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A Note on Warped Product Almost Quasi-Yamabe Solitons

Adara M. Blaga[a]

[a]*Department of Mathematics*
West University of Timisoara

Abstract. We consider almost quasi-Yamabe solitons in Riemannian manifolds, derive a Bochner-type formula in the gradient case and prove that under certain assumptions, the manifold is of constant scalar curvature. We also provide necessary and sufficient conditions for a gradient almost quasi-Yamabe soliton on the base manifold to induce a gradient almost quasi-Yamabe soliton on the warped product manifold.

1. Introduction

The notion of *Yamabe solitons*, which generate self-similar solutions to Yamabe flow [8]:

$$\frac{\partial}{\partial t}g(t) = -scal(t) \cdot g(t), \tag{1}$$

firstly appeared to L. F. di Cerbo and M. N. Disconzi in [3]. In [4], B.-Y. Chen introduced the notion of *quasi-Yamabe soliton* which we shall consider in the present paper for a more general case, when the constants are let to be functions.

Let (M, g) be an n-dimensional Riemannian manifold ($n > 2$), ξ a vector field and η a 1-form on M.

Definition 1.1. *An almost quasi-Yamabe soliton on M is a data* (g, ξ, λ, μ) *which satisfy the equation:*

$$\frac{1}{2}\mathcal{L}_\xi g + (\lambda - scal)g + \mu\eta \otimes \eta = 0, \tag{2}$$

where $\mathcal{L}_\xi$ *is the Lie derivative operator along the vector field* ξ *and* λ *and* μ *are smooth functions on* M.

When the potential vector field of (2) is of gradient type, i.e. $\xi = grad(f)$, then (g, ξ, λ, μ) is said to be a *gradient almost quasi-Yamabe soliton* (or a generalized quasi-Yamabe gradient soliton) [9] and the equation satisfied by it becomes:

$$Hess(f) + (\lambda - scal)g + \mu df \otimes df = 0. \tag{3}$$

In the next section, we shall derive a Bochner-type formula for the gradient almost quasi-Yamabe soliton case and prove that under certain assumptions, the manifold is of constant scalar curvature. In the last section we construct an almost quasi-Yamabe soliton on a warped product manifold. Remark that results on warped product gradient Yamabe solitons for certain types of warping functions f have been obtained by W. I. Tokura, L. R. Adriano and R. S. Pina in [10].

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Email address: adarablaga@yahoo.com (Adara M. Blaga)

2. Gradient almost quasi-Yamabe solitons

Remark that in the gradient case, from (3) we get:

$$\nabla\xi = -(\lambda - scal)I - \mu df \otimes \xi. \tag{4}$$

Therefore, $\nabla_\xi \xi = [\Delta(f) + (n-1)(\lambda - scal)]\xi$, i.e. ξ is a *generalized geodesic vector field* with the potential function $\Delta(f) + (n-1)(\lambda - scal)$ [6].

Also, if $(\lambda, \mu) = (scal - 1, 1)$, then ξ is *torse-forming* and if $\mu = 0$, then ξ is *concircular*.

Now we shall get a condition that μ should satisfy in a gradient almost quasi-Yamabe soliton (g, ξ, λ, μ). Taking the scalar product with $Hess(f)$, from (3) we get:

$$|Hess(f)|^2 + (\lambda - scal)\Delta(f) + \frac{\mu}{2}\xi(|\xi|^2) = 0$$

and tracing (3) we obtain:

$$\Delta(f) + n(\lambda - scal) + \mu|\xi|^2 = 0.$$

From the above relations we deduce the equation:

$$n\lambda^2 + (2n \cdot scal + \mu|\xi|^2)\lambda + n \cdot scal^2 + \mu|\xi|^2 \cdot scal - \frac{\mu}{2}\xi(|\xi|^2) - |Hess(f)|^2 = 0$$

which has solution (in λ) if and only if

$$\mu^2|\xi|^4 + 2n\mu\xi(|\xi|^2) + 4n|Hess(f)|^2 \geq 0$$

(that is always true for ξ of constant length).

The next step is to deduce a Bochner-type formula for the gradient almost quasi-Yamabe soliton case.

