On Δ^m-Asymptotically Deferred Statistical Equivalent Sequences of Order α

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Abstract. In this study we introduce and examine the concepts of Δ^m-asymptotic deferred statistical equivalence of order α and strong Δ_r^m-asymptotic deferred equivalence of order α of sequences of real numbers. Also, we give some relations connected to these concepts.

1. Introduction

The idea of statistical convergence was given by Zygmund [35] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [31] and Fast [19] and later reintroduced by Schoenberg [30] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Çınar et al. ([10], [13]), Çolak [11], Connor [6], Et et al. ([15], [18]), Fridy [20], Işık ([21], [22]), Işık and Et [23], Mursaleen [26], Salat [29], Şengül and Et ([16], [32]), Di Maio and Kočinac [12], Caserta et al. [5], Çakallı ([7], [8]), Çakallı and Savaş [9] and many others.

Marouf [25] introduced definitions for asymptotically equivalent sequences and asymptotic regular matrices. Patterson [28] extended these concepts by presenting an asymptotically statistically equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

The difference sequence spaces was generalized by Et et al.([14], [15], [17], [18]) as follows:

$$\Delta^m (X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one sequence $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m}, \tag{1}$$

$$y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. We use this fact to formulate (3) and (4). Recently the difference sequence spaces have been studied in ([1], [3], [4], [15], [27], [33]).

The main goal of this work is to examine the relations between the concepts of Δ^m-asymptotic deferred statistical equivalence of order α and strong Δ_r^m-asymptotic deferred equivalence of order α of sequences of real numbers, where $\alpha \in (0,1]$ and $r \in \mathbb{R}^+$.

2. Definitions and Preliminaries

The deferred Cesàro mean of sequences of real numbers was introduced by Agnew [2] such as:

$$\left(D_{p,q}x\right)_n = \frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n \to \infty} q(n) = +\infty.$$

The concepts of deferred density and deferred statistical convergence were given by Küçükaslan and Yılmaztürk ([24], [34]) such as:

Let K be a subset of $\mathbb{N}$ and denote the set $\{k : p(n) < k \le q(n), k \in K\}$ by $K_d(n)$. Deferred density of K is defined by

$$\delta_d(K) = \lim_{n \to \infty} \frac{1}{q(n) - p(n)} |K_d(n)|, \text{ provided the limit exists.} \tag{2}$$

The vertical bars in (2) indicate the cardinality of the set $K_d(n)$.

Let $\{p(n)\}$ and $\{q(n)\}$ be two sequences as above and $0 < \alpha \le 1$ be given. We define *deferred $\alpha-$density* of a subset K of $\mathbb{N}$ by

$$\delta_d^\alpha(K) = \lim_{n \to \infty} \frac{1}{(q(n) - p(n))^\alpha} |K_d(n)|, \text{ provided the limit exists.}$$

Deferred $\alpha-$density $\delta_d^\alpha(K)$ reduces to natural density $\delta(K)$ in the special case $\alpha = 1$ and $q(n) = n$, $p(n) = 0$.

It can be clearly seen that every finite subset of $\mathbb{N}$ has zero $\alpha-$deferred density and $\delta_d^\beta(K) \le \delta_d^\alpha(K)$ for $0 < \alpha \le \beta \le 1$. Beside, it does not need to hold $\delta_d^\alpha(K^c) = 1 - \delta_d^\alpha(K)$ for $0 < \alpha < 1$ in general.

Definition 2.1. *[33] Let $\{p(n)\}$ and $\{q(n)\}$ be two sequences of non-negative integers satisfying conditions given above, $m \in \mathbb{N}$ and $\alpha \in (0,1]$ be given. A sequence $x = (x_k)$ is said to be Δ^m-deferred statistically convergent of order α to L if there is a real number L such that for each $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{(q(n) - p(n))^\alpha} \left|\{p(n) < k \le q(n) : |\Delta^m x_k - L| \ge \varepsilon\}\right| = 0.$$

