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Spectral Theory for Polynomially Demicompact Operators

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Abstract. In this article, we introduce the notion of polynomial demicompactness and we use it to give some results on Fredholm operators and to establish a fine description of some essential spectra of a closed densely defined linear operator. Our work is a generalization of many known ones in the literature.

1. Introduction

Throughout this paper, X and Y denote two infinite dimensional complex Banach spaces. We denote by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$) the set of all closed densely defined (resp. bounded) linear operators acting from X into Y. The subspace of all compact (resp. weakly compact) operators of $\mathcal{L}(X,Y)$ is denoted by $\mathcal{K}(X,Y)$ (resp. $\mathcal{W}(X,Y)$). For $T \in \mathcal{C}(X,Y)$, we denote by $\alpha(T)$ the dimension of the kernel $\mathcal{N}(T)$ and by $\beta(T)$ the codimension of the range $\mathcal{R}(T)$ in Y. The next sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from X into Y are, respectively, defined by

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y),$$

and

$$\Phi_\pm(X,Y) := \Phi_-(X,Y) \cup \Phi_+(X,Y).$$

For $T \in \Phi_\pm(X,Y)$, we define the index by the following difference $i(T) := \alpha(T) - \beta(T)$. A complex number λ is in Φ_{+T}, Φ_{-T}, $\Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{W}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_\pm(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\mathcal{W}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively. If $T \in \mathcal{C}(X)$, we denote by $\rho(T)$ the resolvent set of T and by $\sigma(T)$ the spectrum of T. Let $T \in \mathcal{C}(X)$. For $x \in \mathcal{D}(T)$, the graph norm $\|.\|_T$ of x is defined by $\|x\|_T = \|x\| + \|Tx\|$. It follows from the closedness of T that $X_T := (\mathcal{D}(T), \|.\|_T)$ is a Banach space. Clearly, for every $x \in \mathcal{D}(T)$ we have $\|Tx\| \leq \|x\|_T$, so that $T \in \mathcal{L}(X_T, X)$. A linear operator B is said to be T-defined if $\mathcal{D}(T) \subseteq \mathcal{D}(B)$. If $\hat{B}$, the restriction of B to $\mathcal{D}(T)$ is bounded from X_T into X, we say that B is T-bounded.

Remark 1.1. *Notice that if $T \in \mathcal{C}(X)$ and B is T-bounded, then we get the obvious relations*

$$\begin{cases} \alpha(\hat{T}) = \alpha(T), \beta(\hat{T}) = \beta(T), \mathcal{R}(\hat{T}) = \mathcal{R}(T) \\ \\ \alpha(\hat{T} + \hat{B}) = \alpha(T+B), \beta(\hat{T} + \hat{B}) = \beta(T+B), \mathcal{R}(\hat{T} + \hat{B}) = \mathcal{R}(T+B). \end{cases}$$

Hence, $T \in \Phi(X)$, (resp. $\Phi_+(X), \Phi_-(X)$) if, and only if, $\hat{T} \in \Phi(X_T, X)$, (resp. $\Phi_+(X_T, X)$, $\Phi_-(X_T, X)$). ◊

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Definition 1.2. *Let $T \in \mathcal{L}(X,Y)$, where X and Y are two Banach spaces.*
(i) *T is said to have a left Fredholm inverse (resp. left weak-Fredholm inverse) if there exists $T_l \in \mathcal{L}(Y,X)$ and $K \in \mathcal{K}(X)$ (resp. $T_l^w \in \mathcal{L}(Y,X)$ and $W \in \mathcal{W}(X)$) such that $T_lT = I_X - K$ (resp. $T_l^wT = I_X - W$). The operator T_l (resp. T_l^w) is called left Fredholm inverse of T (resp. left weak-Fredholm inverse of T).*
(ii) *T is said to have a right Fredholm inverse (resp. right weak-Fredholm inverse) if there exists $T_r \in \mathcal{L}(Y,X)$ (resp. $T_r^w \in \mathcal{L}(Y,X)$) such that $I_Y - TT_r \in \mathcal{K}(Y)$ (resp. $I_Y - TT_r^w \in \mathcal{W}(Y)$). The operator T_r (resp. T_r^w) is called right Fredholm inverse of T (resp. right weak-Fredholm inverse of T).*
(iii) *T is said to have a Fredholm inverse (resp. weak-Fredholm inverse) if there exists a map which is both a left and a right Fredholm inverse of T (resp. a left and a right weak-Fredholm inverse of T).* ◊

Definition 1.3. *Let $T \in \mathcal{C}(X)$, where X be a Banach space. T is said to have a left Fredholm inverse (resp. right Fredholm inverse, Fredholm inverse, left weak-Fredholm inverse, right weak-Fredholm inverse, weak-Fredholm inverse) if $\hat{T}$ has a left Fredholm inverse (resp. right Fredholm inverse, Fredholm inverse, left weak-Fredholm inverse, right weak-Fredholm inverse, weak-Fredholm inverse).* ◊

The sets of left, right, left weak and right weak-Fredholm inverses are respectively, defined by

$$\begin{aligned}
\Phi_l(X) &:= \{T \in \mathcal{C}(X) \text{ such that T has a left Fredholm inverse}\},\\
\Phi_r(X) &:= \{T \in \mathcal{C}(X) \text{ such that T has a right Fredholm inverse}\},\\
\Phi_l^w(X) &:= \{T \in \mathcal{C}(X) \text{ such that T has a left weak-Fredholm inverse}\},\\
\Phi_r^w(X) &:= \{T \in \mathcal{C}(X) \text{ such that T has a right weak-Fredholm inverse}\}.
\end{aligned}$$

The class of weak-Fredholm operators is $\Phi^w(X) := \Phi_l^w(X) \cap \Phi_r^w(X)$. It is easy to see that $\Phi_l(X) \subset \Phi_l^w(X)$, $\Phi_r(X) \subset \Phi_r^w(X)$ and $\Phi_l(X) \cap \Phi_r X) = \Phi(X) \subset \Phi^w(X)$. A complex number λ is in $\Phi_{lT}(X)$, $\Phi_{rT}(X), \Phi_T(X), \Phi_{lT}^w(X), \Phi_{rT}^w(X)$ or $\Phi_T^w(X)$ if $\lambda - T$ is in $\Phi_l(X)$, $\Phi_r(X), \Phi(X), \Phi_l^w(X), \Phi_r^w(X)$ or $\Phi^w(X)$, respectively. The concept of demicompactness was introduced by W. V. Petryshyn [13], in order to discuss fixed points, as follows

Definition 1.4. *An operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be demicompact if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n - Tx_n \to x \in X$, there exists a convergent subsequence of $(x_n)_n$. The family of demicompact operators on X is denoted by $\mathcal{DC}(X)$.* ◊