Theorem 2.1. *If (3) defines a gradient almost quasi-Yamabe soliton on the n-dimensional Riemannian manifold* (M, g) *and* $\eta = df$ *is the g-dual of the gradient vector field* $\xi := grad(f)$*, then:*

$$\frac{1}{2}\Delta(|\xi|^2) = |\nabla\xi|^2 - \frac{1}{n-1}S(\xi,\xi) - \frac{n-2}{2(n-1)}\mu\nabla_\xi(|\xi|^2) - \tag{5}$$

$$-|\xi|^2[\xi(\mu) - \frac{n}{n-1}\mu^2|\xi|^2 - \frac{n^2}{n-1}\lambda\mu + \frac{n^2}{n-1}\mu \cdot scal].$$

Proof. First remark that:

$$trace(\mu\eta \otimes \eta) = \mu|\xi|^2$$

and

$$div(\mu\eta \otimes \eta) = \frac{\mu}{2}d(|\xi|^2) + \mu\Delta(f)df + d\mu(\xi)df.$$

Taking the trace of the equation (3), we obtain:

$$\Delta(f) + n(\lambda - scal) + \mu|\xi|^2 = 0 \tag{6}$$

and differentiating it:

$$d(\Delta(f)) + nd\lambda - nd(scal) + \mu d(|\xi|^2) + |\xi|^2 d\mu = 0. \tag{7}$$

Now taking the divergence of the same equation, we get:

$$div(Hess(f)) + d\lambda - d(scal) + \frac{\mu}{2}d(|\xi|^2) + \mu\Delta(f)df + d\mu(\xi)df = 0. \tag{8}$$

Substracting the relations (8) and (7) computed in ξ and using [2]:

$$div(Hess(f)) = d(\Delta(f)) + i_{Q\xi}g,$$

$$(div(Hess(f)))(\xi) = \frac{1}{2}\Delta(|\xi|^2) - |\nabla\xi|^2,$$

we obtain (5). □

Remark 2.2. *For the case $\mu = 0$, under the assumptions $S(\xi, \xi) \leq (n-1)|\nabla\xi|^2$ we get $\Delta(|\xi|^2) \geq 0$ and from the maximum principle follows that $|\xi|^2$ is constant in a neighborhood of any local maximum. If $|\xi|$ achieve its maximum, then $S(\xi, \xi) = (n-1)|\nabla\xi|^2$.*

Let us make some remarks on the scalar curvature of M.
From (4) we get:

$$R(\cdot, \cdot)\xi = -[d(\lambda - scal) \otimes I - I \otimes d(\lambda - scal)] - \mu(\lambda - scal)(df \otimes I - I \otimes df) - (d\mu \otimes df - df \otimes d\mu)$$

and

$$R(\cdot, \xi)\cdot = d(\lambda - scal) \otimes I - g \otimes [grad(\lambda - scal) - \mu(\lambda - scal)\xi] + \mu(\lambda - scal)df \otimes I + \tag{9}$$

$$+d\mu \otimes df \otimes \xi - df \otimes df \otimes grad(\mu)$$

which for λ and μ constant become:

$$R(\cdot, \cdot)\xi = [d(scal) \otimes I - I \otimes d(scal)] - \mu(\lambda - scal)(df \otimes I - I \otimes df)$$

and

$$R(\cdot, \xi)\cdot = -d(scal) \otimes I + g \otimes [grad(scal) + \mu(\lambda - scal)\xi] + \mu(\lambda - scal)df \otimes I. \tag{10}$$

Using (9), $R(\xi, \xi)X = 0$ implies:

$$[d(\lambda - scal) + |\xi|^2 d\mu] \otimes \xi = df \otimes [grad(\lambda - scal) + |\xi|^2 grad(\mu)]$$

which for λ and μ constant becomes:

$$d(scal) \otimes \xi = df \otimes grad(scal).$$

Assume further that λ and μ are constant. Computing the previous relation in ξ and choosing an open subset where $\xi \neq 0$, we deduce:

$$grad(scal) = \frac{\xi(scal)}{|\xi|^2}\xi. \tag{11}$$

Denoting by $h =: \frac{\xi(scal)}{|\xi|^2}$, from the symmetry of $Hess(scal)$ we obtain:

$$dh \otimes df = df \otimes dh$$

which implies:

$$|\xi|^2 dh = \xi(h)df \text{ and } |\xi|^2 grad(h) = \xi(h)\xi.$$

A similar result like the one obtained by B.-Y. Chen, S. Deshmukh in [6] for Yamabe solitons can be obtained for quasi-Yamabe solitons, following the same ideas in proving it.