In this case we write $\Delta^m(S_d^\alpha) - \lim x_k = L$. The set of all Δ^m-deferred statistically convergent sequences of order α will be denoted by $\Delta^m(S_d^\alpha)$. If $m = 0$, $q(n) = n$ and $p(n) = 0$, then the concept of Δ^m-deferred statistical convergence of order α coincides with the concept of statistical convergence of order α. If $q(n) = n$, $p(n) = 0$ and $\alpha = 1$ then the concept coincides with Δ^m- statistical convergence denoted by $\Delta^m(S)$. In the special cases $m = 0$, $\alpha = 1$, $q(n) = n$ and $p(n) = 0$, Δ^m-deferred statistical convergence of order α coincides with usual statistical convergence.

Definition 2.2. *Let $\{p(n)\}$ and $\{q(n)\}$ be two sequences as above, α be any real number such that $0 < \alpha \leq 1$ and x, y be two sequences of real numbers such that $\Delta^m x_k > 0$ and $\Delta^m y_k > 0$ for all $k \in \mathbb{N}$. Two sequences x and y are said to be Δ^m-asymptotically deferred statistical equivalent of order α, provided that for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))^\alpha} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon \right\} \right| = 0.$$

In this case we write $x \overset{\Delta^m(S_d^\alpha)}{\sim} y$. If $q(n) = n$ and $p(n) = 0$, then the concept of Δ^m-asymptotic deferred statistical equivalence of order α coincides with the concept of Δ^m-asymptotic statistical equivalence of order α denoted by $x \overset{\Delta^m(S^\alpha)}{\sim} y$. If $q(n) = n$, $p(n) = 0$ and $\alpha = 1$, then the concept of Δ^m-asymptotic deferred statistical equivalence of order α coincides with Δ^m-asymptotic statistical equivalence denoted by $x \overset{\Delta^m(S)}{\sim} y$. If $\alpha = 1$, then the concept of Δ^m-asymptotic deferred statistical equivalence of order α coincides with Δ^m-asymptotic deferred statistical equivalence denoted by $x \overset{\Delta^m(S_d)}{\sim} y$. In the special case $m = 0$, $\alpha = 1$, $q(n) = n$ and $p(n) = 0$, the concept coincides with usual asymptotic statistical equivalence.

Definition 2.3. *Let $\{p(n)\}$ and $\{q(n)\}$ be two sequences as above, α be any real number such that $0 < \alpha \leq 1$ and x, y be two sequences such that $\Delta^m x_k > 0$ and $\Delta^m y_k > 0$ for all $k \in \mathbb{N}$, and r be a positive real number. Two real valued sequences x and y are said to be strong Δ_r^m-asymptotically deferred statistical equivalent of order α provided that*

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))^\alpha} \sum_{p(n)+1}^{q(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r = 0$$

and this is denoted by $x \overset{\Delta^m(w_d^\alpha[r])}{\sim} y$. If $q(n) = n$ and $p(n) = 0$, then the concept of strong Δ_r^m-asymptotic deferred statistical equivalence of order α coincides with the concept of strong Δ_r^m-asymptotic statistical equivalence of order α denoted by $x \overset{\Delta^m(w^\alpha[r])}{\sim} y$. If $q(n) = n$, $p(n) = 0$ and $\alpha = 1$, then the concept of strong Δ_r^m-asymptotic deferred statistical equivalence of order α coincides with the concept of strong Δ_r^m-asymptotic statistical equivalence denoted by $x \overset{\Delta^m(w[r])}{\sim} y$. In the special cases $m = 0$, $\alpha = 1$, $q(n) = n$ and $p(n) = 0$, then the concept of strong Δ_r^m-asymptotic deferred statistical equivalence of order α coincides with the concept of strong asymptotic statistical equivalence denoted by $x \overset{w[r]}{\sim} y$.

3. Main Results

In this section we introduce the concepts of Δ^m-asymptotic deferred statistical equivalence of order α and strong Δ_r^m-asymptotic deferred equivalence of order α of sequences of real numbers. Also some relations between Δ^m-asymptotic deferred statistical equivalence of order α and strong Δ_r^m-asymptotic deferred equivalence of order α are given.

