Higher Order Duality of Multiobjective Constrained Ratio Optimization Problems

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Abstract. A new concept in generalized convexity, called higher order $(C, \alpha, \gamma, \rho, d)$ type-I functions, is introduced. To show the existence of such type of functions, we identify a function lying exclusively in the class of higher order $(C, \alpha, \gamma, \rho, d)$ type-I functions and not in the class of (C, α, ρ, d) type-I functions already existing in the literature. Based upon the higher order $(C, \alpha, \gamma, \rho, d)$ type-I functions, the optimality conditions for a feasible solution to be an efficient solution are derived. A higher order Schaible dual has been then formulated for nondifferentiable multiobjective fractional programs. Weak, strong and strict converse duality theorems are established for higher order Schaible dual model and relevant proofs are given under the aforesaid function.

1. Introduction

It is known that the higher order dual contains a number of parameters which give it a computational advantage over the first order dual. This is due to the reason that higher order dual provides tighter bounds for the value of the objective function, whenever approximations are used. Second and higher order dual programs for nonlinear programs were introduced by Mangasarian [16] by taking the non-linear approximations of the objective function and the constraints. Second order dual models were studied by Ahmad [2] and Gupta [6], whereas in [1, 4, 5, 20] the higher order dual models were discussed.

Fractional programming problems have been of better utility in real life, as ratio optimization often describe an efficiency measure for a system. In last few years, Schaible dual has been extensively used for fractional programs. A dual program of such kind for a nonlinear fractional problem is first developed by Jagannathan [10] and the duality results were further improved by Schaible in [18] and [19]. It was done by converting a fractional program to a convex program, for which the solutions techniques already existed. Multiobjective fractional programming problems are being widely used in optimization theory due to their practicability. A large number of optimality and duality theorems have been established for these problems since their introduction in literature. The Schaible dual for nondifferentiable multiobjective fractional programming problems was discussed in [1, 20].

Due to the fact that not all properties of convex functions are required to set up sufficiency and duality theorems, there has been an increasing interest in generalization of the concept of convexity in view of

optimality and duality results. Keeping this in view several new classes of generalized convex functions have been introduced in [9, 11, 15, 17]. Integrating the above concepts of generalized invexity, (F, α, ρ, d) - convex function where F is a sublinear functional was defined by Liang et al. [12] and their optimality and duality results for a nonlinear fractional programming problem were proved. Later, in [13] three dual models for multiobjective fractional program were proposed. Further generalizing the concept of sublinear functional, the definition of (C, α, ρ, d) - convex function, where C being a convex function, was introduced in [22] and the optimality and duality results for a minimax fractional programming problem were obtained. Chinchuluun et al. [3] studied different dual models for multiobjective fractional programming problems. The Mond-Weir dual for a nondifferentiable multiobjective fractional program was established by Long [14]. Later Dubey et al. [4] defined new type of function called higher order $(C, \alpha, \gamma, \rho, d)$ - convex function and studied the optimality and duality results for a nondifferentiable multiobjective fractional programming problem.

In [8] a new class of generalized (F, α, ρ, d) type-I functions was introduced and sufficient optimality and duality results for nonlinear multiobjective programming problem were obtained. Second order duality results for two dual models of non-differentiable minimax programming problems, which involve second order (F, α, ρ, d) type-I convex functions were established in [2]. These were further generalized by Gupta et al. [6] to second order (C, α, ρ, d) type-I convex functions and second order dual models for nondifferentiable minimax fractional programming problems were formulated.

We are motivated by the earlier work given in [12], [8] and [4] to consider the optimality conditions and duality theorems for nondifferentiable multiobjective fractional programming problems from the viewpoint of $(C, \alpha, \gamma, \rho, d)$ type-I convexity assumptions.

This paper is organized as follows. Section 2 comprises some basic definitions along with the definition of higher order $(C, \alpha, \gamma, \rho, d)$ type-I function, an example of such type of functions and formulation of a nondifferentiable multiobjective fractional program. Necessary and sufficient optimality conditions for this program are given in Section 3. In Section 4, the formulation of higher order dual model along with weak, strong and strict converse duality theorems are given.

2. Preliminaries

Definition 2.1. *[22] A function $C : X \times X \times \mathbb{R}^n \to \mathbb{R}(X \subseteq \mathbb{R}^n)$ is said to be convex on $\mathbb{R}^n$ with respect to third argument if and only if, for any fixed $(x, u) \in X \times X$ and for any $\alpha_1, \alpha_2 \in \mathbb{R}^n$,*

$$C_{(x,u)}(\lambda \alpha_1 + (1 - \lambda)\alpha_2) \leq \lambda C_{(x,u)}(\alpha_1) + (1 - \lambda)C_{(x,u)}(\alpha_2), \qquad \lambda \in (0, 1).$$

Assume that $C_{(x,u)}(0) = 0$, for every $(x, u) \in X \times X$. Now, we introduce the definition of higher order $(C, \alpha, \gamma, \rho, d)$ type-I function. Let $\phi : X \to \mathbb{R}^k$ and $h : X \to \mathbb{R}^m$ be differentiable functions on X. Assume that $F_i, H_j : X \times \mathbb{R}^n \to \mathbb{R}$ are differentiable functions on X. $\alpha = (\alpha_1^1, ..., \alpha_k^1, \alpha_1^2, ..., \alpha_m^2)$, $d = (d_1^1, ..., d_k^1, d_1^2, ..., d_m^2)$, $\rho = (\rho_1^1, ..., \rho_k^1, \rho_1^2, ..., \rho_m^2)$, $\alpha_i^1, \gamma_i^1, \alpha_j^2, \gamma_j^2 : X \times X \to \mathbb{R}_+ \backslash \{0\}$ and $d_i^1, d_j^2 : X \times X \to R_+$ satisfying $d(x, x_0) = 0 \Leftrightarrow x = x_0$ and $p, q, r, s \in \mathbb{R}^n, i = 1, 2, ..., k, j = 1, 2, ..., m$.