Theorem 2.3. *Let (3) define a gradient quasi-Yamabe soliton on the connected n-dimensional Riemannian manifold (M, g) $(n > 1)$ for $\eta = df$ the g-dual of the unitary vector field $\xi := grad(f)$. If $\xi(scal)$ is constant along the integral curves of ξ and $Hess(scal)$ is degenerate in the direction of ξ, then M is of constant scalar curvature.*

Proof. Under these hypotheses, applying divergence to (11) we obtain:

$$\Delta(scal) = \xi(scal)\Delta(f) = -[n(\lambda - scal) + \mu]\xi(scal). \tag{12}$$

Computing the Ricci operator in ξ, $Q\xi = -\sum_{i=1}^{n} R(E_i, \xi)E_i$, for $\{E_i\}_{1\leq i\leq n}$ a local orthonormal frame field on M, and using (10) we get:

$$Q\xi = -(n-1)grad(scal) + (n-1)\mu(\lambda - scal)\xi \tag{13}$$

and

$$S(\xi, \xi) = g(Q\xi, \xi) = -(n-1)\xi(scal) + (n-1)\mu(\lambda - scal). \tag{14}$$

Applying the divergence to (13) we have:

$$div(Q\xi) = -(n-1)\Delta(scal) + (n-1)\mu[(\lambda - scal)\Delta(f) - \xi(scal)]. \tag{15}$$

Computing the same divergence like:

$$div(Q\xi) = div(S)(\xi) + \langle S, Hess(f)\rangle, \tag{16}$$

taking into account the gradient quasi-Yamabe soliton equation, the fact that

$$div(S)(\xi) = \frac{\xi(scal)}{2},$$

the expression of $S(\xi, \xi)$ from (14) and replacing $\Delta(scal)$ from (12), we obtain:

$$\left[\frac{1}{2} - n(n-1)(\lambda - scal) + (n-1)\mu\right]\xi(scal) =$$

$$= (\lambda - scal)[(1 + n(n-1)\mu)scal - n(n-1)\lambda\mu].$$

Differentiating the previous expression along ξ and taking into account the degeneracy of $Hess(scal)(\xi, \xi) = \xi(\xi(scal)) - (\nabla_\xi \xi)(scal)$ in the direction of ξ, after a long computation, we get:

$$\xi(scal)\left[\xi(scal) + scal^2 + k_1 scal + k_2\right] = 0,$$

where the constants k_1 and k_2 are respectively given by:

$$k_1 =: \frac{n+1}{n}\mu + \frac{5}{2n(n-1)}, \quad k_2 =: \lambda^2 - \frac{1}{n}\mu^2 - \frac{n+1}{n}\lambda\mu - \frac{3\lambda + \mu}{2n(n-1)}.$$

Differentiating again the term in the parantheses along ξ we get:

$$\xi(scal)\left[3scal - \lambda + \frac{1}{n}\mu + \frac{5}{2n(n-1)}\right] = 0$$

which completes the proof. □

3. Warped product almost quasi-Yamabe solitons

3.1. *Warped product manifolds*

Consider (B, g_B) and (F, g_F) two Riemannian manifolds of dimensions n and m, respectively. Denote by π and σ the projection maps from the product manifold $B \times F$ to B and F and by $\widetilde{\varphi} := \varphi \circ \pi$ the lift to $B \times F$ of a smooth function φ on B. In this context, we shall call B *the base* and F *the fiber* of $B \times F$, the unique element

$\widetilde{X}$ of $\chi(B \times F)$ that is π-related to $X \in \chi(B)$ and to the zero vector field on F, the *horizontal lift of* X and the unique element $\widetilde{V}$ of $\chi(B \times F)$ that is σ-related to $V \in \chi(F)$ and to the zero vector field on B, the *vertical lift of* V. Also denote by $\mathcal{L}(B)$ the set of all horizontal lifts of vector fields on B, by $\mathcal{L}(F)$ the set of all vertical lifts of vector fields on F, by $\mathcal{H}$ the orthogonal projection of $T_{(p,q)}(B \times F)$ onto its horizontal subspace $T_{(p,q)}(B \times \{q\})$ and by $\mathcal{V}$ the orthogonal projection of $T_{(p,q)}(B \times F)$ onto its vertical subspace $T_{(p,q)}(\{p\} \times F)$.