Theorem 3.1. *Let $\{p(n)\}$ and $\{q(n)\}$ be given as above. If $\lim_n \dfrac{q(n) - p(n)}{n} > 0$, then $x \overset{\Delta^m(S)}{\sim} y$ implies $x \overset{\Delta^m(S_d)}{\sim} y$.*

Proof. Let $x \overset{\Delta^m(S)}{\sim} y$ and $\lim_n \dfrac{q(n) - p(n)}{n} > 0$. For a given $\varepsilon > 0$, we have

$$\left\{ k \leq n : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon \right\} \supseteq \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon \right\}$$

and therefore

$$\frac{1}{n} \left| \left\{ k \leq n : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon \right\} \right| \geq \frac{1}{n} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon \right\} \right|$$

$$= \frac{q(n) - p(n)}{n} \frac{1}{(q(n) - p(n))} \left| \left\{ p(n) < k \leq q(n) : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon \right\} \right|.$$

Taking limit as $n \to \infty$ and using the fact that $\lim\limits_{n} \dfrac{q(n) - p(n)}{n} > 0$, we get $x \overset{\Delta^m(S_d)}{\sim} y$. $\square$

Corollary 3.2. *Under the conditions of Theorem 3.1, if $x \overset{S}{\sim} y$ implies $x \overset{(S_d)}{\sim} y$.*

Theorem 3.3. *Let $\{p(n)\}$ and $\{q(n)\}$ be given as above, α and β be two real numbers such that $0 < \alpha \le \beta \le 1$ and r be a positive real number, then $x \overset{\Delta^m\left(w_d^\alpha[r]\right)}{\sim} y$ implies $x \overset{\Delta^m\left(S_d^\beta\right)}{\sim} y$.*

Proof. Omitted. $\square$

Theorem 3.3 yields the following results.

Corollary 3.4. *Let $\{p(n)\}$ and $\{q(n)\}$ be given as above, α be any real number such that $0 < \alpha \le 1$ and r be a positive real number, then*

i) $x \overset{\Delta^m\left(w_d^\alpha[r]\right)}{\sim} y$ implies $x \overset{\Delta^m(S_d^\alpha)}{\sim} y$,

ii) $x \overset{\Delta^m\left(w_d^\alpha[r]\right)}{\sim} y$ implies $x \overset{\Delta^m(S_d)}{\sim} y$,

iii) $x \overset{\Delta^m(w_d[r])}{\sim} y$ implies $x \overset{\Delta^m(S_d)}{\sim} y$ for $\alpha = 1$,

iv) $x \overset{(w_d[r])}{\sim} y$ implies $x \overset{(S_d)}{\sim} y$ for $\alpha = 1$ and $m = 0$.

Remark 3.5. *Even if x and y are Δ^m-bounded sequences, the converse of Theorem 3.3 does not hold, in general. To show this we must find two sequences that are Δ^m-bounded and Δ^m-asymptotically deferred statistical equivalent of order α, but need not to be strong Δ_r^m-asymptotically deferred statistical equivalent of order β. To show this let $p(n) = 0$ and $q(n) = n$ for all $n \in \mathbb{N}$ and $x = (x_k)$ and $y = (y_k)$ be defined as follow:*

$$\Delta^m x_k = \begin{cases} \frac{1}{\sqrt{k}}, & k \ne m^3 \\ 1, & k = m^3 \end{cases} \tag{3}$$

and

$$\Delta^m y_k = 1, \text{ for all } k \in \mathbb{N}$$

It can be shown that $x, y \in \ell_\infty(\Delta^m)$, x and y are Δ^m-asymptotically deferred statistical equivalent of order α for $\alpha \in \left(\frac{1}{3}, 1\right]$. Now we show that x and y need not to be strong Δ_r^m-asymptotically deferred statistical equivalent of order β. First of all, recall that the inequality $\sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$ is satisfied for $n \ge 2$. Define $H_n = \left\{p(n) < k \le q(n) : k \ne m^3, \; m = 1, 2, 3, ...\right\}$ and take $r = 1$ and $L = 0$. Since

