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A Class of Constrained Inverse Eigenvalue Problem and Associated Approximation Problem for Symmetrizable Matrices

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Abstract. The real symmetric matrix is widely applied in various fields, transforming non-symmetric matrix to symmetric matrix becomes very important for solving the problems associated with the original matrix. In this paper, we consider the constrained inverse eigenvalue problem for symmetrizable matrices, and obtain the solvability conditions and the general expression of the solutions. Moreover, we consider the corresponding optimal approximation problem, obtain the explicit expressions of the optimal approximation solution and the minimum norm solution, and give the algorithm and corresponding computational example.

1. Introduction

Throughout the paper, let R^n denote the set of n-dimensional real vector, and $R_r^{n\times m}$, $SR^{n\times n}$, $SR_+^{n\times n}$, $ASR^{n\times n}$, $OR^{n\times n}$ denote the sets of real $n\times m$ matrices with rank r, real $n\times n$ symmetric matrices, real $n\times n$ symmetric positive definite matrices, real $n\times n$ antisymmetric matrices, real $n\times n$ orthogonal matrices, respectively. Let A^{-1} denote the inverse matrix of $A\in R_n^{n\times n}$, and A^+, A^T, tr$A$ denote the Moore-Penrose generalized inverse, the transpose, the trace of a matrix A, and I stands for an identity matrix. We define a vector inner product $\langle x,y\rangle = y^Tx$ for all $x,y\in R^n$. If $A\in R^{n\times n}$ and $\langle Ax,y\rangle = \langle x,Ay\rangle$ for all $x,y\in R^n$, then A is clled a symmetric matrix. Also we define an inner product $\langle A,B\rangle = \mathrm{tr}(B^TA)$ for all $A,B\in R^{m\times n}$. Then $R^{m\times n}$ is a Hilbert inner product space and the norm generated by this inner product is Frobenius norm. Let $\|A\|_F$ be the Frobenius norm of a matrix A.

Definition 1.1[1] Given $W_1\in R_n^{n\times n}$, $A\in R^{n\times n}$, if $\langle W_1AW_1^{-1}x,y\rangle = \langle x,W_1AW_1^{-1}y\rangle$ for all $x,y\in R^n$, that is, $W_1AW_1^{-1}\in SR^{n\times n}$, then A is called a symmetrizable matrix.

Lemma 1.1 Let $A\in R^{n\times n}$ be a given matrix. Then there exists a nonsingular matrix W_1 such that $W_1AW_1^{-1}\in SR^{n\times n}$ if and only if there exists a symmetric positive definite matrix W such that $W^2A\in SR^{n\times n}$ and $WAW^{-1}\in SR^{n\times n}$.

Proof. (Necessity) Since W_1 is a nonsingular matrix, suppose that the singular value decomposition (SVD) of W_1 is

$$W_1 = R\Gamma Q^T, \tag{1.1}$$

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where $Q, R \in OR^{n\times n}, \Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n), \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n > 0$.

Because $W_1AW_1^{-1} \in SR^{n\times n}$, so we have

$$W_1AW_1^{-1} = (W_1AW_1^{-1})^T, \tag{1.2}$$

substituting (1.1) into (1.2), we have

$$Q\Gamma^2Q^TA = A^TQ\Gamma^2Q^T.$$

Let $W = Q\Gamma Q^T$, then W is a symmetric positive definite matrix, and $W^2A = A^TW^2$, this implies that $WAW^{-1} = W^{-1}A^TW$, so we have $W^2A \in SR^{n\times n}$ and $WAW^{-1} \in SR^{n\times n}$.

(Sufficiency) Because W is a symmetric positive definite matrix, and $WAW^{-1} \in SR^{n\times n}$, let $W_1 = W$, then the conclusion obviously holds. □

Based on Definition 1.1 and Lemma 1.1, the symmetrizable matrix can be defined as follows.

Definition 1.2 Given $W \in SR_+^{n\times n}$, $A \in R^{n\times n}$, if $WAW^{-1} \in SR^{n\times n}$, then A is called a symmetrizable matrix. We denote

$$SR_W^{n\times n} = \{A | A \in R^{n\times n}, WAW^{-1} \in SR^{n\times n}\},$$

and

$$\|A\|_W = \|WAW^{-1}\|_F.$$

Obviously, if $A \in SR_W^{n\times n}$, that is, $B = WAW^{-1} \in SR^{n\times n}$, then B is similar to A, which means that the matrix A have the same eigenvalues as B.