Let $\varphi > 0$ be a smooth function on B and

$$g := \pi^* g_B + (\varphi \circ \pi)^2 \sigma^* g_F \tag{17}$$

be a Riemannian metric on $B \times F$.

Definition 3.1. *[1] The product manifold of B and F together with the Riemannian metric g defined by (17) is called the warped product of B and F by the warping function φ (and is denoted by $(M := B \times_\varphi F, g)$).*

In particular, if $\varphi = 1$, then the warped product becomes the usual product of the Riemannian manifolds.

For simplification, in the rest of the paper we shall simply denote by X the horizontal lift of $X \in \chi(B)$ and by V the vertical lift of $V \in \chi(F)$.

Notice that the lift on M of the gradient and the Hessian satisfy:

$$grad(\widetilde{f}) = \widetilde{grad(f)}, \tag{18}$$

$$(Hess(\widetilde{f}))(X, Y) = (\widetilde{Hess(f))}(X, Y), \quad \text{for any } X, Y \in \mathcal{L}(B), \tag{19}$$

for any smooth function f on B.

Also, the scalar curvatures are connected by the relation [7]:

$$scal = \widetilde{scal_B} + \frac{\widetilde{scal_F}}{\varphi^2} - \pi^*\left(2m\frac{\Delta(\varphi)}{\varphi} + m(m-1)\frac{|grad(\varphi)|^2}{\varphi^2}\right). \tag{20}$$

3.2. Warped product almost quasi-Yamabe solitons

We shall construct a gradient almost quasi-Yamabe soliton on a warped product manifold.

Let (B, g_B) be an n-dimensional Riemannian manifold, $\varphi > 0$ a smooth function on B and f, μ smooth functions on B such that:

$$\Delta(f) + \mu|grad(f)|^2 = n\frac{(grad(f))(\varphi)}{\varphi}. \tag{21}$$

In this case, any gradient almost quasi-Yamabe soliton $(g_B, grad(f), \lambda_B, \mu_B)$ on (B, g_B) is given by $\lambda_B = scal_B - \frac{(grad(f))(\varphi)}{\varphi}$ and $\mu_B = \mu$.

Take (F, g_F) an m-dimensional manifold with

$$scal_F = \pi^*((\lambda - \lambda_B)\varphi^2 + 2m\varphi\Delta(\varphi) + m(m-1)|grad(\varphi)|^2)|_F, \tag{22}$$

where π and σ are the projection maps from the product manifold $B \times F$ to B and F, respectively, $g := \pi^* g_B + (\varphi \circ \pi)^2 \sigma^* g_F$ is a Riemannian metric on $B \times F$, $\lambda_B = scal_B - \frac{(grad(f))(\varphi)}{\varphi}$ and λ is a smooth function on B.

With the above notations, we prove:

Theorem 3.2. *Let (B, g_B) be an n-dimensional Riemannian manifold, $\varphi > 0$, f, μ smooth functions on B satisfying (21) and (F, g_F) an m-dimensional Riemannian manifold with the scalar curvature given by (22). Then $(g, \xi, \pi^*(\lambda), \pi^*(\mu))$, where $\xi = grad(\widetilde{f})$, is a gradient almost quasi-Yamabe soliton on the warped product manifold $(B \times_\varphi F, g)$ if and only if $(g_B, grad(f), \lambda_B = scal_B - \frac{(grad(f))(\varphi)}{\varphi}, \mu)$ is a gradient almost quasi-Yamabe soliton on (B, g_B).*

Proof. The gradient almost quasi-Yamabe soliton $(g, \xi, \pi^*(\lambda), \pi^*(\mu))$ on $(B \times_\varphi F, g)$ is given by:

$$Hess(\widetilde{f}) + (\pi^*(\lambda) - scal)g + \pi^*(\mu)d\widetilde{f} \otimes d\widetilde{f} = 0. \tag{23}$$

Notice that from (20), (21) and (22) we deduce that

$$\pi^*(\lambda) - scal = \pi^*(\lambda_B) - \widetilde{scal_B},$$

hence for any $X, Y \in \mathcal{L}(B)$ we get:

$$H^f(X, Y) + (\lambda_B - scal_B)g_B(X, Y) + \mu df(X)df(Y) = 0 \tag{24}$$

i.e. $(g_B, grad(f), \lambda_B, \mu)$ is a gradient almost quasi-Yamabe soliton on (B, g_B), where H^f denotes the lift of $Hess(f)$.