$$\sum_{p(n)+1}^{q(n)} \left|\frac{\Delta^m x_k}{\Delta^m y_k}\right|^r = \sum_{k=1}^{n} \left|\frac{\Delta^m x_k}{\Delta^m y_k}\right| = \sum_{k \in H_n}^{n} \left|\frac{\Delta^m x_k}{\Delta^m y_k}\right| + \sum_{k \notin H_n}^{n} \left|\frac{\Delta^m x_k}{\Delta^m y_k}\right|$$

$$= \sum_{k \in H_n}^{n} \frac{1}{\sqrt{k}} + \sum_{k \notin H_n}^{n} 1 > \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$

For $p(n) = 0$ and $q(n) = n$, we have

$$\frac{1}{(q(n) - p(n))^\alpha} \sum_{p(n)+1}^{q(n)} \left|\frac{\Delta^m x_k}{\Delta^m y_k}\right| > \frac{1}{n^\alpha} \sum_{k=1}^{n} \left|\frac{\Delta^m x_k}{\Delta^m y_k}\right| > \frac{1}{n^\alpha} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

$$> \frac{1}{n^\alpha} \sqrt{n} = \frac{1}{n^{\alpha - \frac{1}{2}}} \to \infty \text{ as } n \to \infty.$$

So x and y are not strong Δ_r^m-asymptotically deferred statistical equivalent of order α for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

Theorem 3.6. *Let $\{p(n)\}$ and $\{q(n)\}$ be given as above, if x and y are Δ^m−bounded then $x \overset{\Delta^m(S_d)}{\sim} y$ implies $x \overset{\Delta^m(w_d[r])}{\sim} y$.*

Proof. Omitted. □

The following result a consequence of Theorem 3.6.

Corollary 3.7. *If x and y are Δ^m−bounded sequences, then $x \overset{\Delta^m(S)}{\sim} y$ implies $x \overset{\Delta^m(w[r])}{\sim} y$.*

Theorem 3.8. *Let $\{p(n)\}$ and $\{q(n)\}$ be given as above. If the sequence $\left(\dfrac{q(n)}{q(n)-p(n)}\right)$ is bounded, then $x \overset{\Delta^m(S^\alpha)}{\sim} y$ implies $x \overset{\Delta^m(S_d^\alpha)}{\sim} y$.*

Proof. Since $p(n) < q(n)$ and $\lim\limits_{n\to\infty} q(n) = +\infty$, if

$$\lim_{n\to\infty} \frac{1}{n^\alpha}\left|\left\{k \le n : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right| = 0$$

then

$$\lim_{n\to\infty} \frac{1}{[q(n)]^\alpha}\left|\left\{k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right| = 0.$$

Since the sequence $\left(\dfrac{q(n)}{q(n)-p(n)}\right)$ is bounded, there exists a number M such that $\dfrac{q(n)}{q(n)-p(n)} \le M$. For a given $\varepsilon > 0$, we have

$$\left\{p(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\} \subseteq \left\{k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}$$

and the inequality

$$\left|\left\{p(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right| \le \left|\left\{k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|$$

holds. Therefore

$$\frac{1}{(q(n)-p(n))^\alpha}\left|\left\{p(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|$$
$$\le \frac{[q(n)]^\alpha}{(q(n)-p(n))^\alpha}\frac{1}{[q(n)]^\alpha}\left|\left\{k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|.$$

Taking limit as $n \to \infty$, we get $x \overset{\Delta^m(S_d^\alpha)}{\sim} y$. □

Remark 3.9. *The converse of Theorem 3.8 does not hold even if $\left(\dfrac{q(n)}{q(n)-p(n)}\right)$ is bounded. For this, consider two sequences $x = (x_k)$ and $y = (y_k)$ defined by*

$$\Delta^m x_k = \begin{cases} \frac{k+1}{2}, & k \text{ is odd} \\ -\frac{k}{2}, & k \text{ is even} \end{cases}, \tag{4}$$

and

$$\Delta^m y_k = 1, \text{ for all } k \in \mathbb{N}.$$

Take $p(n) = 2n$, $q(n) = 4n$ and choose $r = 1$. It is clear that $x \overset{\Delta^m(S_d^\alpha)}{\sim} y$, but $x \overset{\Delta^m(S^\alpha)}{\not\sim} y$.