Clearly, if A and B are demicompact and λ is a complex number, then $A+B$, AB and λA are not necessarily demicompact. The first purpose of this work is to pursue the analysis started in [10] and to extend it to more general classes, using the concept of demicompactness. This class, defined in Section 3, is the class of polynomially demicompact operators. When dealing with essential spectra of closed, densely defined linear operators on Banach spaces, one of the main problems consists in studying the invariance of the essential spectra of these operators subjected to various kinds of perturbation. Among the works in this direction we quote, for example [1, 4, 7]. More precisely, it was shown in [1] the invariance of the Schechter essential spectrum on Banach spaces by means of polynomially compact perturbations. The same result has been proved by W. Chaker et al. in [4] for the class Λ_X, where

$$\Lambda_X := \{J \in \mathcal{L}(X) \text{ such that } \mu J \in \mathcal{DC}(X)\ \forall \mu \in [0,1]\}.$$

In the same work, the Schechter essential spectrum was characterized by

$$\sigma_{e_5}(T) = \bigcap_{K \in \Upsilon_T(X)} \sigma(T+K),$$

where,

$$\Upsilon_T(X) = \{S \in \mathcal{L}(X) \text{ such that } -S(\lambda - T - S)^{-1} \in \Lambda_X,\ \forall \lambda \in \rho(T+S)\}.$$

W. V. Petryshyn has proved, in [13], that $I-T$ is a Fredholm operator and $i(I-T) = 0$ for every condensing operator T. The same result has been proved by W. Chaker et all in [4], for $T \in \Lambda_X$. For more results in this direction the reader can refers to [8, 9]. Our main purpose is to refine the characterization of essential spectra. More precisely, we shall introduce the class Θ_X, which is defined by

$$\Theta_X = \bigcup_{n \in \mathbb{N}\setminus\{0,1\}, B \in \mathcal{L}(X)} \Omega_{n,B}$$

where

$$\Omega_{n,B} := \left\{ J \in \mathcal{L}(X) \text{ such that } J = B^n \text{ and } -\sum_{k=1}^{n-1} B^k \in \Lambda_X \right\}.$$

We note that Θ_X contains the set Λ_X. In order to state our results, we need to fix some notations and assumptions that we are using. Throughout this note, let X be a Banach space and $T \in \mathcal{C}(X)$, various notions of essential spectrum appear in application of spectral theory. In this work, we are concerned, respectively, with the Weidmann, the Kato, the Wolf, the Schechter, the approximate point or the Schmoëger, the defect or the Rakocević, the left and the right Weyl essential spectra which are, respectively, defined as follows

$$\begin{aligned}
\sigma_{e_2}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_-(X)\} := \mathbb{C}\backslash\Phi_{-T},\\
\sigma_{e_3}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_\pm(X)\} := \mathbb{C}\backslash\Phi_{\pm T},\\
\sigma_{e_4}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi(X)\} := \mathbb{C}\backslash\Phi_{T},\\
\sigma_{e_5}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma(T+K),\\
\sigma_{e_7}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap}(T+K),\\
\sigma_{e_8}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{\delta}(T+K),\\
\sigma_{e_{wl}}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{l}(T+K),\\
\sigma_{e_{wr}}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{r}(T+K),
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{ap}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \inf_{x\in D(T);\|x\|=1} \|(\lambda - T)x\| = 0\},\\
\sigma_{\delta}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \text{ is not surjective}\},\\
\sigma_{l}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_l(X)\} := \mathbb{C}\backslash\Phi_{lT},\\
\sigma_{r}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_r(X)\} := \mathbb{C}\backslash\Phi_{rT},\\
&.
\end{aligned}$$

(See for instance [7, 14–16, 18]). Note that for $T \in \mathcal{C}(X)$, we have

$$\sigma_{e_2}(T) \subset \sigma_{e_4}(T) \subset \sigma_{e_5}(T) = \sigma_{e_7}(T) \cup \sigma_{e_8}(T),$$

$$\sigma_{e_3}(T) \subset \sigma_{e_4}(T) \quad \text{and} \quad \sigma_{e_2}(T) \subset \sigma_{e_8}(T).$$

This paper is organized in the following way. In Section 2, we recall some definitions and results needed in the rest of the paper. In Section 3, we study the relationship between polynomially demicompact and demicompact operators and we give an example for Theorem 3.1. In Section 4, we give a fine description of some essential spectra by means of polynomial demicompactness. In Section 5, we give some perturbation results and some relations between essential spectra of the sum of two bounded linear operators and essential spectra of each these operators. In Section 6, we investigate the left and the right Weyl essential spectra of matrix operators defined on a Banach space which posses the Dunford Pettis property (see definition in Section 2).

2. Preliminary results

We start this section by recalling some Fredholm results related with demicompact operators.

Theorem 2.1. *[4] Let $T \in \mathcal{C}(X)$. If T is demicompact, then $I - T$ is an upper semi-Fredholm operator.* ◊

Theorem 2.2. *[4] Let $T \in \mathcal{C}(X)$. If μT is demicompact for each $\mu \in [0,1]$, then $I - T$ is a Fredholm operator of index zero.* ◊

Theorem 2.3. *[4] Let $T : D(T) \subseteq X \longrightarrow X$ be a closed linear operator. If T is a 1-set-contraction, then μT is demicompact for each $\mu \in [0,1)$.* ◊

In the following Theorem, we recall some known results that we will need in the sequel.

Theorem 2.4. *[16] Let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ where X, Y and Z are Banach spaces.*
(i) If $AB \in \Phi_+(X, Z)$, then $B \in \Phi_+(X, Y)$.
(ii) If $X = Y = Z$, $AB \in \Phi(X)$ and $BA \in \Phi(X)$, then $A \in \Phi(X)$ and $B \in \Phi(X)$.
(iii) If $AB \in \Phi_l(X, Z)$, then $B \in \Phi_l(X, Y)$.
(iv) If $AB \in \Phi_r(X, Z)$, then $A \in \Phi_r(X, Y)$. ◊

The following theorems give a characterization of the Schechter, the Schmoëger, the Rakocević, the left and the right essential spectra by means of Fredholm, upper, lower semi-Freholm, left and right Fredholm operators, respectively. The proof can be found in [7].