The real symmetric matrix is widely applied in various fields, transforming non-symmetric matrix to symmetric matrix becomes very important for solving the problems associated with the original matrix. For example, in 1922, Stenel[2] obtained a symmetric matrix by multiplying non-singular matrix and the original matrix. In 1936, to solve some problems in probability theory, Kolmogorov obtained a symmetric matrix by multiplying a positive symmetric matrix and the original matrix. And those methods are widely used in later papers[3–5]. Also, in 2000, Sun[6] defined the sets of symmetrizable positive definite matrices.

The inverse eigenvaue problem has broad application background in theoretical physics, molecular structure, vibration design, vibration control and so on[7–9]. The results on the unconstrained inverse eigenvalue problem with several sets of matrices have been discussed[10–15]. Pan[7–8] and Peng[16] discussed the constrained inverse eigenvalue problem and associated approximation problems of antisymmetric matrices, skew symmetric and centrosymmetric matrices and normal matrices, respectively.

However, the constrained inverse eigenvalue problems on symmetrizable or anti-symmetrizable sets have not been resolved yet. We will study these problems in this paper.

Given interval $[a, b]$, real matrices $W \in SR_+^{n\times n}$, $X \in R^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in R^{m\times m}$. $\lambda(\Lambda)$ and $\lambda(A)$ denote the sets of eigenvalues of Λ and A, respectively. $\lambda(A)\backslash\lambda(\Lambda)$ denotes the difference of $\lambda(A)$ and $\lambda(\Lambda)$. And we denote $SR_{W[a,b]}^{n\times n} = \{A | A \in SR_W^{n\times n}, \lambda(A) \subset [a, b]\}$.

In this paper, we mainly consider the following two problems.

Problem I. Given interval $[a, b]$, the matrices X, Λ, W, find $A \in SR_W^{n\times n}$ such that

$$AX = X\Lambda, \tag{1.3}$$

and all the remaining eigenvalues of any matrix A that satisfies (1.3) are located in interval $[a, b]$, that is, $\lambda(A) \setminus \lambda(\Lambda) \subset [a, b]$.

Denote the solution set of Problem I by S_E. If S_E is nonempty, we consider the associated optimal approximation problem.

Problem II. Given $A^* \in R^{n\times n}$, find an $n \times n$ matrix $\hat{A} \in S_E$ such that

$$\|\hat{A} - A^*\|_W = \min_{A\in S_E} \|A - A^*\|_W. \tag{1.4}$$

The paper is organized as follows. In Section 2, we establish the solvability conditions for Problem I and give the expression of the general solution to Problem I. In Section 3, we prove the existence and uniqueness of the solution for Problem II, and we give an algorithm and a corresponding computational example to illustrate the theoretical results.

2. The solution of Problem I

Given $X \in R^{n\times m}$, $W \in SR_{+}^{n\times n}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in R^{m\times m}$, find $A \in SR_{W}^{n\times n}$ such that $AX = X\Lambda$, then the i-th column of X is an eigenvector of A corresponding to the eigenvalue λ_i if and only if the i-th column of WX is an eigenvector of WAW^{-1} corresponding to the eigenvalue λ_i, that is, $WAW^{-1}WX = WX\Lambda$. Suppose that the SVD of WX is

$$WX = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^T, \tag{2.1}$$

where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r), \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$, $U = (U_1, U_2), U_1 \in OR^{n\times r}, U_2 \in OR^{n\times(n-r)}$, $V = (V_1, V_2), V_1 \in R^{m\times r}, V_2 \in R^{m\times(m-r)}$.

Based on Lemma 1 in [10], we can obtain the following conclusion.

