Superharmonicity of Logarithm of Jacobian of Harmonic Mappings Between Surfaces

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Abstract. We prove that the logarithm of the Jacobian of a sense preserving harmonic mapping between surfaces is superharmonic, provided that the Gaussian curvature of the image domain is non-negative.

0.1. Harmonic mappings between Riemannian surfaces

Let $(\mathcal{N}, \sigma)$ and $(\mathcal{M}, \rho)$ be Riemannian surfaces with smooth conformal metrics σ and ρ, respectively. Here the conformal metric σ induces the metric $d_\sigma = \sigma^2 dz d\bar{z}$. If a mapping $f : (\mathcal{N}, \sigma) \to (\mathcal{M}, \rho)$ is C^2, then f is said to be harmonic (or ρ-harmonic) if

$$f_{z\bar{z}} + (\log \rho^2)_w \circ f \cdot f_z f_{\bar{z}} = 0, \tag{1}$$

where z and w are the local parameters on $\mathcal{M}$ and $\mathcal{N}$ respectively (see [5]). Also f satisfies (1) if and only if f is C^1 and its Hopf differential

$$\Psi = \rho^2 \circ f \cdot f_z \overline{f_{\bar{z}}} \tag{2}$$

is a holomorphic quadratic differential on N. It follows from (2), that (i) every holomorphic mapping is harmonic and (ii) the composition of a harmonic mapping and a holomorphic mapping is harmonic.

From now on, we will assume that $\mathcal{M}$ is parameterized by a complex domain D. Let D be a domain in $\mathbf{C}$ and ρ be a conformal metric in D. The Gaussian curvature of a double diferentiable metric ρ is given by

$$K_\rho = -\frac{2\Delta \log \rho}{\rho^2}.$$

Let h be a solution of (1).
Let

$$\partial h = \rho(h(z))h_z, \quad \bar{\partial} h = \rho(h(z))h_{\bar{z}},$$

and

$$\partial_\sigma h = \frac{\rho(h(z))}{\sigma(z)}h_z, \quad \bar{\partial}_\sigma h = \frac{\rho(h(z))}{\sigma(z)}h_{\bar{z}},$$

and define

$$J_h = |\partial h|^2 - |\bar{\partial}h|^2,$$
$$D_h = |\partial h|^2 + |\bar{\partial}h|^2$$

and

$$\Delta f = f_{xx} + f_{yy}.$$

Further define

$$J_h^\sigma = |\partial_\sigma h|^2 - |\bar{\partial}_\sigma h|^2$$

and

$$\Delta^\sigma f = \frac{1}{\sigma^2}\Delta f.$$

Then we have

Proposition 0.1 (Bochner's formulas). *(See e.g. [5, p. 113]). If h is a harmonic mapping then*

$$\Delta^\sigma \log |\partial_\sigma h|^2 = K_1 - K_2 J_h^\sigma$$

and

$$\Delta^\sigma \log |\bar{\partial}_\sigma h|^2 = K_1 + K_2 J_h^\sigma.$$

Here K_1 is the Gaussian curvature of σ and K_2 is the Gaussian curvature of ρ.

For our approach it will be useful to use the following direct corollary of Bochner's formulas.

Corollary 0.2. *If h is a harmonic mapping then*

$$\Delta \log |\partial h|^2 = -K_2 J_h$$

and

$$\Delta \log |\bar{\partial}h|^2 = K_2 J_h.$$

Proposition 0.3. *[10, p. 10-11] The functions $|\partial h|$ and $|\bar{\partial}u|$ are identically zero or they have the isolated zeros with well defined orders.*

Let us conclude this introduction with the following observation concerning harmonic maps.

For $g : M \mapsto N$ the energy integral is defined by

$$E_\rho[g] = \int_M (|\partial_\sigma g|^2 + |\bar{\partial}_\sigma g|^2)dV_\sigma, \tag{3}$$

where $\partial_\sigma g$, and $\bar{\partial}_\sigma g$ are the partial derivatives taken with respect to the metrics ϱ and σ, and dV_σ is the volume element on (M, σ). Assume that energy integral of h is bounded. Then h is harmonic if and only if h is a critical point of the corresponding functional where the homotopy class of h is the range of this functional. For this definition and some important properties of harmonic maps see [10].