Theorem 2.5. *[16] Let $T \in \mathcal{C}(X)$, then*

$$\lambda \notin \sigma_{e_5}(T) \text{ if, and only if, } \lambda - T \in \Phi(X) \text{ and } i(\lambda - T) = 0. \quad \diamond$$

Theorem 2.6. *[7] Let $T \in \mathcal{C}(X)$, then*
(i) $\lambda \notin \sigma_{e_7}(T)$ if, and only if, $\lambda - T \in \Phi_+(X)$ and $i(\lambda - T) \leq 0$.
(ii) $\lambda \notin \sigma_{e_8}(T)$ if, and only if, $\lambda - T \in \Phi_-(X)$ and $i(\lambda - T) \geq 0$. ◊

Theorem 2.7. *[2] Let $T \in \mathcal{C}(X)$, then,*
(i) $\lambda \notin \sigma_{e_{wl}}(T)$ if, and only if, $\lambda - T \in \Phi_l(X)$ and $i(\lambda - T) \leq 0$.
(ii) $\lambda \notin \sigma_{e_{wr}}(T)$ if, and only if, $\lambda - T \in \Phi_r(X)$ and $i(\lambda - T) \geq 0$. ◊

Now, we recall that a bounded linear operator T is said to be power compact if $T^m \in \mathcal{K}(X)$ for some $m \in \mathbb{N}\backslash\{0\}$. Clearly, every power compact operator is demicompact. In fact, we have the following more general result.

Proposition 2.8. *Let $T \in \mathcal{L}(X)$. Then, T^m is demicompact for some $m \in \mathbb{N}\backslash\{0\}$ if, and only if, T is a demicompact operator.* ◊

Proof. We assume that the assumption holds and we take $(x_n)_n$ a bounded sequence in $\mathcal{D}(T)$ such that $x'_n := (I - T)(x_n)$ converges. Obviously,

$$x_n = T^m x_n + \sum_{k=0}^{m-1} T^k x'_n,$$

which allows us to write

$$(I - T^m)x_n = \sum_{k=0}^{m-1} T^k x'_n. \tag{1}$$

Since $(x'_n)_n$ is convergent and $T^k \in \mathcal{L}(X)$ for each $0 \leq k \leq m-1$, it follows from Eq. (1) that $(I - T^m)$ is a convergent sequence. Using the demicompactness of T^m, we infer that there exists $(x_{\varphi(n)})_n$ a convergent subsequence of $(x_n)_n$. Conversely, it suffices to take $m = 1$. Q.E.D.

Proposition 2.9. *(i) $\mathcal{F}_+(X) \subset \mathcal{DC}(X)$.*
(ii) $\mathcal{F}_-(X) \subset \mathcal{DC}(X)$. ◊

Proof. (*i*) Let T be an upper semi-Fredholm perturbation. Since I is an upper semi-Fredholm operator, $I-T$ is such too. Using Theorem 2.1, we deduce that T is demicompact.
(*ii*) Let T be a lower semi-Fredholm perturbation operator. Using the fact that I is lower semi-Fredholm, we deduce that the operator $I-T$ has also this property. Therefore, $\beta(I-T)=\alpha(I-T^*)<\infty$, where T^* denotes the adjoint of T. Since $\mathcal{R}(I-T)$ is closed, $I-T^*$ is an upper semi-Fredholm operator. Applying Theorem 2.1, this implies that T^* is demicompact. Q.E.D.

Remark 2.10. *As an immediate consequence of Proposition 2.9, for every $T\in\mathcal{F}(X)$, T and T^* are demicompact.* ◊

Definition 2.11. *A Banach space X is said to have the Dunford-Pettis property (in short DP property) if every bounded weakly compact operator T from X into another Banach space Y transforms weakly compact sets on X into norm-compact sets on Y.* ◊

Remark 2.12. *It was proved in [11] that if X is Banach space with DP property, then*

$$\mathcal{W}(X)\subset\mathcal{F}(X).$$ ◊

3. Polynomially demicompact operators

In this section, we will generalize the following result proved in [10] for the class of polynomially demicompact operators on X. In fact, the authors showed that a polynomially compact operator T, element of $\mathcal{P}(X):=\{T\in\mathcal{L}(X)$ such that there exists a nonzero complex polynomial $P(z)=\sum_{r=0}^{p}a_rz^r$ satisfying $P(1)\neq 0$, $P(1)-a_0\neq 0$, and $P(T)\in\mathcal{K}(X)\}$, is demicompact. In order to state our results, we need to introduce the set $\mathcal{PDC}(X)$ which defined by

$$\mathcal{PDC}(X)=\bigcup_{P\in\mathbb{C}[z]\backslash\{0\},P(1)\neq 0}\mathcal{H}_P,$$

where

$$\mathcal{H}_P:=\left\{T\in\mathcal{L}(X)\text{ such that }\frac{1}{P(1)}P(T)\in\mathcal{DC}(X)\right\}.$$

We note that $\mathcal{PDC}(X)$ contains the set $\mathcal{P}(X)$.

Theorem 3.1. *$T\in\mathcal{PDC}(X)$ if, and only if, T is demicompact.* ◊

Proof. We first establish the following relation that we are using in the proof. Since $I-T$ commutes with I, Newton's binomial formula allows us to write the following relation

$$T^j=I+\sum_{i=1}^{j}(-1)^iC_j^i(I-T)^i.$$

By making some simple calculations, we may write

$$P(T)=P(1)I+\sum_{j=1}^{p}a_j\left(\sum_{i=1}^{j}(-1)^iC_j^i(I-T)^i\right).$$

Since $P(1)\neq 0$, we have

$$I-\frac{1}{P(1)}P(T)=\frac{1}{P(1)}\sum_{j=1}^{p}a_j\left(\sum_{i=1}^{j}(-1)^iC_j^i(I-T)^i\right). \tag{2}$$

Now, let $(x_n)_n$ be a bounded sequence in X satisfying $(I-T)x_n\to x_0$. Using the continuity of T together with the relation (2), we infer that there exists x such that

$$\left(I-\frac{1}{P(1)}P(T)\right)x_n\to x.$$

By demicompactness of $\frac{1}{P(1)}P(T)$, we conclude that $(x_n)_n$ admits a convergent subsequence. The converse can be checked by taking $P(z)=z$. Q.E.D. Before giving the example we recall the following definition and Theorems

Definition 3.2. *The Caputo derivative of fractional order α of function $x \in C^m$, $m \in \mathbb{N}$ is defined as*

$$ {}_C D_{0,t}^{(\alpha)} x(t) = D_{0,t}^{(m-\alpha)} \frac{d^m}{dt^m} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) d\tau $$

in which $m-1 < \alpha < m$ and Γ is the well-known Euler Gamma function. $\diamond$

Theorem 3.3. *[12] If $x(t) \in C^1[0,T]$, for $T > 0$ then*

$$ {}_C D_{0,t}^{(\alpha_2)}\, {}_C D_{0,t}^{(\alpha_1)} x(t) = {}_C D_{0,t}^{(\alpha_1)}\, {}_C D_{0,t}^{(\alpha_2)} x(t) = {}_C D_{0,t}^{(\alpha_1+\alpha_2)} x(t),\ t \in [0,T], $$

where α_1 and $\alpha_2 \in \mathbb{R}_+$ and $\alpha_1 + \alpha_2 \leq 1$. $\diamond$