Definition 2.2. *The function (ϕ, h) is said to be higher order (strictly) $(C, \alpha, \gamma, \rho, d)$ type-I at u with respect to F, p and r and H, q and s respectively, if for each $x \in X$,*

$$\frac{1}{\alpha^1(x,u)}[\phi(x) - \phi(u)] \geqq (>) C_{(x,u)}(\nabla \phi(u) + \nabla_p F(u, p)) + \frac{1}{\gamma^1(x,u)}[F(u, r) - r^T \nabla_r F(u, r)] + \frac{\rho^1 d^1(x, u)}{\alpha^1(x, u)},$$

$$\frac{1}{\alpha^2(x,u)}(-h(u)) \geqq (>) C_{(x,u)}(\nabla h(u) + \nabla_q H(u, q)) + \frac{1}{\gamma^2(x,u)}[H(u, s) - s^T \nabla_s H(u, s)] + \frac{\rho^2 d^2(x, u)}{\alpha^2(x, u)}.$$

Note that, for $a, b, c \in \mathbb{R}^k$.

1. *the symbol $\frac{ab}{c}$ denotes $(\frac{a_1 b_1}{c_1}, \frac{a_2 b_2}{c_2}, ..., \frac{a_k b_k}{c_k})^T$*

2. $\frac{a+b}{c}$ denotes $(\frac{a_1+b_1}{c_1}, \frac{a_2+b_2}{c_2}, ..., \frac{a_k+b_k}{c_k})^T$. And $ab = (a_1b_1, a_2b_2, ..., a_kb_k)^T$

3. the symbol $C_{(x,u)}(\nabla\phi(u) + \nabla_p F(u,p))$ means vector

$$\left(C_{(x,u)}(\nabla\phi_1(u) + \nabla_p F_1(u,p)), C_{(x,u)}(\nabla\phi_2(u) + \nabla_p F_2(u,p)), ..., C_{(x,u)}(\nabla\phi_k(u) + \nabla_p F_k(u,p))\right)^T.$$

Remark 2.3.

(i) If $F_i(u,\cdot) = 0$ and $H_j(u,\cdot) = 0$ then the definition (2.2) becomes that of (C, α, ρ, d) type-I as defined by Yuan et al. [21].

(ii) For $F_i(u,\cdot) = \frac{1}{2}(\cdot)^t\nabla^2 f_i(u)(\cdot)$, $\alpha(x,u) = \gamma(x,u)$, $H_j(u,\cdot) = \frac{1}{2}(\cdot)^t\nabla^2 h_j(u)(\cdot)$, $p = r$ and $q = s$.

   (a) If $k = 1, m = 1$ then definition (2.2) becomes second order (C, α, ρ, d) type-I function given in [6].

   (b) If C is sublinear with respect to third variable then the above definition becomes the definition of second order (F, α, ρ, p, d) type-I given by [7]. In addition to that, if $p = 0$ and $q = 0$ the function defined above becomes (F, α, ρ, d) type-I function given in [8].

Example 2.4. Let $k = 1, m = 1, X = \mathbb{R}^+ \subset \mathbb{R}$, $\phi : X \to \mathbb{R}$, $h : X \to \mathbb{R}$, $C : X \times X \times \mathbb{R} \to \mathbb{R}$, $F, H : X \times \mathbb{R} \to \mathbb{R}$, $d^1, d^2 : X \times X \to \mathbb{R}^+$ be defined as follows

$$\phi(x) = \frac{(x+1)^4 + x^2 - 3\sin^2(x)}{x+1}, \quad h(x) = exp(x) - x^2, \quad d^1, d^2 = (x-u)^2,$$

$$F(u,\cdot) = \frac{(\cdot)^2}{u+1}, \quad H(u,\cdot) = 2(\cdot)^2(u+1), \quad C_{(x,u)}(a) = \frac{a^2}{10}.$$

For $\alpha = (\alpha^1, \alpha^2), \rho = (\rho^1, \rho^2), \gamma = (\gamma^1, \gamma^2)$. Let $\alpha = (3/4, 1)$, $\gamma = (3/4, 1)$, $\rho = (1, -1)$, $p = r = q = s = -1$. At a point $u = 0$, for all $x \in X$,

$$\Psi = \frac{1}{\alpha^1(x,u)}[\phi(x) - \phi(u)] - C_{(x,u)}(\nabla\phi(u) + \nabla_p F(u,p)) - \frac{1}{\gamma^1(x,u)}[F(u,r) - r^T\nabla_r F(u,r)]$$

$$-\frac{\rho^1 d^1(x,u)}{\alpha^1(x,u)}$$

$$= \frac{4}{3}\left[\frac{(x+1)^4 + x^2 - 3\sin^2(x)}{x+1} - \frac{(0+1)^4 + 0^2 - 3\sin^2(0)}{0+1}\right] - C_{(x,0)}\left(3 + \frac{2p}{0+1}\right)$$

$$-\frac{4}{3}\left[\frac{r^2}{0+1} - r\cdot\frac{2r}{0+1}\right] - \frac{4x^2}{3}$$

$$= \frac{4}{3}\left[\frac{(x+1)^4 + x^2 - 3\sin^2(x) - x - 1}{(x+1)}\right] - C_{(x,0)}(3 - 2) - \left(\frac{4}{3}\right)\left(\frac{-r^2}{1}\right) - \frac{4x^2}{3}$$

$$= \frac{4}{3}\left[\frac{(x+1)^4 + x^2 - 3\sin^2(x) - x - 1}{(x+1)}\right] - \frac{1}{10} + \frac{4}{3} - \frac{4x^2}{3}$$

$$= \frac{40x^4 + 120x^3 + 240x^2 + 157x - 120\sin^2(x) + 37}{30(x+1)} \geq 0, \quad \forall x \in X,$$

as can be seen from figure 1(insert figure, named Fig1).