Conversely, notice that the left-hand side term in (23) computed in (X, V), for $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$ vanishes identically and for each situation (X, Y) and (V, W), we can recover the equation (23) from (21) and the fact that $(g_B, grad(f), \lambda_B, \mu)$ is a gradient almost quasi-Yamabe soliton on (B, g_B). Indeed, taking the trace of (24) we get

$$\Delta(f) + n(\lambda_B - scal_B) + \mu|grad(f)|^2 = 0$$

and using (21) we obtain

$$\pi^*(\lambda_B) - \widetilde{scal_B} = -\frac{(grad(f))(\varphi)}{\varphi}.$$

We know that for any $V, W \in \mathcal{L}(F)$:

$$H^f(V, W) = (Hess(\widetilde{f}))(V, W) = g(\nabla_V(grad(\widetilde{f})), W) =$$

$$= \pi^*\left[\frac{(grad(f))(\varphi)}{\varphi}\right]|_F\widetilde{\varphi}^2|_F g_F(V, W)$$

and we deduce that

$$H^f(V, W) + (\pi^*(\lambda_B) - \widetilde{scal_B})|_F g(V, W) = 0.$$

□

Example 3.3. *Consider $M = \{(x, y, z) \in \mathbb{R}^3, z > 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$,*

$$g_M := \frac{1}{z^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz) \text{ and } \xi_M := -z\frac{\partial}{\partial z}.$$

Let $(g_M, \xi_M, -8, 2)$ be the gradient quasi-Yamabe soliton on the Riemannian manifold (M, g_M) and let S^3 be the 3-sphere with the round metric g_S (which is Einstein with the Ricci tensor equals to $2g_S$). Thus we obtain the gradient quasi-Yamabe soliton $(g, \xi, -2, 2)$ on the "generalized cylinder" $M \times S^3$, where $g = g_M + g_S$ and ξ is the lift on $M \times S^3$ of the gradient vector field $\xi_M = grad(f)$, where $f(x, y, z) := -\ln z$.

3.3. *Some consequences of condition (21)*

Let us make some remarks on the class of manifolds that satisfy the condition:

$$\Delta(f) + \mu|\xi|^2 = n\frac{d\varphi(\xi)}{\varphi}, \tag{25}$$

for $\varphi > 0$, f and μ smooth functions on the oriented and compact Riemannian manifold (B, g_B) and $\xi := grad(f)$.

Remark that if

$$Hess(f) - \frac{n}{2\varphi}(df \otimes d\varphi + d\varphi \otimes df) + \mu df \otimes df = 0, \tag{26}$$

then (25) is satisfied. Computing $Hess(f)(X, Y) := g_B(\nabla_X \xi, Y)$ we get

$$\nabla\xi = \frac{n}{2\varphi}(df \otimes grad(\varphi) + d\varphi \otimes \xi) - \mu df \otimes \xi.$$

Also notice that in this case, if $(g_B, \xi, \lambda_B, \mu)$ is a gradient almost quasi-Yamabe soliton on (B, g_B), then the metric g_B is precisely

$$g_B = -\frac{n}{2\varphi(\lambda_B - scal_B)}(df \otimes d\varphi + d\varphi \otimes df)$$

and $scal_B = \lambda_B + \frac{d\varphi(\xi)}{\varphi}$.

In what follows, we shall focus on condition (26). We've checked that [2]:

$$|Hess(f) - \frac{\Delta(f)}{n} g_B|^2 = |Hess(f)|^2 - \frac{(\Delta(f))^2}{n},$$

$$(div(Hess(f)))(\xi) = div(Hess(f)(\xi)) - |Hess(f)|^2,$$

therefore:

$$(div(Hess(f)))(\xi) = div(Hess(f)(\xi)) - |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 - \frac{(\Delta(f))^2}{n}. \tag{27}$$