Theorem 3.4. *[12] If $x(t) \in C^m[0,T]$, $m \in \mathbb{N}$ for $T > 0$ then*

$$ {}_C D_{0,t}^{(\alpha)} x(t) = {}_C D_{0,t}^{(\alpha_n)} \cdots {}_C D_{0,t}^{(\alpha_2)}\, {}_C D_{0,t}^{(\alpha_1)} x(t);\ t \in [0,T], $$

where $\alpha = \sum_{i=1}^{n} \alpha_i$; $\alpha_i \in (0,1]$, $m-1 \leq \alpha < m$ and there exists $i_k < n$, such that $\sum_{j=1}^{i_k} \alpha_j = k$, and $k = 1, 2, \cdots, m-1$. $\diamond$

Example 3.5. *Let C_ω be the space of continuous ω-periodic functions $x : \mathbb{R} \longrightarrow \mathbb{R}$ and C'_ω the space of continuously differentiable ω-periodic functions $x : \mathbb{R} \longrightarrow \mathbb{R}$. C_ω equipped with the maximum norm $\|.\|_\infty$ and C'_ω with the norm given by $\|u\|_\infty^1 = \max\{\|u\|_\infty, \|u'\|_\infty\}$ for $u \in C'_\omega$ are Banach spaces. Let us consider the following differential equation:*

$$ x'(t) = a(t)x'(t-h_1) + b(t)x(t-h_2) + f(t). $$

Here, a and b are continuous ω-periodic functions such that $|a(t)| < k$, $(k < \infty)$, where $k < \frac{1}{\omega}$ if $\omega > 2$ or $k < \frac{1}{2}$ if $\omega \leq 2$; $f \in C_\omega$ is a given function and $x \in C'_\omega$ is an unknown function. This equation can be rewritten in the operator from

$$ Gx - Ax = f, $$

where $G : C'_\omega \to C_\omega$ is given by the formula

$$ (Gx)(t) = x'(t), $$

and the operator $A : C'_\omega \to C_\omega$ by the formula

$$ (Ax)(t) = a(t)x'(t-h_1) + b(t)x(t-h_2). $$

Let us consider the polynomial $P(z) = z^n$ and the operator $T = {}_C D_\omega^{(\frac{1}{n})}$; $n \in \mathbb{N}\backslash\{0\}$, where ${}_C D_\omega^{(\frac{1}{n})}$ is the Caputo derivative of fractional order $\frac{1}{n}$. Applying Theorem 3.4, we get

$$ P(T) = T^n(x) = [{}_C D_\omega^{(\frac{1}{n})}]^n x(t) = x'(t). $$

Clearly, $P(T)$ is a bounded linear operator with $\|P(T)\| = 1$ and therefore, $P(T)$ is 1-set-contractive. Hence, using Theorem 2.3, we get

$$ \mu\, {}_C D_\omega^{(\frac{1}{n})} \in \mathcal{DC}(X)\ \forall \mu \in [0,1[. $$ $\diamond$

The following result is an extention of Theorem 3.1, under other assumptions.

Theorem 3.6. *Let $T \in \mathcal{L}(X)$. Suppose that there exists a complex polynomial P such that $P(0) = 0$. Then, $I - P(T)$ is demicompact if, and only if, $I - T$ is demicompact.* $\diamond$

Proof. Assume that $I - P(T)$ is demicompact operator, it follows from Theorem 2.1 that $P(T) \in \Phi_+(X)$. Now, take $x \in \mathcal{N}(T)$, then $Tx = 0$ which implies that for all $j \geq 1$, $T^j = 0$. Hence,

$$ P(T)x = \sum_{j=0}^{m} a_j T^j x = P(0)x + \sum_{j=1}^{m} a_j T^j x = 0, $$

where $P(z) = \sum_{j=0}^{m} a_j z^j$. Then, $\mathcal{N}(T) \subset \mathcal{N}(P(T))$ is obvious and this shows that $\alpha(T) < \infty$. Next, since $\mathcal{R}(P(T))$ is closed, we deduce from Theorem 3.12 in [16] that there exists $k > 0$ such that $\forall y \in X$, $\|y\| \leq k\|P(T)y\|$. In particular,

$$\begin{aligned}\|x\| &\leq k\|P(T)x\| \\ &\leq k\sum_{j=1}^{m}|a_j|\|T\|^{j-1}\|Tx\|.\end{aligned}$$

The use of Theorem 3.12 in [16] shows that $\mathcal{R}(T)$ is closed. Therefore, $T \in \Phi_+(X)$ and we conclude by Theorem 2.1 in [5] that $I - T$ is demicompact. Conversely, the result can be obtained by taking $P(z) = z$. Q.E.D.

Remark 3.7. *Using Theorem 3.1 in [4] and Theorem 2.1 in [5], we deduce that if P is a complex polynomial such that $P(0) = 0$ and $T \in \mathcal{L}(X)$, then $P(T) \in \Phi_+(X)$ if, and only if, $T \in \Phi_+(X)$.* ◇

Theorem 3.8. *If $T \in \Theta_X$, then $I - T \in \Phi(X)$ and $i(I - T) = 0$.* ◇

Proof. Since $T \in \Theta_X$, then there exist $n \in \mathbb{N}\backslash\{0, 1\}$ and $B \in \mathcal{L}(X)$ such that $-t\sum_{k=1}^{n-1} B^k \in \mathcal{DC}(X)$, for all $t \in [0, 1]$. Hence, from Theorem 2.2,

$$I + \sum_{k=1}^{n-1} B^k \in \Phi(X) \text{ and } i(I + \sum_{k=1}^{n-1} B^k) = 0.$$

which is equivalent to

$$\sum_{k=0}^{n-1} B^k \in \Phi(X) \text{ and } i(\sum_{k=0}^{n-1} B^k) = 0.$$

In the other hand, since $\forall t \in [0, 1]$, $\frac{1}{n-1}t \in [0, 1]$, then $\frac{1}{n-1}\sum_{k=1}^{n-1}(t^{\frac{1}{k}}B)^k \in \mathcal{DC}(X)$. Hence, applying Theorem 3.1 for the polynomial $P(z) = -\sum_{k=1}^{n-1} z^k$, which verifies $P(1) = 1 - n \neq 0$, we get $t^{\frac{1}{k}}B \in \mathcal{DC}(X)$ $\forall t \in [0, 1]$. It follows that $tB \in \mathcal{DC}(X)$ $\forall t \in [0, 1]$. Thus, thanks to Theorem 2.2, we deduce that

$$I - B \in \Phi(X) \text{ and } i(I - B) = 0.$$

Now, using the equality

$$I - T = I - B^n = (I - B)\sum_{k=0}^{n-1} B^k,$$

we infer from the Atkinson's theorem that

$$I - T \in \Phi(X) \text{ and } i(I - T) = 0.$$

Q.E.D.