$$\frac{1}{\alpha^2(x,u)}(-h(u)) - C_{(x,u)}(\nabla h(u) + \nabla_q H(u,q)) - \frac{1}{\gamma^2(x,u)}[H(u,s) - s^T\nabla_s H(u,s)] - \frac{\rho^2 d^2(x,u)}{\alpha^2(x,u)}$$

$$= 0.1 + x^2.$$

It is clear that the above function is always positive. So we have (ϕ, h) is higher order $(C, \alpha, \gamma, \rho, d)$ type-I at $u = 0$. But for $x = 0.1$ the above function is not (C, α, ρ, d) type-I at $u = 0$.
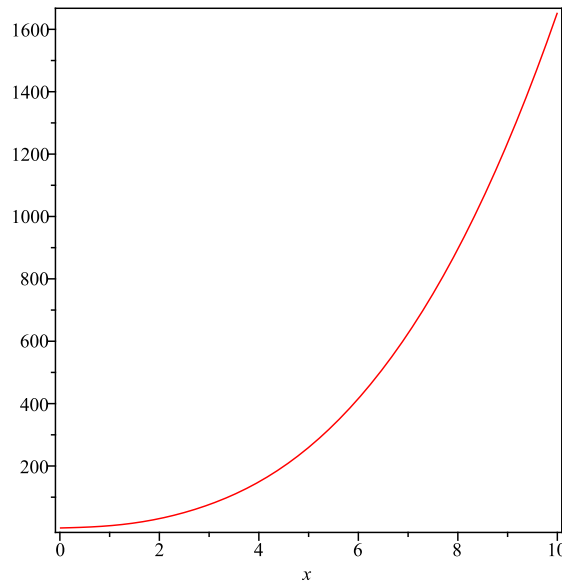
Figure 1: Graph of Ψ

Definition 2.5. *Let C be a compact convex set in $\mathbb{R}^n$. The support function $S(x|C)$ of C at x is defined by*

$$S(x|C) = max\,\{x^T y | y \in C\}.$$

Such functions are convex and everywhere finite. So its subdifferential is given as

$$\partial S(x|C) = \{z \in C : x^T z = S(x|C)\}.$$

Let us consider a nondifferentiable multiobjective fractional programming problem

$$(MFP) \qquad Min \quad F(x) = \left\{ \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \frac{f_2(x) + S(x|C_2)}{g_2(x) - S(x|D_2)}, \cdots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right\}$$

subject to

$$h_j(x) + S(x|E_j) \le 0, \ \ j = 1, 2, ..., m.$$

Feasible set is defined as $X^0 = \{x \in X \subseteq \mathbb{R}^n \mid h_j(x) + S(x|E_j) \le 0, \ j = 1, 2, ..., m\}$.
Where C_i, D_i and E_j for $i = 1, 2, ..., k, j = 1, 2, ..., m$ are compact convex sets in $\mathbb{R}^n$. $S(x|C_i)$, $S(x|D_i)$, $S(x|E_j)$ denote support functions of convex sets C_i, D_i and E_j respectively, for $i = 1, 2, ..., k$, $j = 1, 2, ..., m$. And f_i, g_i, h_j are continuously differentiable functions in $\mathbb{R}^n$. $f_i(\cdot) + S(\cdot|C_i) \ge 0$ and $g_i(\cdot) - S(\cdot|D_i) > 0$.

Definition 2.6. *[4] A point $u \in X^0$ is weakly efficient solution of (MFP), if there exists no $x \in X^0$ such that for every $i = 1, 2, ...k$,*

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} < \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}.$$

Definition 2.7. *[4] A point $u \in X^0$ is said to be an efficient solution of (MFP), if there exists no $x \in X^0$ such that for every $i = 1, 2, ..., k$,*

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \le \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}$$

and for some $r = 1, 2, ..., k,$

$$\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(u) + S(u|C_r)}{g_r(u) - S(u|D_r)}.$$

Lemma 2.8. *[20] If u is efficient solution of (MFP), then u solves (FP_β) for each $r = 1, 2, ..., k$ where (FP_β) is given as*

$$(FP_\beta) \qquad Min \quad \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)}$$

$$subject\ to \qquad \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \beta_i, \quad i = 1, 2, ..., k,\ i \neq r,$$
$$h_j(x) + S(x|E_j) \leq 0,\ j = 1, 2, ..., m,$$

where $\beta_i = \dfrac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}.$
Now since $g_i(\cdot) - S(\cdot|D_i) > 0,$ *for all* $i = 1, 2, ..., k$ *therefore* (FP_β) *can be rewritten as*

$$(FP_\beta^1) \qquad Min \quad \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)}$$

$$subject\ to \qquad f_i(x) + S(x|C_i) - \beta_i(g_i(x) - S(x|D_i)) \leq 0, \quad i = 1, 2, ..., k,\ i \neq r,$$
$$h_j(x) + S(x|E_j) \leq 0,\ j = 1, 2, ..., m.$$

Lemma 2.9. *[20] u is efficient solution of (MFP) if and only if u solves (FP_β^1) for each $r = 1, 2, ..., k$, where β_i is defined as above.*