Applying the divergence to (26), computing it in ξ and taking into account that

$$div\left(\frac{1}{\varphi} df \otimes d\varphi\right) = \left(\frac{\Delta(f)}{\varphi} - \frac{d\varphi(\xi)}{\varphi^2}\right) d\varphi + \frac{1}{\varphi} i_{\nabla_\xi grad(\varphi)} g_B$$

and

$$div(\mu df \otimes df) = \frac{\mu}{2} d(|\xi|^2) + \mu\Delta(f) df + d\mu(\xi) df,$$

we get:

$$(div(Hess(f)))(\xi) = n\left(\frac{\Delta(f)}{\varphi} - \frac{d\varphi(\xi)}{\varphi^2}\right) d\varphi(\xi) + \frac{n}{\varphi} g_B(\nabla_\xi grad(\varphi), \xi) - \tag{28}$$

$$-\frac{\mu}{2} d(|\xi|^2)(\xi) - \mu\Delta(f)|\xi|^2 - d\mu(\xi)|\xi|^2$$

and we obtain:

$$div(Hess(f)(\xi)) = |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 + \frac{(\Delta(f))^2}{n} + n\left(\frac{\Delta(f)}{\varphi} - \frac{d\varphi(\xi)}{\varphi^2}\right) d\varphi(\xi) + \tag{29}$$

$$+\frac{n}{\varphi} g_B(\nabla_\xi grad(\varphi), \xi) - \frac{\mu}{2} d(|\xi|^2)(\xi) - \mu\Delta(f)|\xi|^2 - d\mu(\xi)|\xi|^2.$$

Integrating with respect to the canonical measure on B, we have:

$$\int_B d(|\xi|^2)(\xi) = \int_B \langle grad(|\xi|^2), \xi \rangle = -\int_B \langle |\xi|^2, div(\xi) \rangle = -\int_B |\xi|^2 \cdot \Delta(f).$$

Using:

$$|\xi|^2 \cdot \Delta(f) = |\xi|^2 \cdot div(\xi) = div(|\xi|^2 \xi) - |\xi|^2$$

taking μ constant and integrating (29) on B, from the above relations and the divergence theorem, we obtain:

$$\int_B |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 + (n+1)\int_B \Delta(f)\cdot \frac{d\varphi(\xi)}{\varphi} + \frac{(2-\mu)n+2}{2n}\int_B |\xi|^2 - \tag{30}$$

$$-n\int_B \frac{(d\varphi(\xi))^2}{\varphi^2} + n\int_B g_B(\frac{1}{\varphi}\nabla_\xi grad(\varphi), \xi) = 0.$$

Assume now that μ is constant and consider the product manifold $B\times F$, in which case (26) and (25) (for $\varphi = 1$) become:

$$Hess(f) + \mu df \otimes df = 0 \ \ and \ \ \Delta(f) + \mu|\xi|^2 = 0. \tag{31}$$

Remark 3.4. *i) In the case of product manifold (for $\varphi = 1$), the chosen manifold (F, g_F) is of scalar curvature $scal_F = \pi^*(\lambda - scal_B)|_F$. In particular, for $\lambda = scal_B$, (F, g_F) is locally isometric to an Euclidean space. Moreover, $\nabla_\xi \xi = -\mu|\xi|^2\xi$, therefore, ξ is a generalized geodesic vector field with the potential function $\Delta(f)$.*

ii) For $\varphi = 1$ and μ constant, we obtain:

$$\mu^2 \int_B |\xi|^4 = 0$$

and we can state:

Corollary 3.5. *Let (B, g_B) be an oriented and compact n-dimensional Riemannian manifold, f a smooth function on B and μ a real constant satisfying (31). Then $\mu = 0$ hence, f is harmonic and $\nabla\xi = 0$.*

Proposition 3.6. *Let (B, g_B) be an oriented, compact and complete n-dimensional ($n > 1$) Riemannian manifold, f a smooth function on B and μ a real constant satisfying (31). Then B is conformal to a sphere in the $(n+1)$-dimensional Euclidean space.*

Proof. From the above observations, we have:

$$\int_B |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 = \int_B |Hess(f)|^2 - \int_B \frac{(\Delta(f))^2}{n} = \frac{n-1}{n}\mu^2 \int_B |\xi|^4 = 0,$$

so $Hess(f) = \frac{\Delta(f)}{n} g_B$ which implies by [11] that B is conformal to a sphere in the $(n+1)$-dimensional Euclidean space. □

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