4. Characterization of essential spectra

In this section we will give a fine description of some essential spectra of a closed densely defined linear operator by means of Θ_X. To do this, the following notation will be convenient

$$\chi_T(X) = \Big\{S \in \mathcal{L}(X) \text{ such that } -(\lambda - T - S)^{-1}S \in \Theta_X,\ \forall \lambda \in \rho(T + S)\Big\}.$$

Notice that

$$\mathcal{K}(X) \subset \Upsilon_T(X) \subset \chi_T(X).$$

Theorem 4.1. *Let $T \in \mathcal{C}(X)$, then*

$$\sigma_{e_5}(T) = \bigcap_{S \in \chi_T(X)} \sigma(T + S).$$ ◇

Proof. We first claim that $\sigma_{e_5}(T) \subset \bigcap_{S \in \chi_T(X)} \sigma(T + S)$. Indeed, if $\lambda \notin \bigcap_{S \in \chi_T(X)} \sigma(T + S)$, then there exists $S \in \chi_T(X)$ such that $\lambda \notin \sigma(T + S)$. Therefore, we have

$$\lambda - T - S \in \Phi(X) \text{ with } i(\lambda - T - S) = 0.$$

Moreover, from the fact that $-(\lambda - T - S)^{-1}S \in \Theta_X$ together with Theorem 3.8, we infer that

$$I + (\lambda - T - S)^{-1}S \in \Phi(X) \text{ and } i[I + (\lambda - T - S)^{-1}S] = 0.$$

Now, using the equality

$$\lambda - T = (\lambda - T - S)[I + (\lambda - T - S)^{-1}S],$$

together with the Atkinson's theorem one gets

$$\lambda - T \in \Phi(X) \text{ and } i(\lambda - T) = 0,$$

which shows that $\lambda \notin \sigma_{e_5}(T)$. Then,

$$\sigma_{e_5}(T) \subset \bigcap_{S \in \chi_T(X)} \sigma(T + S).$$

For the inverse inclusion, since $\mathcal{K}(X) \subset \chi_T(X)$, then

$$\bigcap_{S \in \chi_T(X)} \sigma(T + S) \subset \sigma_{e_5}(T).$$

Q.E.D.

Corollary 4.2. *Let $T \in \mathcal{C}(X)$ and let $\Sigma(X)$ be a subset of $\chi_T(X)$ containing $\mathcal{K}(X)$. Then,*

$$\sigma_{e_5}(T) = \bigcap_{S \in \Sigma(X)} \sigma(T + S).$$

◊

Proof. From the following inclusions $\mathcal{K}(X) \subset \Sigma(X) \subset \chi_T(X)$, we infer that

$$\bigcap_{S \in \chi_T(X)} \sigma(T + S) \subset \bigcap_{S \in \Sigma(X)} \sigma(T + S) \subset \bigcap_{S \in \mathcal{K}(X)} \sigma(T + S).$$

Applying Theorem 4.1, we get

$$\sigma_{e_5}(T) = \bigcap_{S \in \Sigma(X)} \sigma(T + S).$$

Q.E.D.

Corollary 4.3. *Let $T \in \mathcal{C}(X)$ and $\mathcal{G}(X)$ be a subset of $\chi_T(X)$ containing $\mathcal{K}(X)$. If for all $K, K' \in \mathcal{G}(X)$, we have $K \pm K' \in \mathcal{G}(X)$, then for every $K \in \mathcal{G}(X)$*

$$\sigma_{e_5}(T) = \sigma_{e_5}(T + K).$$

◊

Proof. From Corollary 4.2, we have

$$\sigma_{e_5}(T) = \bigcap_{K \in \mathcal{G}(X)} \sigma(T + K).$$

Moreover, we have $\mathcal{G}(X) + K = \mathcal{G}(X)$ for every $K \in \mathcal{G}(X)$. Then, $\sigma'(T + K) = \sigma'(T)$, where $\sigma'(T) = \bigcap_{K \in \mathcal{G}(X)} \sigma(T + K)$. Hence, we get the desired result. Q.E.D.

In the next, we will give a characterization of the Schmoëger and the Rakocević essential spectra by means of polynomial demicompactness. To this end, we start by defining the following sets

$$\sigma_{ri}(T) = \bigcap_{S \in \chi_T(X)} \sigma_{ap}(T + S) \text{ and } \sigma_{le}(T) = \bigcap_{S \in \chi_T(X)} \sigma_\delta(T + S).$$

Theorem 4.4. *For each $T \in \mathcal{C}(X)$,*

$$\sigma_{e_7}(T) = \sigma_{ri}(T) \ \textit{and} \ \ \sigma_{e_8}(T) = \sigma_{le}(T).$$

◊

Proof. We start by showing that $\sigma_{e_7}(T) \subset \sigma_{ri}(T)$ (resp. $\sigma_{e_8}(T) \subset \sigma_{le}(T)$). For $\lambda \notin \sigma_{ri}(T)$, (resp. $\lambda \notin \sigma_{le}(T)$), there exists $S \in \chi_T(X)$ such that $\lambda - T - S$ is injective (resp. surjective), it follows from Theorem 2.6 that

$$\lambda - T - S \in \Phi_+(X) \text{ and } i(\lambda - T - S) \le 0,$$

$$(\text{resp. } \lambda - T - S \in \Phi_-(X) \text{ and } i(\lambda - T - S) \ge 0).$$

Similarly to the proof of Theorem 4.1 we show that

$$[I + (\lambda - T - S)^{-1}S] \in \Phi(X) \text{ and } i[I + (\lambda - T - S)^{-1}S] = 0.$$

which implies that

$$[I + (\lambda - T - S)^{-1}S] \in \Phi_+(X) \text{ and } i[I + (\lambda - T - S)^{-1}S] \le 0,$$

$$(\text{resp, } [I + (\lambda - T - S)^{-1}S] \in \Phi_-(X) \text{ and } i[I + (\lambda - T - S)^{-1}S] \ge 0.)$$

Thus, we get from the Atkinson's theorem

$$\lambda - T \in \Phi_+(X) \text{ and } i(\lambda - T) \le 0$$

$$(\text{resp. } \lambda - T \in \Phi_-(X) \text{ and } i(\lambda - T) \ge 0).$$

Thanks to Theorem 2.6, we conclude that $\lambda \notin \sigma_{e_7}(T)$ (resp. $\lambda \notin \sigma_{e_8}(T)$). Conversely, observe that $\mathcal{K}(X) \subset \chi_T(X)$, we deduce that $\sigma_{ri}(T) \subset \sigma_{e_7}(T)$ (resp. $\sigma_{le}(T) \subset \sigma_{e_8}(T)$). Hence, we get the desired result. Q.E.D.