3. Optimality Conditions

Theorem 3.1. *(Necessary Optimality Condition) [4] Assume that u is an efficient solution of (MFP) and Slater's constraint qualification is satisfied on X. Then there exist $\bar{\lambda} \in \mathbb{R}^k$, $\mu \in \mathbb{R}^m$, $z_i \in \mathbb{R}^n$, $v_i \in \mathbb{R}^n$, $w_j \in \mathbb{R}^n$, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$, such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla\left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\right) + \sum_{j=1}^{m} \mu_j \nabla(h_j(u) + u^T w_j) = 0,$$

$$\sum_{j=1}^{m} \mu_j(h_j(u) + u^T w_j) = 0,$$

$$u^T z_i = S(u|C_i),\ u^T v_i = S(u|D_i),\ u^T w_j = S(u|E_j),$$
$$z_i \in C_i,\ v_i \in D_i,\ w_j \in E_j,\ \bar{\lambda}_i > 0,\ \mu_j \geq 0.$$

Theorem 3.2. *(Equivalent Necessary Optimality Condition) Assume that u is an efficient solution of (MFP) and Slater's constraint qualification is satisfied on X. Then there exist $\lambda \in \mathbb{R}^k$, $\mu \in \mathbb{R}^m$, $z_i \in \mathbb{R}^n$, $v_i \in \mathbb{R}^n$, $w_j \in \mathbb{R}^n$, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$, such that*

$$\sum_{i=1}^{k} \lambda_i(\nabla(f_i(u) + u^T z_i) - \beta_i \nabla(g_i(u) - u^T v_i))$$

$$+ \sum_{j=1}^{m} \mu_j \nabla(h_j(u) + u^T w_j) = 0, \tag{1}$$

$$f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i) = 0, \tag{2}$$

$$\sum_{j=1}^{m} \mu_j(h_j(u) + u^T w_j) = 0, \tag{3}$$

$$u^T z_i = S(u|C_i), \ u^T v_i = S(u|D_i), u^T w_j = S(u|E_j), \tag{4}$$

$$z_i \in C_i, \ v_i \in D_i, \ w_j \in E_j, \ \lambda_i > 0, \ \mu_j \geq 0. \tag{5}$$

Theorem 3.3. (*Sufficient Optimality Condition*) *Let u be a feasible solution of (MFP). Assume that there exist $\lambda_i > 0$ and $\mu_j \geq 0$, for $i = 1, 2, ..., k, \ j = 1, 2, ..., m,$ such that equivalent necessary optimality conditions (1)-(5) hold. Let for any $i = 1, 2, ..., k, \ j = 1, 2, ..., m$, the following hold*

(i) $(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$ *is higher order $(C, \alpha, \gamma, \rho, d)$ type-I with respect to K, p, r and H, q, s respectively,*

(ii) $(-(g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$ *is higher order $(C, \alpha, \gamma, \rho, d)$ type-I with respect to $-G, p, r$ and H, q, s respectively,*

(iii) $\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}(1 + \beta_i) + \sum_{j=1}^{m} \frac{\mu_j \rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \geq 0,$

(iv) $\gamma_i^1(x, u) = \xi(x, u), \ \alpha_j^2(x, u) = \zeta(x, u)$ *and* $\gamma_j^2(x, u) = \sigma(x, u),$

(v) $\sum_{j=1}^{m} \mu_j(H_j(u, s) - s^T \nabla_s H_j(u, s)) \geq 0, \ \sum_{i=1}^{k} \lambda_i(K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r))) \geq 0$ *and* $\sum_{i=1}^{k} \lambda_i(\nabla_p K_i(u, p) - \beta_i \nabla_p G_i(u, p)) + \sum_{j=1}^{m} \mu_j \nabla_q H_j(u, q) = 0.$

Then u is an efficient solution of (MFP).

Proof If u is not an efficient solution, then there exists some $x \in X^0$ such that the following holds.

$$\begin{aligned} \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} &\leq \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}, & \text{for all } i = 1, 2, ..., k, \\ \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} &< \frac{f_r(u) + S(u|C_r)}{g_r(u) - S(u|D_r)} & \text{for some } r = 1, 2, ..., k. \end{aligned} \tag{6}$$

For some $z_i \in C_i$ and $v_i \in D_i$, it is given that $u^T z_i = S(u|C_i), \ u^T v_i = S(u|D_i), \ i = 1, 2, ..., k,$ gives

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)} = \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} = \beta_i, \qquad \text{for all } i = 1, 2, ..., k,$$

$$\frac{f_r(x) + x^T z_r}{g_r(x) - x^T v_r} \leq \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(u) + S(u|C_r)}{g_r(u) - S(u|D_r)} = \frac{f_r(u) + u^T z_r}{g_r(u) - u^T v_r} = \beta_r, \qquad \text{for some } r = 1, 2, ..., k,$$

so that

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \beta_i, \qquad \text{for all } i = 1, 2, ..., k$$

and

$$\frac{f_r(x) + x^T z_r}{g_r(x) - x^T v_r} < \beta_r, \qquad \text{for some } r = 1, 2, ..., k.$$

As $g_i(x) - S(x|D_i) > 0$, so we get

$$f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) \leq 0, \qquad \text{for all } i = 1, 2, ..., k$$

and

$$f_r(x) + x^T z_r - \beta_r(g_r(x) - x^T v_r) < 0, \qquad \text{for some } r = 1, 2, ..., k.$$