Lemma 2.1 Given $X \in R_r^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1 I_{m_1}, \lambda_2 I_{m_2}, \ldots, \lambda_t I_{m_t}) \in R^{m\times m}$, $\lambda_i \neq \lambda_j (i \neq j)$, $W \in SR_{+}^{n\times n}$, $WX = (WX_1, WX_2, \ldots, WX_t) \in R^{n\times m}$. Then there exists $E \in SR^{n\times n}$ such that

$$EWX = WX\Lambda, \tag{2.2}$$

if and only if

$$X_i^T W^2 X_j = 0, i \neq j, i, j = 1, 2, \ldots, t.$$

Moreover, the solutions of Equation (2.2) can be expressed as

$$E = WX\Lambda(WX)^+ + U_2 G U_2^T, \forall G \in SR^{(n-r)\times(n-r)},$$

where $U_2 \in R^{n\times(n-r)}$, $U_2^T U_2 = I_{n-r}$, $N((WX)^T) = R(U_2)$, $r(WX_i) = r_i (i = 1, 2, \ldots, t)$, $r = \sum_{i=1}^{t} r_i, t \leq r$.

Theorem 2.1 Given $X \in R_r^{n\times m}$, $W \in R_{+}^{n\times n}$, $\Lambda = \mathrm{diag}(\lambda_1 I_{m_1}, \lambda_2 I_{m_2}, \ldots, \lambda_t I_{m_t}) \in R^{m\times m}$, $\lambda_i \neq \lambda_j (i \neq j)$, $X = (X_1, X_2, \ldots, X_t) \in R^{n\times m}$, then there exists $A \in SR_{W[a,b]}^{n\times n}$ such that $AX = X\Lambda$ if and only if

$$X_i^T W^2 X_j = 0, i \neq j, i, j = 1, 2, \ldots, t. \tag{2.3}$$

Moreover, the solutions of Equation (1.3) can be expressed as

$$A = X\Lambda(WX)^+ W + W^{-1} U_2 G U_2^T W, \forall G \in SR_{[a,b]}^{(n-r)\times(n-r)}, \tag{2.4}$$

where $U_2 \in OR^{n\times(n-r)}$, $U_2^T U_2 = I_{n-r}$, $N((WX)^T) = R(U_2)$, $r(X_i) = r(WX_i) = r_i (i = 1, 2, \ldots, t)$, $r = \sum_{i=1}^{t} r_i, t \leq r$.

Proof. $AX = X\Lambda$ is equivalent to

$$WAW^{-1}WX = WX\Lambda.$$

According to $WAW^{-1} \in SR^{n\times n}$ and Lemma 2.1, the solution set of $AX = X\Lambda$ is nonempty if and only if $X_i^T W^2 X_j = 0$. The expression of solutions can be expressed as

$$A = X\Lambda(WX)^+ W + W^{-1} U_2 G U_2^T W, \forall G \in SR^{(n-r)\times(n-r)}. \tag{2.5}$$

Now we study the eigenvalues of A in (2.5). $\lambda_1, \lambda_2, \ldots, \lambda_t$ are eigenvalues of A. Assume that the eigenvalues of G are $\mu_1, \mu_2, \ldots, \mu_{n-r}$, and corresponding eigenvectors are $y_1, y_2, \ldots, y_{n-r}$, respectively.

Let

$$\mu = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{n-r}), Y = (y_1, y_2, \ldots, y_{n-r}),$$

we have

$$GY = Y\mu.$$

Due to $(WX)^+U_2 = 0$ and $U_2^TU_2 = I_{n-r}$, we get

$$AW^{-1}U_2Y = X\Lambda(WX)^+U_2Y + W^{-1}U_2GY = W^{-1}U_2GY = W^{-1}U_2Y\mu.$$

It shows that $\mu_1, \mu_2, \ldots, \mu_{n-r}$ are eigenvalues of A. Therefore, we only restrict $\mu_1, \mu_2, \ldots, \mu_{n-r} \in [a, b]$, then the general expression of the solutions of (1.3) is (2.4). □

Denote the solution set of Problem I by S_E. If S_E is nonempty, we can obtain the following conclusion.

Corollary 2.1 If solution set S_E of Problem I is nonempty, then S_E is a closed convex set.

