteq S^oS^\diamond,\ a^o b^\diamond = a^o(a^o)^* b^\diamond \in S^oS^oS^\diamond \subseteq S^oSS^\diamond,$$

where $b^\diamond\widetilde{\mathcal{R}}e_{b^\diamond} \in E$ and $e_{b^\diamond}\mathcal{L}(\overline{b^\diamond})^+ \in E^o$. Thus $S^oSS^\diamond = S^oS^\diamond$ and so $S^oS^\square S^\diamond \subseteq S^oSS^\diamond = S^oS^\diamond$. For every $a^o \in S^o, b^\diamond \in S^\diamond$, then

$$a^o b^\diamond = a^o f_{a^o}(f_{a^o})^\square f_{a^o} b^\diamond \in S^oSS^\square SS^\diamond = S^oS^\square S^\diamond,$$

with $(f_{a^o})^\square$ is an inverse in $S^\square$ of f_{a^o}. Thus $S^oS^\square S^\diamond = S^oS^\diamond$ and therefore all quasi-ideal quasi-Ehresmann transversals of S form a rectangular band. □

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