1. Background and statement of the main result

Manojlović in [8] proved that a sense preserving Euclidean harmonic mapping between two planar domains has superharmonic logarithm of Jacobian and used this fact to prove the minimum principle for the Jacobian of Euclidean harmonic mappings between planar domains. Then she proved that the q.c. harmonic mapping between two planar domains is a quasi-isometry. The minimum principle of the Jacobian has been proved previously by the author in [6, p. 8], but in [6] has not been treated the superharmonicity of the logarithm of Jacobian. This fact has been re-discovered by Iwaniec and Onninen in [2] and generalized by Iwaniec, Koski and Onninen in [3]. The aim of this note is to obtain a similar results for the class of harmonic mappings between Riemannian surfaces with certain curvature conditions. We prove the following theorem

Theorem 1.1. *If h is orientation preserving ρ-harmonic mapping between Riemannian surfaces (D, σ) and (Ω, ρ) such that the Gauss curvature of ρ is K_2 and $J = J_h = \rho^2(h(z))(|h_z|^2 - |h_{\bar{z}}|^2)$ is the Jacobian then*

$$-\Delta \log J = K_2 D_h + \frac{4\rho^4}{J^2}\left(\frac{|h_z|^2}{|h_{\bar{z}}|^2}|B|^2 + \frac{|h_{\bar{z}}|^2}{|h_z|^2}|A|^2 - 2\Re(A\bar{B})\right) \tag{4}$$

where

$$A = h_{zz}\bar{h}_{\bar{z}} + \bar{h}_{z\bar{z}}h_z$$

and

$$B = h_{z\bar{z}}\bar{h}_z + h_{\bar{z}}\bar{h}_{zz},$$

and

$$D_h = |\partial h|^2 + |\bar{\partial} h|^2.$$

In particular $-\log J_h$ is subharmonic, if h is a sense-preserving homeomorphism and $K_2 \geq 0$.

Proof. In the sequel we make use of the formula

$$\Delta \log R = \frac{R\Delta R - |\nabla R|^2}{R^2}, \tag{5}$$

or what is the same

$$\Delta R = \frac{|\nabla R|^2}{R} + R\Delta \log R. \tag{6}$$

Here $\nabla R = (R_x, R_y)$ is the Euclidean gradient of a differentiable non-vanishing real function R. Let $J = J_h$. Then by Bochner's formula and the formula (5) for $|\partial h|^2$ and $|\bar{\partial} h|^2$ we obtain

$$\Delta J = \Delta|\partial h|^2 - \Delta|\bar{\partial} h|^2$$
$$= |\partial h|^{-2}|\nabla|\partial h|^2|^2 - |\bar{\partial} h|^{-2}|\nabla|\bar{\partial} h|^2|^2 - K_2 J_h D_h,$$

where

$$D_h = |\partial h|^2 + |\bar{\partial} h|^2.$$

Thus

$$\Delta J = 4|\nabla|\partial h||^2 - 4|\nabla|\bar{\partial} h||^2 - K_2 J_h D_h.$$

Hence

$$J^2\Delta \log J = J\Delta J - |\nabla J|^2$$
$$= J(4|\nabla|\partial h||^2 - 4|\nabla|\bar{\partial} h||^2 - K_2 J_h D_h) - |\nabla J|^2.$$

On the sequel we use the formulas

$$|h_z|^2 = h_z\bar{h}_{\bar{z}}, \text{ and, } |h_{\bar{z}}|^2 = h_{\bar{z}}\bar{h}_z$$

and for a real function R the formula

$$|\nabla R| = 2|R_z|.$$

Since

$$J = |\partial h|^2 - |\bar{\partial} h|^2 = \rho^2(|h_z|^2 - |h_{\bar{z}}|^2),$$

we obtain that

$$|\nabla J|^2 = 4|J_z|^2 = 4\left|\left(\rho^2(|h_z|^2 - |h_{\bar{z}}|^2)\right)_z\right|^2$$

which yields

$$|\nabla J|^2 = 4|\rho^2\left(h_{zz}\bar{h}_{\bar{z}} + h_z\bar{h}_{z\bar{z}} - h_{z\bar{z}}\bar{h}_z - h_{\bar{z}}\bar{h}_{zz}\right) + 2\rho\rho_z(|h_z|^2 - |h_{\bar{z}}|^2)|^2$$

and

$$|\nabla|\partial h||^2 = 4\||\partial h|_z\|^2 = 4|\rho_z|h_z| + \rho|h_z|_z|^2 = 4\left|\rho_z|h_z| + \frac{\rho}{2|h_z|}(h_{zz}\bar{h}_{\bar{z}} + h_z\bar{h}_{z\bar{z}})\right|^2,$$

$$|\nabla|\bar{\partial}h||^2 = 4\||\bar{\partial}h|_z\|^2 = 4|\rho_z|h_{\bar{z}}| + \rho|h_{\bar{z}}|_z|^2 = 4\left|\rho_z|h_{\bar{z}}| + \frac{\rho}{2|h_{\bar{z}}|}(h_{z\bar{z}}\bar{h}_z + h_{\bar{z}}\bar{h}_{zz})\right|^2$$