Proof. For matrix sequence $\{A_k\}, A_k \in S_E$ satisfying $\lim\limits_{k\to\infty} A_k = A$, we have

$$A_k = X\Lambda(WX)^+W + W^{-1}U_2G_kU_2^TW, G_k \in SR_{[a,b]}^{(n-r)\times(n-r)},$$

$$\lim_{k\to\infty} G_k = G,$$

$$A = X\Lambda(WX)^+W + W^{-1}U_2GU_2^TW.$$

Because that the eigenvalues are continuous functions of matrix elements, so we can obtain $G \in SR_{[a,b]}^{(n-r)\times(n-r)}$, then $A \in S_E$, that is, S_E is a closed set.

Let

$$A_1 = X\Lambda(WX)^+W + W^{-1}U_2G_1U_2^TW \in S_E,$$

$$A_2 = X\Lambda(WX)^+W + W^{-1}U_2G_2U_2^TW \in S_E,$$

where $\forall G_1, G_2 \in SR_{[a,b]}^{(n-r)\times(n-r)}$.

Then we have $c_1G_1 + c_2G_2 \in SR_{[a,b]}^{(n-r)\times(n-r)}, c_1 \geq 0, c_2 \geq 0, c_1 + c_2 = 1$, (see [17]) and $c_1A_1 + c_2A_2 \in S_E$. So S_E is a closed convex set. □

3. The solution of Problem II

Given interval $[a, b]$ and $A^* \in R^{n\times n}$, let $A_1 = \frac{WA^*W^{-1}+W^{-1}A^{*T}W}{2} \in SR^{n\times n}$ and $A_2 = \frac{WA^*W^{-1}-W^{-1}A^{*T}W}{2} \in ASR^{n\times n}$. By the spectral decomposition theorem, A_1 can be expressed as

$$A_1 = \sum_{i=1}^{n} \mu_i u_i u_i^T.$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > b \geq \mu_{k+1} \geq \cdots \geq \mu_{k+l} \geq a > \mu_{k+l+1} \geq \cdots \geq \mu_n$, u_i is an eigenvector of matrix A_1 corresponding to the eigenvalue μ_i satisfying $\|u_i\| = 1$.

Denote

$$[A_1]_{[a,b]} = \sum_{i=1}^{k} bu_iu_i^T + \sum_{i=k+1}^{k+l} \mu_iu_iu_i^T + \sum_{i=k+l+1}^{n} au_iu_i^T, \tag{3.1}$$

and

$$[A^*]_{[a,b]}^W = W^{-1}[A_1]_{[a,b]}W \in SR_{W[a,b]}^{n\times n}. \tag{3.2}$$

Obviously, $[A^*]_{[a,b]}^W$ is only determined by a, b, W and A^*. Next, we give some properties of $[A^*]_{[a,b]}^W$ as follows.

Property 3.1 If $A^* \in SR_{W[a,b]}^{n\times n}$, then $[A^*]_{[a,b]}^W = A^*$.

Proof. If $A^* \in SR_{W[a,b]}^{n\times n}$, then $WA^*W^{-1} \in SR^{n\times n}, \lambda(A^*) \subset [a, b]$, by (3.2), we have $[A^*]_{[a,b]}^W = A^*$. □

Property 3.2 If $A^* = 0$, then

$$[A^*]^W_{[a,b]} = \begin{cases} 0, & if \ \ a \le 0 \le b, \\ aI_n, & if \ \ a > 0, \\ bI_n, & if \ \ b < 0. \end{cases}$$

Property 3.3 If $A^* \in ASR^{n\times n}_W$, that is, $WA^*W^{-1} \in ASR^{n\times n}$, then $[A^*]^W_{[a,b]} = [0]^W_{[a,b]}$.

Lemma 3.1 Let $WA^*W^{-1} = D$ be a diagonal matrix. If

$$\|\hat{A} - A^*\|_W = \min_{A\in SR^{n\times n}_{W[a,b]}} \|A - A^*\|_W$$

then $\hat{A} = [A^*]^W_{[a,b]} = W^{-1}[D]_{[a,b]}W$.

Lemma 3.2 Let $A^* \in R^{n\times n}$. If

$$\|\hat{A} - A^*\|_W = \min_{A\in SR^{n\times n}_{W[a,b]}} \|A - A^*\|_W,$$

then $\hat{A} = [A^*]^W_{[a,b]} = W^{-1}[A_1]_{[a,b]}W$.