Corollary 4.5. *Let $T \in \mathcal{C}(X)$ and let $\Gamma(X)$ be a set such that $\mathcal{K}(X) \subset \Gamma(X) \subset \chi_T(X)$. Then,*

$$\sigma_{e_7}(T) = \bigcap_{K \in \Gamma(X)} \sigma_{ap}(T + K) \text{ and } \sigma_{e_8}(T) = \bigcap_{K \in \Gamma(X)} \sigma_\delta(T + K). \qquad \diamond$$

Proof. Since $\mathcal{K}(X) \subset \Gamma(X) \subset \chi_T(X)$, we obtain

$$\bigcap_{K \in \chi_T(X)} \sigma_{ap}(T + K) \subset \bigcap_{K \in \Gamma(X)} \sigma_{ap}(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap}(T + K) := \sigma_{e_7}(T).$$

$$(\text{resp. } \bigcap_{K \in \chi_T(X)} \sigma_\delta(T + K) \subset \bigcap_{K \in \Gamma(X)} \sigma_\delta(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_\delta(T + K) := \sigma_{e_8}(T)).$$

The use of Theorem 4.4 allows us to conclude that

$$\sigma_{e_7}(T) = \bigcap_{K \in \Gamma(X)} \sigma_{ap}(T + K),$$

and

$$\sigma_{e_8}(T) = \bigcap_{K \in \Gamma(X)} \sigma_\delta(T + K).$$

Hence, we get the desired result. Q.E.D.

In the rest of the section, we give a fine description of the left and the right Weyl essential spectra. To do this, we need to define, for $T \in \mathcal{C}(X)$, the following sets

$$\sigma_e^l(T) = \bigcap_{S \in \chi_T(X)} \sigma_l(T + S) \text{ and } \sigma_e^r(T) = \bigcap_{S \in \chi_T(X)} \sigma_r(T + S).$$

Theorem 4.6. *Let $T \in \mathcal{C}(X)$, then*

$$\sigma_{e_{wl}}(T) = \sigma_e^l(T) \quad \textit{and} \quad \sigma_{e_{wr}}(T) = \sigma_e^r(T). \qquad \diamond$$

Proof. We first prove that $\sigma_{e_{wl}}(T) \subset \sigma_e^l(T)$ (resp. $\sigma_{e_{wr}}(T) \subset \sigma_e^r(T)$). Indeed, for $\lambda \notin \sigma_e^l(T)$ (resp. $\sigma_e^r(T)$), there exists $S \in \chi_T(X)$ such that $\lambda \notin \sigma_l(T + S)$ (resp. $\sigma_r(T + S)$). Hence,

$$\lambda - T - S \in \Phi_l(X) \text{ and } i(\lambda - T - S) \le 0,$$

$$(\text{resp. } \lambda - T - S \in \Phi_r(X) \text{ and } i(\lambda - T - S) \ge 0).$$

Next, since $-(\lambda - T - S)^{-1}S \in \Theta_X$. Thus, applying Theorem 3.8, one has

$$I + (\lambda - T - S)^{-1}S \in \Phi(X) \text{ and } i[I + (\lambda - T - S)^{-1}S] = 0,$$

which implies that

$$I + (\lambda - T - S)^{-1}S \in \Phi_l(X) \text{ and } i[I + (\lambda - T - S)^{-1}S] \leq 0,$$

$$(\text{resp. } I + (\lambda - T - S)^{-1}S \in \Phi_r(X) \text{ and } i[I + (\lambda - T - S)^{-1}S] \geq 0).$$

Using the equality

$$\lambda - T = (\lambda - T - S)[I + (\lambda - T - S)^{-1}S],$$

we deduce from Theorem 2.5 in [6] that

$$\lambda - T \in \Phi_l(X) \text{ and } i(\lambda - T) \leq 0.$$

$$(\text{resp. } \lambda - T \in \Phi_r(X) \text{ and } i(\lambda - T) \geq 0).$$

We conclude from Theorem 2.7 that $\lambda \notin \sigma_{e_{wl}}(T)$ (resp. $\sigma_{e_{wr}}(T)$). The inverse inclusion follows from the fact that $\mathcal{K}(X) \subset \chi_T(X)$. Q.E.D.

5. Some perturbation results

In this section, basing on the last results, we give some perturbation results.

Theorem 5.1. *If for every $\lambda \notin \sigma_{e_i}(T)$, where $i \in \{2, 3, 4, 5, 7, 8\}$, the operator $\lambda - T$ has a left (resp. right) Fredholm inverse $T_{\lambda l}$ (resp. $T_{\lambda r}$) such that $ST_{\lambda l}$ (resp. $T_{\lambda r}S$)$\in \Theta_X$, then*

$$\sigma_{e_i}(T + S) \subset \sigma_{e_i}(T).$$

◇

Proof. We give the proof for $i = 5$. Note that the other cases can be checked in the same manner. Let $\lambda \notin \sigma_{e_5}(T)$, then by Theorem 2.5, $\lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0$. Let $T_{\lambda l}$ (resp. $T_{\lambda r}$) be a left (resp. right) Fredholm inverse of $\lambda - T$, then $T_{\lambda l}(\lambda - T) = I - K$, (resp. $(\lambda - T)T_{\lambda r} = I - K'$), where $K \in \mathcal{K}(X)$ (resp.$K' \in \mathcal{K}(X)$). By making some simple calculations, we get

$$\lambda - T - S = (I - ST_{\lambda l})(\lambda - T) - SK. \tag{3}$$

$$(\text{resp. } \lambda - T - S = (\lambda - T)(I - T_{\lambda r}S) - K'S). \tag{4}$$

Since $ST_{\lambda l}$ (resp. $T_{\lambda r}S$) $\in \Theta_X$, then, using Theorem 3.8, we get $I - ST_{\lambda l}$ (resp. $I - T_{\lambda r}S$)) $\in \Phi(X)$ and $i(I - ST_{\lambda l}) = 0$, (resp. $i(I - T_{\lambda r}S) = 0$). Thus applying the Atkinson's theorem and Eq. (3) (resp. (4)), we infer that $\lambda - T - S \in \Phi(X)$ and $i(\lambda - T - S) = 0$. Consequently, we get from Theorem 2.5 $\lambda \notin \sigma_{e_5}(T + S)$. This allows us to conclude that $\sigma_{e_5}(T + S) \subset \sigma_{e_5}(T)$. Q.E.D.