For $\lambda_i > 0$ and $\alpha_i^1(x, u) \in \mathbb{R}_+ \setminus \{0\}$,

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) < 0 \tag{7}$$

Using hypotheses (i) and (ii) we obtain

$$\frac{f_i(x) + x^T z_i - (f_i(u) + u^T z_i)}{\alpha_i^1(x, u)} \geq C_{(x,u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) +$$

$$\frac{1}{\gamma_i^1(x, u)}(K_i(u, r) - r^T \nabla_r K_i(u, r)) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} \tag{8}$$

$$\frac{-(g_i(x) - x^T v_i - (g_i(u) - u^T v_i))}{\alpha_i^1(x, u)} \geq C_{(x,u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)) +$$

$$\frac{1}{\gamma_i^1(x, u)}(-(G_i(u, r) - r^T \nabla_r G_i(u, r))) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} \tag{9}$$

$$\frac{-(h_j(u) + u^T w_j)}{\alpha_j^2(x, u)} \geq C_{(x,u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, \underline{q})) +$$

$$\frac{1}{\gamma_j^2(x, u)}(H_j(u, s) - s^T \nabla_s H_j(u, s)) + \frac{\rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)} \tag{10}$$

Adding the two inequalities (8) and (9) after multiplying (9) by $\beta_i, (\beta_i \geq 0)$ $i = 1, 2, ..., k$, we get

$$\frac{1}{\alpha_i^1(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)))$$

$$\geq C_{(x,u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \frac{1}{\gamma_i^1(x, u)}(K_i(u, r) - r^T \nabla_r K_i(u, r))$$

$$+ \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)} + \beta_i C_{(x,u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p))$$

$$- \beta_i \frac{1}{\gamma_i^1(x, u)}(G_i(u, r) - r^T \nabla_r G_i(u, r)) + \beta_i \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}. \tag{11}$$

Multiply inequality (10) by $\mu_j(\mu_j \geq 0)$, and (11) by $\lambda_i(\lambda_i > 0)$, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$, then adding over their ranges, we have

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)))$$

$$+ \sum_{j=1}^{m} \frac{\mu_j}{\alpha_j^2(x, u)}(-(h_j(u) + u^T w_j))$$

$$\geq \sum_{i=1}^{k} \lambda_i [C_{(x,u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \beta_i C_{(x,u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p))]$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\gamma_i^1(x, u)}(K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r)))$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d_i^1(x,u)}{\alpha_i^1(x,u)}(1+\beta_i) + \sum_{j=1}^{m} \mu_j C_{(x,u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u,q))$$

$$+ \sum_{j=1}^{m} \frac{\mu_j}{\gamma_j^2(x,u)}(H_j(u,s) - s^T \nabla_s H_j(u,s)) + \sum_{j=1}^{m} \frac{\mu_j \rho_j^2 d_j^2(x,u)}{\alpha_j^2(x,u)}. \tag{12}$$

Taking $\tau = \sum_{i=1}^{k} \lambda_i(1+\beta_i) + \sum_{j=1}^{m} \mu_j$, so $\tau > 0$, divide the equation by τ and using convexity of C we obtain

$$\sum_{i=1}^{k} \frac{\lambda_i}{\tau \alpha_i^1(x,u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)))$$

$$+ \sum_{j=1}^{m} \frac{\mu_j}{\tau \alpha_j^2(x,u)}(-(h_j(u) + u^T w_j))$$

$$\geq C_{(x,u)}\left[\frac{1}{\tau}\sum_{i=1}^{k}\lambda_i\left(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u,p) - \beta_i\nabla(g_i(u) - u^T v_i) - \beta_i\nabla_p G_i(u,p)\right)\right.$$

$$\left. + \frac{1}{\tau}\sum_{j=1}^{m}\mu_j\left(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u,q)\right)\right] + \sum_{j=1}^{m} \frac{\mu_j}{\tau \gamma_j^2(x,u)}(H_j(u,s) - s^T \nabla_s H_j(u,s))$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\tau \gamma_i^1(x,u)}(K_i(u,r) - r^T \nabla_r K_i(u,r) - \beta_i(G_i(u,r) - r^T \nabla_r G_i(u,r)))$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d_i^1(x,u)}{\tau \alpha_i^1(x,u)}(1+\beta_i) + \sum_{j=1}^{m} \frac{\mu_j \rho_j^2 d_j^2(x,u)}{\tau \alpha_j^2(x,u)}. \tag{13}$$

It follows from hypotheses (iii) and (iv) that

$$\sum_{i=1}^{k} \frac{\lambda_i}{\tau \alpha_i^1(x,u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i))$$

$$\geq C_{(x,u)}\left[\frac{1}{\tau}\sum_{i=1}^{k}\lambda_i\left(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u,p) - \beta_i\nabla(g_i(u) - u^T v_i) - \beta_i\nabla_p G_i(u,p)\right)\right.$$

$$\left. + \frac{1}{\tau}\sum_{j=1}^{m}\mu_j\left(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u,q)\right)\right] + \sum_{j=1}^{m} \frac{\mu_j}{\tau \sigma(x,u)}(H_j(u,s) - s^T \nabla_s H_j(u,s))$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\tau \alpha_i^1(x,u)}(f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)) + \sum_{j=1}^{m} \frac{\mu_j}{\tau \zeta(x,u)}(h_j(u) + u^T w_j)$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\tau \xi(x,u)}(K_i(u,r) - r^T \nabla_r K_i(u,r) - \beta_i(G_i(u,r) - r^T \nabla_r G_i(u,r))). \tag{14}$$

Further, using hypothesis (v) we have

$$\sum_{i=1}^{k} \frac{\lambda_i}{\tau \alpha_i^1(x,u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i))$$

$$\geq C_{(x,u)}\left[\frac{1}{\tau}\left(\sum_{i=1}^{k}\lambda_i(\nabla(f_i(u) + u^T z_i) - \beta_i\nabla(g_i(u) - u^T v_i)) + \sum_{j=1}^{m}\mu_j\nabla(h_j(u) + u^T w_j)\right)\right]$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\tau \alpha_i^1(x,u)}(f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)) + \frac{1}{\tau \zeta(x,u)} \sum_{j=1}^{m} \mu_j(h_j(u) + u^T w_j). \tag{15}$$