Proof. Let $A^* \in R^{n\times n}$, WA^*W^{-1} can be uniquely decomposed into the sum of the symmetric matrices A_1 and the antisymmetry matrix A_2, that is, $WA^*W^{-1} = A_1 + A_2, A_1 = \frac{WA^*W^{-1}+W^{-1}A^{*T}W}{2} \in SR^{n\times n}$ and $A_2 = \frac{WA^*W^{-1}-W^{-1}A^{*T}W}{2} \in ASR^{n\times n}$, then

$$\|\hat{A} - A^*\|^2_W = \|W\hat{A}W^{-1} - A_1 - A_2\|^2_F = \|W\hat{A}W^{-1} - A_1\|^2_F + \|A_2\|^2_F.$$

Suppose that the spectral decomposition of A_1 is

$$A_1 = P^TDP,$$

where P and D are orthogonal and diagonal matrices, respectively.

From the orthogonal invariance of the Frobenius norm, we have

$$\|W\hat{A}W^{-1} - A_1\|^2_F = \|PW\hat{A}W^{-1}P^T - D\|^2_F,$$

by Lemma 3.1, we have $PW\hat{A}W^{-1}P^T = [D]_{[a,b]}$ and $W\hat{A}W^{-1} = P^T[D]_{[a,b]}P = [A_1]_{[a,b]}$, it follows that $\hat{A} = W^{-1}[A_1]_{[a,b]}W = [A^*]^W_{[a,b]}$. □

Theorem 3.1 Given $A^* \in R^{n\times n}$, if (2.3) holds, then there exists unique $\hat{A} \in S_E$ such that

$$\|\hat{A} - A^*\|_W = \min_{A\in S_E} \|A - A^*\|_W.$$

Moreover, $\hat{A}$ can be expressed as

$$\hat{A} = X\Lambda(WX)^+W + W^{-1}U_2\hat{G}U_2^TW,$$

where $\hat{G} = [U_2^TA_1U_2]_{[a,b]}$.

Proof. Let $A^* \in R^{n\times n}$, by Theorem 2.1, we have

$$W\hat{A}W^{-1} = U\begin{pmatrix} \Sigma V_1^T\Lambda V_1\Sigma^{-1} & 0 \\ 0 & \hat{G} \end{pmatrix}U^T,$$

then

$$\begin{aligned}\|\hat{A}-A^*\|_W^2 &= \|W\hat{A}W^{-1}-WA^*W^{-1}\|_F^2 = \|W\hat{A}W^{-1}-A_1\|_F^2+\|A_2\|_F^2\\ &= \left\|\begin{pmatrix}\Sigma V_1^T\Lambda V_1\Sigma^{-1} & 0\\ 0 & \hat{G}\end{pmatrix}-U^TA_1U\right\|_F^2+\|A_2\|_F^2\\ &= \left\|\begin{pmatrix}\Sigma V_1^T\Lambda V_1\Sigma^{-1} & 0\\ 0 & \hat{G}\end{pmatrix}-\begin{pmatrix}U_1^TA_1U_1 & U_1^TA_1U_2\\ U_2^TA_1U_1 & U_2^TA_1U_2\end{pmatrix}\right\|_F^2+\|A_2\|_F^2\\ &= \left\|\begin{pmatrix}U_1^TA_1U_1-\Sigma V_1^T\Lambda V_1\Sigma^{-1} & U_1^TA_1U_2\\ U_2^TA_1U_1 & U_2^TA_1U_2-\hat{G}\end{pmatrix}\right\|_F^2+\|A_2\|_F^2\\ &= \|U_1^TA_1U_1-\Sigma V_1^T\Lambda V_1\Sigma^{-1}\|_F^2+\|U_1^TA_1U_2\|_F^2+\|U_2^TA_1U_1\|_F^2\\ &\quad+\|U_2^TA_1U_2-\hat{G}\|_F^2+\|A_2\|_F^2,\end{aligned}$$

clearly,

$$\|\hat{A}-A^*\|_W = \min_{A\in S_E}\|A-A^*\|_W$$

is equivalent to

$$\|U_2^TA_1U_2-\hat{G}\|_F = \min_{G\in SR^{n\times n}_{[a,b]}}\|U_2^TA_1U_2-G\|_F,$$