Theorem 5.2. *Let X be a Banach space with the DP property.*
(i) If $T \in \mathcal{C}(X)$ and S be T-bounded on X such that T has a right weak Fredholm inverse T_r^w such that $-\hat{S}T_r^w \in \Theta_X$. Then,

$$T + S \in \Phi_r(X) \textit{ and } i(T + S) = i(T).$$

Moreover, if $T \in \Phi_-(X)$, then $T + S \in \Phi(X)$.
(ii) If $T \in \mathcal{C}(X)$ and S be T-bounded on X such that T has a left weak Fredholm inverse T_l^w such that $-T_l^w\hat{S} \in \Theta_X$. Then,

$$T + S \in \Phi_l(X) \textit{ and } i(T + S) = i(T).$$

Moreover, if $T \in \Phi_+(X)$, then $T + S \in \Phi(X)$. ◇

Proof. (i) Since T_r^w is a right weak Fredholm inverse of T, then there exists $W \in \mathcal{W}(X)$ such that $\hat{T}T_r^w = I - W$, then

$$(\hat{T} + \hat{S})T_r^w = (I + \hat{S}T_r^w) - W.$$

Now, since $-\hat{S}T_r^w \in \Theta_X$, we get from Theorem 3.8

$$I + \hat{S}T_r^w \in \Phi(X) \text{ and } i(I + \hat{S}T_r^w) = 0.$$

Hence, we infer from Remark 2.12 that

$$(\hat{T} + \hat{S})T_r^w \in \Phi(X) \text{ and } i((\hat{T} + \hat{S})T_r^w) = 0.$$

Then, we deduce that $(\hat{T} + \hat{S})T_r^w \in \Phi_r(X)$ and this implies that $\hat{T} + \hat{S} \in \Phi_r(X_T)$. It follows from Remark 1.1 that $T + S \in \Phi_r(X)$. Moreover, since $i((\hat{T} + \hat{S})T_r^w) = i(\hat{T}T_r^w) = 0$, then $i(\hat{T} + \hat{S}) = -i(T_r^w) = i(\hat{T})$. Hence, from Remark 1.1 we infer that $i(T + S) = i(T)$. Next, if $T \in \Phi_+(X)$, then Remark 1.1 implies that $\hat{T} \in \Phi_+(X)$. Recalling that $\hat{T}T_r^w \in \Phi(X)$, we deduce from Theorem 7.14 in [16] that $T_r^w \in \Phi(X, X_T)$. Now, using the fact that $(\hat{T} + \hat{S})T_r^w \in \Phi(X)$, we infer from Theorem 7.12 in [16] together with Remark 1.1 that $T + S \in \Phi(X)$.
(ii) A similar reasoning allows us to reach the result (ii). Q.E.D.

This brings us to introduce the following subsets of $\chi_T(X)$.

$$\mathcal{M}(X) = \{S \in \mathcal{L}(X) \text{ such that } -ST \in \Theta_X \text{ for all } T \in \mathcal{L}(X)\}$$

and

$$\mathcal{E}(X) = \{S \in \mathcal{L}(X) \text{ such that } -TS \in \Theta_X \text{ for all } T \in \mathcal{L}(X)\}.$$

Proposition 5.3. *Let $T \in \mathcal{C}(X)$, then*
(i) If $T \in \Phi_l(X)$ and $S \in \mathcal{E}(X)$, then $T + S \in \Phi_l(X)$ and $i(T + S) = i(T)$.
(ii) If $T \in \Phi_r(X)$ and $S \in \mathcal{M}(X)$, then $T + S \in \Phi_r(X)$ and $i(T + S) = i(T)$. ◊

Proof. Let $T \in \Phi_l(X)$ (resp. $\Phi_r(X)$), then there exist T_l (resp. T_r) $\in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that

$$T_l(T + S) = I - K + T_lS,$$

$$(\text{resp. } (T + S)T_r = I - K + ST_r).$$

Now, since $S \in \mathcal{E}(X)$ (resp. $\mathcal{M}(X)$), then $-T_lS$ (resp. $-ST_r$) $\in \Theta_X$. The rest of the proof is given in the same way of Theorem 5.2 for the case of the strong topology on the Banach space X. Q.E.D.

Theorem 5.4. *Let $T \in \mathcal{C}(X)$, then*
(i) If $S \in \mathcal{M}(X)$, then $\sigma_{e_{wl}}(T) = \sigma_{e_{wl}}(T + S)$.
(ii) If $S \in \mathcal{M}(X)$, then $\sigma_{e_{wr}}(T) = \sigma_{e_{wr}}(T + S)$. ◊

Proof. (i) Let $\lambda \in \mathbb{C}$ be such that $\lambda \notin \sigma_{e_{wl}}(T)$, then we get from Theorem 2.7

$$\lambda - T \in \Phi_l(X) \text{ and } i(\lambda - T) \leq 0.$$

Using Proposition 5.3, we have

$$\lambda - T - S \in \Phi_l(X) \text{ and } i(\lambda - T - S) \leq 0.$$

Reusing Theorem 2.7, we deduce that $\lambda \notin \sigma_{e_{wl}}(T + S)$. The opposite inclusion follows from symmetry.
(ii) Similarly to (i), the proof of (ii) may be checked. Q.E.D.

6. Essential spectra of matrix operators

In this section, we will describe the left and the right Weyl essential spectra of the matrix operator L, the closure of L_0, acting on the space $X \times X$, where X is a Banach space with the DP property. In the product space $X \times X$, we consider the following operator which is formally defined by a matrix

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5}$$

where the operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(D)$) and acts on X. In the following, it is always assumed that the entries of this matrix satisfy the following conditions, introduced in [17].

($H1$) A is closed, densely defined linear operator on X with nonempty resolvent set $\rho(A)$.

($H2$) The operator B is a densely defined linear operator on X and for (hence all) $\mu \in \rho(A)$, the operator $(A-\mu)^{-1}B$ is closable. (In particular, if B is closable, then $(A-\mu)^{-1}B$ is closable).

($H3$) The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence all) $\mu \in \rho(A)$, the operator $C(A-\mu)^{-1}$ is bounded. (In particular, if C is closable, then $C(A-\mu)^{-1}$ is bounded).

($H4$) The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in X and for some (hence all) $\mu \in \rho(A)$, the operator $D - C(A-\mu)^{-1}B$ is closable. We will denote by $S(\mu)$ its closure.

Remark 6.1. (i) *Under the assumptions* ($H1$) *and* ($H2$), *we infer that for each* $\mu \in \rho(A)$ *the operator* $G(\mu) := \overline{(A-\mu)^{-1}B}$ *is bounded on* X.
(ii) *From the assumption* ($H3$), *it follows that the operator:* $F(\mu) := C(A-\mu)^{-1}$ *is bounded on* X. ◇

The following result give a sufficient condition for which the operator L_0 is closable and describes its closure L.