Finally, using $f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i) = 0$, $\sum_{j=1}^{m} \mu_j(h_j(u) + u^T w_j) = 0$ and $C_{(x,u)}(0) = 0$, it follows that

$$\sum_{i=1}^{k} \frac{\lambda_i}{\tau \alpha_i^1(x,u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \geq 0, \tag{16}$$

which contradicts (7). Hence u is an efficient solution of (MFP). $\square$

4. Duality Model

In this section, we consider the higher order Schaible dual to (MFP):

(MFD)   Max $\beta = (\beta_1, \beta_2, ..., \beta_k)$
subject to

$$\sum_{i=1}^{k} \lambda_i[\nabla(f_i(u) + u^T z_i) - \beta_i \nabla(g_i(u) - u^T v_i)] + \sum_{j=1}^{m} \mu_j \nabla(h_j(u) + u^T w_j) + \sum_{i=1}^{k} \lambda_i(\nabla_p K_i(u,p)$$

$$-\beta_i \nabla_p G_i(u,p)) + \sum_{j=1}^{m} \mu_j \nabla_q H_j(u,q) = 0, \tag{17}$$

$$\sum_{i=1}^{k} \lambda_i(f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i) + (K_i(u,r) - r^T \nabla_r K_i(u,r)) - \beta_i(G_i(u,r)$$

$$-r^T \nabla_r G_i(u,r))) \geq 0, \tag{18}$$

$$\sum_{j=1}^{m} \mu_j(h_j(u) + u^T w_j + H_j(u,s) - s^T \nabla_s H_j(u,s)) \geq 0, \tag{19}$$

$\lambda_i > 0, \quad \mu_j \geq 0, \quad$ and $\quad \beta_i \geq 0, \quad$ for $\quad i = 1,2...,k \quad$ and $\quad j = 1,2,...,m,$
$z_i \in C_i, \quad v_i \in D_i, \quad w_j \in E_j. \tag{20}$

Theorem 4.1. *(Weak Duality Theorem) Let $x \in X^0$ and $(u, \beta, \lambda, \mu, z, v, w, p, q, r, s)$ be feasible solutions of (MFP) and (MFD) respectively. Suppose that*

(i) *$(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$ is higher order $(C, \alpha, \gamma, \rho, d)$ type-I with respect to K, p, r and H, q, s respectively,*

(ii) *$(-(g(\cdot)-(\cdot)^T v), h(\cdot)+(\cdot)^T w)$ is higher order $(C, \alpha, \gamma, \rho, d)$ type-I with respect to $-G, p, r$ and H, q, s respectively,*

(iii) *$\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d_i^1(x,u)}{\alpha_i^1(x,u)}(1 + \beta_i) + \sum_{j=1}^{m} \frac{\mu_j \rho_j^2 d_j^2(x,u)}{\alpha_j^2(x,u)} \geq 0,$*

(iv) *$\alpha_i^1(x,u) = \gamma_i^1(x,u) = a(x,u)$ and $\alpha_j^2(x,u) = \gamma_j^2(x,u) = b(x,u).$*

Then the following can not hold:

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \beta_i, \quad \text{for all, } i = 1,2,...,k,$$

$$\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \beta_r, \quad \text{for some } r = 1,2,...,k. \tag{21}$$

Proof Let if possible (21) hold. Then for $\lambda_i > 0$ and $a(x, u) \in \mathbb{R}_+ \setminus \{0\}$, for $z_i \in C_i$ and $v_i \in D_i$ for $i = 1, 2, ..., k$ and using the definition of support function $x^T z_i \leq S(x|C_i), x^T v_i \leq S(x|D_i)$, we get the following

$$\sum_{i=1}^{k} \frac{\lambda_i}{a(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) < 0. \tag{22}$$

Using higher order $(C, \alpha, \gamma, \rho, d)$ type-I convexity of $(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$ with respect to K, p, r and H, q, s respectively and that of $(-(g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$ with respect to $-G, p, r$ and H, q, s respectively, we have the following

$$\frac{f_i(x) + x^T z_i - (f_i(u) + u^T z_i)}{\alpha_i^1(x, u)} \geq C_{(x,u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) +$$

$$\frac{1}{\gamma_i^1(x, u)}(K_i(u, r) - r^T \nabla_r K_i(u, r)) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}, \tag{23}$$

$$\frac{-(g_i(x) - x^T v_i - (g_i(u) - u^T v_i))}{\alpha_i^1(x, u)} \geq C_{(x,u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)) +$$

$$\frac{1}{\gamma_i^1(x, u)}(-(G_i(u, r) - r^T \nabla_r G_i(u, r))) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}, \tag{24}$$

$$\frac{-(h_j(u) + u^T w_j)}{\alpha_j^2(x, u)} \geq C_{(x,u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)) +$$

$$\frac{1}{\gamma_j^2(x, u)}(H_j(u, s) - s^T \nabla_s H_j(u, s)) + \frac{\rho_j^2 d_j^2(x, u)}{\alpha_j^2(x, u)}. \tag{25}$$

Multiply (24) by $\beta_i(\beta_i \geq 0)$, $i = 1, 2, ..., k$, then adding (23) and (24), we get

$$\frac{1}{\alpha_i^1(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)))$$

$$\geq C_{(x,u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \frac{1}{\gamma_i^1(x, u)}(K_i(u, r) - r^T \nabla_r K_i(u, r)) + \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}$$

$$+ \beta_i C_{(x,u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)) - \beta_i \frac{1}{\gamma_i^1(x, u)}(G_i(u, r) - r^T \nabla_r G_i(u, r))$$

$$+ \beta_i \frac{\rho_i^1 d_i^1(x, u)}{\alpha_i^1(x, u)}. \tag{26}$$

Adding the inequalities (25) and (26) over their ranges after multiplying (25) by $\mu_j(\mu_j \geq 0)$ and (26) by $\lambda_i(\lambda_i > 0)$, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$, we have