and

$$(|\nabla|\partial h||^2 - |\nabla|\bar{\partial}h||^2) = \frac{|\nabla\rho|^2 J}{\rho^2} + \rho^2\left(\frac{|h_{zz}\bar{h}_{\bar{z}} + \bar{h}_{z\bar{z}}h_z|^2}{|h_z|^2} - \frac{|h_{z\bar{z}}\bar{h}_z + h_{\bar{z}}\bar{h}_{zz}|^2}{|h_{\bar{z}}|^2}\right)$$
$$+ 4\Re(\rho\rho_z(h_{zz}\bar{h}_{\bar{z}} + \bar{h}_{z\bar{z}}h_z - h_{z\bar{z}}\bar{h}_z - h_{\bar{z}}\bar{h}_{zz})).$$

Hence

$$|\nabla J|^2 - J\Delta J = 4|\rho^2\left(h_{zz}\bar{h}_{\bar{z}} + h_z\bar{h}_{z\bar{z}} - h_{z\bar{z}}\bar{h}_z - h_{\bar{z}}\bar{h}_{zz}\right) + 2\rho\rho_z(|h_z|^2 - |h_{\bar{z}}|^2)|^2$$
$$- 4J\Big(\frac{|\nabla\rho|^2 J}{\rho^2} + \rho^2\left(\frac{|h_{zz}\bar{h}_{\bar{z}} + \bar{h}_{z\bar{z}}h_z|^2}{|h_z|^2} - \frac{|h_{z\bar{z}}\bar{h}_z + h_{\bar{z}}\bar{h}_{zz}|^2}{|h_{\bar{z}}|^2}\right)$$
$$+ 4\Re(\rho\rho_{\bar{z}}(h_{zz}\bar{h}_{\bar{z}} + \bar{h}_{z\bar{z}}h_z - h_{z\bar{z}}\bar{h}_z - h_{\bar{z}}\bar{h}_{zz}))\Big) + K_2 J^2 D_h.$$

Thus

$$|\nabla J|^2 - J\Delta J = 4|\rho^2\,(A - B) + 2\rho\rho_z(|h_z|^2 - |h_{\bar{z}}|^2)|^2$$
$$- 4J\Big(\frac{|\nabla\rho|^2 J}{\rho^2} + \rho^2\left(\frac{|A|^2}{|h_z|^2} - \frac{|B|^2}{|h_{\bar{z}}|^2}\right) + 4\Re(\rho\rho_{\bar{z}}(A - B))\Big) + K_2 J^2 D_h.$$

After some easy manipulations we arrive at the equation

$$|\nabla J|^2 - J\Delta J = K_2 J^2 D_h + 4\rho^4\frac{|h_z|^2}{|h_{\bar{z}}|^2}|B|^2 + 4\rho^4\frac{|h_{\bar{z}}|^2}{|h_z|^2}|A|^2 - 8\rho^4\Re(A\bar{B}) \tag{7}$$

where

$$A = h_{zz}\bar{h}_{\bar{z}} + \bar{h}_{z\bar{z}}h_z$$

and

$$B = h_{z\bar{z}}\bar{h}_z + h_{\bar{z}}\bar{h}_{zz}.$$

From (7) and formula (5) we obtain (4) for the points on the domain where ∂h and $\bar{\partial}h$ does not vanish.

But Proposition 0.3 says that the isolated zeros of $|\partial h|$ and $|\bar{\partial}h|$ are isolated, and thus (4) is valid except for a set of isolated points of D.

It remains to observe that

$$\frac{|h_z|^2}{|h_{\bar{z}}|^2}|B|^2 + \frac{|h_{\bar{z}}|^2}{|h_z|^2}|A|^2 - 2\Re(A\bar{B})$$
$$= \left(\frac{|h_z|}{|h_{\bar{z}}|}|B| - \frac{|h_{\bar{z}}|}{|h_z|}|A|\right)^2 + 2(|A||B| - \Re(A\bar{B})) \geq 0.$$

Since $\log J_h$ is continuous, we conclude that $-\log J_h$ is subharmonic under the constraint $K_2 \geq 0$. $\square$

Since a harmonic homeomorphism has the non-vanishing Jacobian (see e.g. [9] or [1]), we know that $\log J_h$ is well defined. By maximum principle for subharmonic functions we obtain

Corollary 1.2. *Assume that h is a homeomorphism between surfaces (D, σ) and (S, ρ) and let $\gamma \subset D$ be a Jordan curve surrounding a domain $G \subset D$. Assume further that the curvature of S is non-negative. Then*

$$\min\{J_h(z) : z \in G \cup \gamma\} = \min_{z \in \gamma} J_h(z).$$

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