Theorem 3.8 yields the following result.

Corollary 3.10. *Let $\{q(n)\}$ be an arbitrary sequence with $q(n) < n$ for all $n \in \mathbb{N}$ and $\left(\dfrac{n}{q(n)-p(n)}\right)$ be bounded. Then $x \overset{\Delta^m(S)}{\sim} y$ implies $x \overset{\Delta^m(S_d)}{\sim} y$.*

Theorem 3.11. *Let $\{p(n)\}$ and $\{q(n)\}$ be given as above, α be any real number such that $0 < \alpha \leq 1$. If the sequence $\left(\dfrac{p(n)}{q(n)-p(n)}\right)$ is bounded, then $x \overset{\Delta^m(w^\alpha[r])}{\sim} y$ implies $x \overset{\Delta^m(w^\alpha_d[r])}{\sim} y$.*

Proof. Since the sequence $\left(\dfrac{p(n)}{q(n)-p(n)}\right)$ is bounded there exists a positive number K such that $\dfrac{p(n)}{q(n)-p(n)} \leq K$. Let us assume that $x \overset{\Delta^m(w^\alpha[r])}{\sim} y$, then we have

$$
\frac{1}{(q(n)-p(n))^\alpha} \sum_{p(n)+1}^{q(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r
$$

$$
= \frac{1}{(q(n)-p(n))^\alpha} \left[\sum_{k=1}^{q(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r - \sum_{k=1}^{p(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r \right]
$$

$$
\leq \frac{1}{(q(n)-p(n))^\alpha} \left[\sum_{k=1}^{q(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r + \sum_{k=1}^{p(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r \right]
$$

$$
= \frac{[p(n)]^\alpha}{(q(n)-p(n))^\alpha} \frac{1}{[p(n)]^\alpha} \sum_{k=1}^{p(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r + \frac{[q(n)]^\alpha}{[q(n)-p(n)]^\alpha} \frac{1}{[q(n)]^\alpha} \sum_{k=1}^{q(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r
$$

$$
= K^\alpha \frac{1}{[p(n)]^\alpha} \sum_{k=1}^{p(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r + (1+K)^\alpha \frac{1}{[q(n)]^\alpha} \sum_{k=1}^{q(n)} \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right|^r .
$$

Hence we get $x \overset{\Delta^m(w^\alpha_d[r])}{\sim} y$. $\square$

The following result is easily derivable from Theorem 3.11.

Corollary 3.12. *Let $\{p(n)\}$ and $\{q(n)\}$ be given as above. If $\left(\dfrac{q(n)+p(n)}{q(n)-p(n)}\right)$ is bounded, then $x \overset{\Delta^m(w^\alpha[r])}{\sim} y$ implies $x \overset{\Delta^m(w^\alpha_d[r])}{\sim} y$.*

Theorem 3.13. *Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be sequences of non-negative integers satisfying*

$$
p(n) \leq p'(n) < q'(n) \leq q(n), \text{for all } n \in \mathbb{N} \tag{5}
$$

such that the sets $\{k : p(n) < k \leq p'(n)\}, \{k : q'(n) < k \leq q(n)\}$ are finite, then $x \overset{\Delta^m(S_{d'})}{\sim} y$ implies $x \overset{\Delta^m(S_d)}{\sim} y$, where if the following equality is satisfied then we say that $x \overset{\Delta^m(S_{d'})}{\sim} y$,

$$
\lim_{n\to\infty} \frac{1}{(q'(n)-p'(n))} \left| \left\{ p'(n) < k \leq q'(n) : \left| \frac{\Delta^m x_k}{\Delta^m y_k} - L \right| \geq \varepsilon \right\} \right| = 0.
$$