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x, u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)))$$

$$+ \sum_{j=1}^{m} \frac{\mu_j}{\alpha_j^2(x, u)}(-(h_j(u) + u^T w_j))$$

$$\geq \sum_{i=1}^{k} \lambda_i [C_{(x,u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)) + \beta_i C_{(x,u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p))]$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\gamma_i^1(x, u)}(K_i(u, r) - r^T \nabla_r K_i(u, r) - \beta_i(G_i(u, r) - r^T \nabla_r G_i(u, r)))$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d_i^1(x,u)}{\alpha_i^1(x,u)}(1+\beta_i) + \sum_{j=1}^{m} \mu_j C_{(x,u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u,q))$$

$$+ \sum_{j=1}^{m} \frac{\mu_j}{\gamma_j^2(x,u)}(H_j(u,s) - s^T \nabla_s H_j(u,s)) + \sum_{j=1}^{m} \frac{\mu_j \rho_j^2 d_j^2(x,u)}{\alpha_j^2(x,u)}. \tag{27}$$

Using (iii) and taking $\tau = \sum_{i=1}^{k} \lambda_i(1+\beta_i) + \sum_{j=1}^{m} \mu_j$, so $\tau > 0$, divide the equation by τ and using convexity of C

$$\sum_{i=1}^{k} \frac{\lambda_i}{\tau \alpha_i^1(x,u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i) - (f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)))$$

$$+ \sum_{j=1}^{m} \frac{\mu_j}{\tau \alpha_j^2(x,u)}(-(h_j(u) + u^T w_j))$$

$$\geq C_{(x,u)}\left[\frac{1}{\tau} \sum_{i=1}^{k} \lambda_i\left(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u,p) - \beta_i \nabla(g_i(u) - u^T v_i) - \beta_i \nabla_p G_i(u,p) \right) \right.$$

$$\left. + \frac{1}{\tau} \sum_{j=1}^{m} \mu_j\left(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u,q) \right) \right] + \sum_{j=1}^{m} \frac{\mu_j}{\tau \gamma_j^2(x,u)}(H_j(u,s) - s^T \nabla_s H_j(u,s))$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\tau \gamma_i^1(x,u)}(K_i(u,r) - r^T \nabla_r K_i(u,r) - \beta_i(G_i(u,r) - r^T \nabla_r G_i(u,r))). \tag{28}$$

Using (17), hypothesis (iv) and the fact that $C_{(x,u)}(0) = 0$, the last inequality yields

$$\sum_{i=1}^{k} \frac{\lambda_i}{\tau a(x,u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \geq$$

$$\frac{1}{\tau b(x,u)} \sum_{j=1}^{m} \mu_j(H_j(u,s) - s^T \nabla_s H_j(u,s)) + \frac{1}{\tau b(x,u)} \sum_{j=1}^{m} \mu_j((h_j(u) + u^T w_j))$$

$$+ \frac{1}{\tau a(x,u)} \sum_{i=1}^{k} \lambda_i(f_i(u) + u^T z_i - \beta_i(g_i(u) - u^T v_i)) + \frac{1}{\tau a(x,u)} \sum_{i=1}^{k} \lambda_i(K_i(u,r)$$

$$- r^T \nabla_r K_i(u,r) - \beta_i(G_i(u,r) - r^T \nabla_r G_i(u,r))). \tag{29}$$

Finally using the dual feasibility conditions (18) and (19), the equation (29) becomes

$$\sum_{i=1}^{k} \frac{\lambda_i}{a(x,u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \geq 0,$$

which contradicts (22). Hence the result. $\square$

Theorem 4.2. *Let $x \in X^0$ and $(u, \beta, \lambda, \mu, z, v, w, p, q, r, s)$ be feasible solutions of (MFP) and (MFD) respectively. Suppose that*

(i) *$(f(\cdot) + (\cdot)^T z - \beta(g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$ is higher order $(C, \alpha, \gamma, \rho, d)$ type-I with respect to $(K - \beta G), p, r$ and H, q, s respectively*

(ii) *$\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d_i^1(x,u)}{\alpha_i^1(x,u)} + \sum_{j=1}^{m} \frac{\mu_j \rho_j^2 d_j^2(x,u)}{\alpha_j^2(x,u)} \geq 0$*

(iii) *$\alpha_i^1(x,u) = \gamma_i^1(x,u) = a(x,u)$ and $\alpha_j^2(x,u) = \gamma_j^2(x,u) = b(x,u)$*

Then the following can not hold

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \le \beta_i, \qquad \text{for all } i = 1, 2, ..., k,$$

$$\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \beta_r, \qquad \text{for some } r = 1, 2, ..., k. \tag{30}$$

Proof It follows on the lines of Theorem 4.1.   □

Theorem 4.3. *(**Strong Duality Theorem**) If u is an efficient solution of (MFP) and let Slater's constraint qualification is satisfied on X. Also if*

$$H_j(u, 0) = 0, \qquad \nabla_s H_j(u, 0) = 0, \qquad \nabla_q H_j(u, 0) = 0,$$
$$K_i(u, 0) = 0, \qquad \nabla_p K_i(u, 0) = 0, \qquad \nabla_r K_i(u, 0) = 0,$$
$$G_i(u, 0) = 0, \qquad \nabla_p G_i(u, 0) = 0, \qquad \nabla_r G_i(u, 0) = 0.$$

Then $\exists\ \lambda > 0,\ \lambda \in \mathbb{R}^k,\ \mu \in \mathbb{R}^m,\ \mu \ge 0,\ \beta \ge 0,\ \beta \in \mathbb{R}^k,\ z \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\ w \in \mathbb{R}^n,\ such\ that\ (u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)\ is\ a\ feasible\ solution\ of\ (MFD)\ and\ corresponding\ values\ of\ the\ objective\ functions\ are\ equal. Further\ if,\ the\ conditions\ of\ weak\ duality\ theorem\ hold\ for\ all\ feasible\ solutions\ of\ (MFP)\ and\ each\ feasible\ solution $(u', \beta', \lambda', \mu'z', v', w', p', q', r', s')$ of (MFD) then $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$ is an efficient solution of (MFD).