Theorem 6.2. *[3] Let conditions* ($H1$)-($H3$) *be satisfied and the lineal* $\mathcal{D}(B) \cap \mathcal{D}(D)$ *be dense in* X. *Then, the operator* L_0 *is closable and the closure* L *of* L_0 *is given by*

$$L = \mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}, \tag{6}$$

where $\mu \in \rho(A)$. *Or, spelled out,*

$$\begin{aligned} L : \mathcal{D}(L) \subset (X \times X) &\longrightarrow X \times X \\ \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow L \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} A(x + G(\mu)y) - \mu G(\mu) \\ C(x + G(\mu)y) - S(\mu)y \end{pmatrix}, \end{aligned}$$

with

$$\mathcal{D}(L) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times X \text{ such that } x + G(\mu)y \in \mathcal{D}(A) \text{ and } y \in \mathcal{D}(S(\mu)) \right\}.$$

Note that the description of the operator L *does not depend on the choice of the point* $\mu \in \rho(A)$. ◇

Remark 6.3. *Let* $\lambda \in \mathbb{C}$. *It follows from* (6) *that*

$$\begin{aligned} \lambda - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu)M(\mu) \\ &:= UV(\lambda)W - (\lambda - \mu)M(\mu), \end{aligned} \tag{7}$$

where

$$M(\mu) = \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}.$$ ◇

Theorem 6.4. *Let $\lambda \in \mathbb{C}$.*
(i) Suppose that $A_\lambda \in \Phi_l^w(X)$ and $S_\lambda(\mu) \in \Phi_l^w(X)$. If $F(\mu) \in \mathcal{W}(X)$, then

$$\lambda - L \in \Phi_l(X) \text{ and } i(\lambda - L) = i(V(\lambda)).$$

(ii) Suppose that $A_\lambda \in \Phi_r^w(X)$ and $S_\lambda(\mu) \in \Phi_r^w(X)$. If $F(\mu) \in \mathcal{W}(X)$, then

$$\lambda - L \in \Phi_r(X) \text{ and } i(\lambda - L) = i(V(\lambda)).$$ ◇

Proof. (i) Denote by $T_\lambda = UV(\lambda)W$ and $V_{\lambda l}^w = \begin{pmatrix} A_{\lambda l}^w & W_1 \\ W_2 & S_{\lambda l}^w(\mu) \end{pmatrix}$, where W_1 and W_2 are weakly compact operators. It is easy to see that $V_{\lambda l}^w$ is a left weak-Fredholm inverse of $V(\lambda)$. Thus, $T_l^w = W^{-1}V_{\lambda l}^w U^{-1}$ is a left weak-Fredholm inverse of T_λ. On the other hand, we have

$$T_{\lambda l}^w M(\mu) = \begin{pmatrix} W_1F(\mu) - G(\mu)S_{\lambda l}^w F(\mu) & A_{\lambda l}^w G(\mu) - G(\mu)W_2G(\mu) \\ S_{\lambda l}^w F(\mu) & W_2G(\mu) \end{pmatrix}.$$

Now, take the following bounded sequence $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n \in X \times X$ such that

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} := (I + \alpha T_{\lambda l}^w M(\mu)) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

where $\alpha \in [0, 1]$. Hence, we get the following system

$$\begin{cases} x_n' = (I - \alpha G(\mu)S_{\lambda l}(\mu)F(\mu))\, x_n + \alpha W_1F(\mu)x_n + \alpha(G(\mu)W_2 - A_{\lambda l})G(\mu)y_n. \\ \\ y_n' = \alpha S_{\lambda l}(\mu)F(\mu)x_n + (I + \alpha W_2G(\mu))y_n. \end{cases} \tag{8}$$

We first notice that since $-\alpha W_2G(\mu)$ is weakly compact, then $(-\alpha W_2G(\mu))^2$ is compact and so, demicompact. This implies from Theorem 3.1 that $-\alpha W_2G(\mu)$ is also demicompact. In the same manner, we prove the demicompactness of the operator $\alpha G(\mu)S_{\lambda l}(\mu)F(\mu)$. In the other hand, remark that the operators $\alpha S_{\lambda l}(\mu)F(\mu)$ and $\alpha W_1F(\mu)$ are both weakly compact. It results from the fact that X has the DP property, that $\alpha S_{\lambda l}(\mu)F(\mu)x_n$ and $\alpha W_1F(\mu)x_n$ have convergent subsequences. Hence, from the second equation of system (8), we infer that $(I + \alpha W_2G(\mu))y_n$ has a convergent subsequence. By demicompactness of $-\alpha W_2G(\mu)$, we deduce that $(y_n)_n$ admits a convergent subsequence. In the same way, we infer from the first equation of system (8) that $(I - \alpha G(\mu)S_{\lambda l}(\mu)F(\mu))x_n$ has a convergent subsequence. This together with the fact that $\alpha G(\mu)S_{\lambda l}(\mu)F(\mu)$ is demicompact allows us to conclude that $(x_n)_n$ has a convergent subsequence. Therefore, there exists a subsequence of $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n$ which converges on X. Thus, $-\alpha T_{\lambda l}^w M(\mu) \in \mathcal{DC}(X)$, $\forall \alpha \in [0, 1]$. Finally, the result follows from Theorem 5.2.
(ii) The result can be checked similarly to (i). Q.E.D.

Corollary 6.5. (i) *Suppose that, for each $\lambda \in \Phi_{lA}^w(X) \cap \Phi_{lS(\mu)}^w(X)$, we have $F(\mu) \in \mathcal{W}(X)$. Then,*

$$\sigma_{e_{wl}}(L) \subset \sigma_{e_{wl}}(A) \cup \sigma_{e_{wl}}(S(\mu)).$$

(ii) *Suppose that, for each $\lambda \in \Phi_{rA}^w(X) \cap \Phi_{rS(\mu)}^w(X)$, we have $F(\mu) \in \mathcal{W}(X)$. Then,*

$$\sigma_{e_{wr}}(L) \subset \sigma_{e_{wr}}(A) \cup \sigma_{e_{wr}}(S(\mu)).$$ ◇

Proof. (i) Let $\lambda \notin \sigma_{e_{wl}}(A) \cap \sigma_{e_{wl}}(S(\mu))$, then

$$\lambda - A \in \Phi_l(X) \text{ and } i(\lambda - A) \leq 0,$$

and

$$\lambda - S(\mu) \in \Phi_l(X) \text{ and } i(\lambda - S(\mu)) \leq 0.$$

This implies that $\lambda \in \Phi_{lA}^w(X) \cap \Phi_{lS(\mu)}^w(X)$. The result follows from Theorem 6.4 (i).
(ii) Similarly to (i), the result follows from Theorem 6.4 (ii). Q.E.D.

Remark 6.6. *Notice that in Theorem 6.4 and Corollary 6.5, the hypothesis $F(\mu) \in \mathcal{W}(X)$ can be replaced by $G(\mu) \in \mathcal{W}(X)$.* ◇

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