then we have

$$\hat{G} = [U_2^TA_1U_2]_{[a,b]},$$

where $A_1 = \frac{WA^*W^{-1}+W^{-1}A^{*T}W}{2} \in SR^{n\times n}$. □

Corollary 3.1(Minimum norm solution) If solution set S_E of Problem I is nonempty, then problem

$$\|\hat{A}\|_W = \min_{A\in SR^{n\times n}_{W[a,b]}}\|A\|_W$$

has unique solution

$$\hat{A} = \begin{cases} X\Lambda(WX)^+W, & if\ \ a\le 0\le b,\\ X\Lambda(WX)^+W+a(I_n-X(WX)^+W), & if\ \ a>0,\\ X\Lambda(WX)^+W+b(I_n-X(WX)^+W), & if\ \ b<0.\end{cases} \tag{3.3}$$

Proof. Suppose that the SVD of WX is (2.1), then $U_2U_2^T = I_n - WX(WX)^+$. By $A^* = 0$ and Property 3.2 and Theorem 3.1, we get (3.3). □

Corollary 3.2 If solution set S_E of Problem I is nonempty, and $A^* \in ASR_W^{n\times n}$, then problem II has unique solution (3.2).

Algorithm 3.1
step 1. Input real $a, b, A^* \in R^{n\times n}$, $X \in R^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in R^{m\times m}$, $W \in SR_+^{n\times n}$;
step 2. Calculate W^{-1}, WX, and U_2, $(WX)^+$ by (2.1);
step 3. If (2.3) holds, then go to step 4, else go to step 8;
step 4. Calculate A_1, and $U_2^TA_1U_2$;
step 5. Calculate the spectral decomposition of $U_2^TA_1U_2$;
step 6. Calculate $[U_2^TA_1U_2]_{[a,b]}$;
step 7. Calculate $\hat{A}$ by Theorem 3.1;
step 8. Stop.

According to Theorem 3.1, the above algorithm shows that $\hat{A}$ is the solution of problem II. Next, we give a corresponding computational example to illustrate our theoretical results.

Example 3.1 Consider the constrained inverse eigenvalue problem and associated approximation problem

$$AX = X\Lambda,\ \ s.t.\ \ A \in SR_W^{n\times n}$$

with

$$X = \begin{pmatrix} -7 & -7 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 13 & 0 & 13 & 7 \\ -4 & 0 & -4 & -2 \\ -1 & 1 & -2 & 0 \\ 3 & -2 & 5 & 0 \end{pmatrix}, \Lambda = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, W = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix},$$

$$A^* = \begin{pmatrix} 11.0619 & 20.8073 & -6.4230 & -22.4805 & -7.7881 & -4.3998 \\ -4.0594 & -7.4180 & 2.5407 & 8.8923 & 4.2723 & 2.2171 \\ 10.5755 & 22.0783 & -23.2170 & -77.7595 & 72.5335 & 32.3700 \\ -3.2540 & -6.7933 & 6.8360 & 22.9260 & -22.3180 & -9.9600 \\ 6.3703 & 14.2401 & 3.9650 & 13.8775 & 19.2891 & 7.7086 \\ -14.3676 & -31.8768 & -10.5120 & -36.7920 & -45.7372 & -18.3973 \end{pmatrix},$$

and $a = 1,\ b = 3$.

By Algorithm 3.1, then the optimal approximation solution $\hat{A}$ of Problem II is

$$\hat{A} = \begin{pmatrix} 7.7674 & 13.7790 & -5.4165 & -18.9578 & -15.6592 & -7.3470 \\ -3.0727 & -5.3995 & 2.1296 & 7.4537 & 7.7079 & 3.5091 \\ 9.9261 & 21.0541 & -22.5582 & -75.4535 & 66.4853 & 30.0825 \\ -3.0542 & -6.4782 & 6.6333 & 22.2165 & -20.4570 & -9.2562 \\ 6.5048 & 14.8976 & 4.3944 & 15.3803 & 14.9925 & 6.0759 \\ -14.5367 & -33.0343 & -11.4721 & -40.1523 & -36.2135 & -14.7798 \end{pmatrix}.$$

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