Proof. Let us assume that the sets $\left\{k : p(n) < k \le p'(n)\right\}$ and $\left\{k : q'(n) < k \le q(n)\right\}$ are finite and $x \overset{\Delta^m(S_{d'})}{\sim} y$. Then for any $\varepsilon > 0$ we have

$$
\left\{k : p(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\} = \left\{k : p(n) < k \le p'(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}
$$

$$
\cup \left\{k : p'(n) < k \le q'(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}
$$

$$
\cup \left\{k : q'(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}
$$

and so

$$
\frac{1}{q(n) - p(n)} \left|\left\{k : p(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|
$$

$$
\le \frac{1}{q'(n) - p'(n)} \left|\left\{k : p(n) < k \le p'(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|
$$

$$
+ \frac{1}{q'(n) - p'(n)} \left|\left\{k : p'(n) < k \le q'(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|
$$

$$
+ \frac{1}{q'(n) - p'(n)} \left|\left\{k : q'(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|.
$$

Taking limit as $n \to \infty$, we get $x \overset{\Delta^m(S_d)}{\sim} y$. $\square$

Theorem 3.14. *Let* $\{p(n)\}, \{q(n)\}, \left\{p'(n)\right\}$ *and* $\left\{q'(n)\right\}$ *be sequences of non-negative integers satisfying* (5) *such that*

$$
\lim\left(\frac{q(n) - p(n)}{q'(n) - p'(n)}\right) > 0, \tag{6}
$$

then $x \overset{\Delta^m(S_d^\alpha)}{\sim} y$ *implies* $x \overset{\Delta^m(S_{d'}^\alpha)}{\sim} y$.

Proof. It is easy to see that the inclusion

$$
\left\{k : p'(n) < k \le q'(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\} \subset \left\{k : p(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}
$$

holds and so the following inequality too

$$
\left|\left\{k : p'(n) < k \le q'(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right| \le \left|\left\{k : p(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|.
$$

Therefore we have

$$
\frac{1}{(q'(n) - p'(n))^\alpha} \left|\left\{k : p'(n) < k \le q'(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|
$$

$$
\le \left(\frac{q(n) - p(n)}{q'(n) - p'(n)}\right)^\alpha \frac{1}{(q(n) - p(n))^\alpha} \left|\left\{k : p(n) < k \le q(n) : \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \ge \varepsilon\right\}\right|.
$$

Taking limit as $n \to \infty$, we get $x \overset{\Delta^m(S_{d'}^\alpha)}{\sim} y$. $\square$

Theorem 3.15. *Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be sequences of non-negative integers satisfying* (5) *such that the sets $\{k : p(n) < k \leq p'(n)\}$ and $\{k : q'(n) < k \leq q(n)\}$ are finite. If x, y are Δ^m-bounded then $x \overset{\Delta^m\left(w_{d'}[r]\right)}{\sim} y$ implies $x \overset{\Delta^m(w_d[r])}{\sim} y$.*

Proof. Let x and y are Δ^m-bounded sequences, then there exists a positive real numbers M such that $\left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right| \leq M$. The we can write

$$
\frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right|^r = \frac{1}{q(n) - p(n)} \left[\begin{array}{c} \sum_{p(n)+1}^{p'(n)} \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right|^r \\ + \sum_{p'(n)+1}^{q'(n)} \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right|^r + \sum_{q'(n)+1}^{q(n)} \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right|^r \end{array} \right]
$$

$$
\leq \frac{2}{q'(n) - p'(n)} M^r O(1) + \frac{1}{q'(n) - p'(n)} \sum_{q'(n)+1}^{q(n)} \left|\frac{\Delta^m x_k}{\Delta^m y_k} - L\right|^r.
$$

So we have $x \overset{\Delta^m(w_d[r])}{\sim} y$. □

Theorem 3.16. *Let $\{p(n)\}, \{q(n)\}, \{p'(n)\}$ and $\{q'(n)\}$ be sequences of non-negative integers satisfying* (5) *and* (6), *then $x \overset{\Delta^m\left(w_d^\alpha[r]\right)}{\sim} y$ implies $x \overset{\Delta^m\left(w_{d'}^\alpha[r]\right)}{\sim} y$.*

Proof. Omitted. □

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