Proof If u is an efficient solution of (MFP) and Slater's constraint qualification is satisfied on X then the Theorem [3.2] gives that $\exists\ \lambda \in \mathbb{R}^k,\ \mu \in \mathbb{R}^m,\ z_i \in \mathbb{R}^n,\ v_i \in \mathbb{R}^n,\ w_j \in \mathbb{R}^n$, such that

$$\sum_{i=1}^{k} \lambda_i (\nabla(f_i(u) + u^T z_i) - \beta_i \nabla(g_i(u) - u^T v_i))$$

$$+ \sum_{j=1}^{m} \mu_j \nabla(h_j(u) + u^T w_j) = 0, \tag{31}$$

$$f_i(u) + u^T z_i - \beta_i (g_i(u) - u^T v_i) = 0, \tag{32}$$

$$\sum_{j=1}^{m} \mu_j (h_j(u) + u^T w_j) = 0, \tag{33}$$

$$u^T z_i = S(u|C_i),\ u^T v_i = S(u|D_i),\ u^T w_j = S(u|E_j), \tag{34}$$

$$z_i \in C_i, \quad v_i \in D_i, \quad w_j \in E_j,$$

$$\lambda_i > 0,\ \mu_j \ge 0, \qquad i = 1, 2, ..., k, \qquad j = 1, 2, ..., m. \tag{35}$$

So we have $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$ as a feasible solution of (MFD). Due to (32) and (34) the objective values of both programs are equal.

Now we claim that $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$ is efficient solution of (MFD).

If not, then $\exists$ some $(u', \beta', \lambda', \mu', z', v', w', p', q', r', s')$ such that,
$$\beta \le \beta'.$$

But this contradicts weak duality theorem. Hence $(u, \beta, \lambda, \mu, z, v, w, p = 0, q = 0, r = 0, s = 0)$ is an efficient solution of (MFD).   □

Theorem 4.4. *(**Strict Converse Duality Theorem**) Let x be an efficient solution of (MFP) and $(u, \beta, \lambda, \mu, z, v, w, p, q, r, s)$ be an efficient solution of (MFD). Suppose that:*

(i) $(f(\cdot) + (\cdot)^T z, h(\cdot) + (\cdot)^T w)$ *is higher order strictly $(C, \alpha, \gamma, \rho, d)$ type-I with respect to K, p, r and H, q, s respectively,*

(ii) $(-(g(\cdot) - (\cdot)^T v), h(\cdot) + (\cdot)^T w)$ *is higher order strictly $(C, \alpha, \gamma, \rho, d)$ type-I with respect to $-G, p, r$ and H, q, s respectively,*

(iii) $\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d_i^1(x,u)}{\alpha_i^1(x,u)}(1+\beta_i) + \sum_{j=1}^{m} \frac{\mu_j \rho_j^2 d_j^2(x,u)}{\alpha_j^2(x,u)} \geq 0,$

(iv) $\alpha_i^1(x,u) = \gamma_i^1(x,u) = a(x,u)$ and $\alpha_j^2(x,u) = \gamma_j^2(x,u) = b(x,u),$

(v) $\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \beta_i.$

Then $x = u.$

Proof Suppose $x \neq u.$ Then due to hypotheses (i) and (ii) we have

$$\frac{f_i(x) + x^T z_i - (f_i(u) + u^T z_i)}{\alpha_i^1(x,u)} > C_{(x,u)}(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u,p)) +$$

$$\frac{1}{\gamma_i^1(x,u)}(K_i(u,r) - r^T \nabla_r K_i(u,r)) + \frac{\rho_i^1 d_i^1(x,u)}{\alpha_i^1(x,u)} \quad (36)$$

$$\frac{-(g_i(x) - x^T v_i - (g_i(u) - u^T v_i))}{\alpha_i^1(x,u)} > C_{(x,u)}(-\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u,p)) +$$

$$\frac{1}{\gamma_i^1(x,u)}(-(G_i(u,r) - r^T \nabla_r G_i(u,r))) + \frac{\rho_i^1 d_i^1(x,u)}{\alpha_i^1(x,u)} \quad (37)$$

$$\frac{-(h_j(u) + u^T w_j)}{\alpha_j^2(x,u)} > C_{(x,u)}(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u,q)) +$$

$$\frac{1}{\gamma_j^2(x,u)}(H_j(u,s) - s^T \nabla_s H_j(u,s)) + \frac{\rho_j^2 d_j^2(x,u)}{\alpha_j^2(x,u)} \quad (38)$$

Following the similar manner as in Theorem 4.1 we are left with

$$\sum_{i=1}^{k} \frac{\lambda_i}{a(x,u)}(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) > 0. \quad (39)$$

It follows from hypothesis (v) that

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \beta_i,$$

which further imply that

$$\sum_{i=1}^{k} \lambda_i(f_i(x) + x^T z_i - \beta_i(g_i(x) - x^T v_i)) \leq 0 \quad \text{for } \lambda_i > 0, \quad (40)$$

contradicting (39). Hence $x = u.$ $\square$

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