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A Process of Project Quality Improvement

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Abstract. Measuring and managing project quality is one of the fundamental problems in project management. In this paper, once project is completed, we define the measure of task's quality and the measure of overall project quality. We construct mathematical model of the process of quality improvement as the sequence of mutually dependent projects, where every project in the sequence is the revision of its previous one. We prove that it is possible, at certain point, to obtain a project with the highest quality measure and with failures less than the initially given level. The purpose of this paper is, to help companies to achieve the satisfying level of project quality by using the proposed model. According to our knowledge, paper offers an original connection between project management and measure theory potentially interesting to a reader for further research.

1. Introduction

There are large number of papers devoted to project's improvement solutions (see for example Baccarini [1] and Prabhakar [8]). The way of determining a quality of a completed project has been analyzed by many authors and widely discussed in the literature, including Bryde [2]. After reviewing the scientific papers in the area of project management, we can easily detect the trend of defining project success, determining the factors and criteria of success, mostly summarized in Ika [5]. Contrary to those papers, we investigate a problem of introducing the use of real numbers in project management for the purpose of measuring a task quality, a quality of realized project and to build a mathematical model of quality improvement.

Primarily in the paper, we define the way of measuring a quality of realized tasks, in order to be able to define a measure of overall project quality. In the literature there are various project and task definitions. See for example Elearn [3], Kähkönen [7] and Whelton and Ballard [9]. In this paper, each project is identified with the project goal that needs to be accomplished by the project management team. Also, each task is considered as a partial goal that needs to be executed by suitable team of experts. The completion of all tasks (partial goals) is, naturally, a condition for a completion of a project goal. Every goal (task) can be executed in various ways. Each of those ways of task completion, we define as the option of a task. The quality of each option is measured by some real number, which we call a quality index. Defining each of the options and evaluating their quality indexes are necessary actions that need to be done by experts at the beginning of a project design. That way, we have a possibility of comparing task options according

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to quality. Classical approach to project planning design does not include the determination of all task's options in a given project environment. Consequently, in the paper, we define a project as a collection of all possible project realizations, where every project realization presents a set of options belonging to different tasks.

After determination of all quality indexes, in Section 2., we compare the measure of quality of a realized task with other tasks in the realized project. For each task, we define the Difference indicator, which measures the influence of the given task on the overall project quality. In Definition 2.6., we introduce the real number μ as the measure of failure of the realized project.

Further in the paper we explore how to define a process of project quality improvement. We construct a mathematical model of the process by starting with some realized project with estimated measure of failure μ and we build its revision as new project, as we explained in detail in Subsection 2.2. We point that, in the same section we give certain conditions that ensure higher quality of the revision comparing with the initial project. We continue the procedure by constructing a sequence of revisions and hence, we obtain a decreasing sequence of corresponding measures of failure. Theorems 3.9 and 3.11 show that it is possible to achieve a project with failures bellow the initially given level $\epsilon > 0$. Following this context, we introduce two basic assumptions:

1. Reliability principle
2. Revision principle.

First principle ensures that the process of the quality improvement needs to begin from one starting project which is already completed. Really, every enterprize has its own activity in a specific field of expertise. For example, the building housing projects are performed in the concrete political, economical, geographical environment and with the defined human and financial resources. All limitations produced by the environment strongly influence the success of projects performed by this company. Projects of other companies, of the same type are being realized in its own environments, with its own specific limitations. Because of this reason, it is not reliable to use the experiences from other companies. Therefore, in order to be more successful in the project constructions the best solution for the company is to be focused on its own realized projects. Second principle is previously explained.

2. A model of project quality improvement

Let $\mathcal{P}$ be a project, defined as a collection of all possible project realizations, where every project realization presents a set of options belonging to different tasks. Also, let $\mathcal{Z}$ be a set of all its tasks. If $x, y \in \mathcal{Z}$ and the realization of x is the condition for the realization of y, we say that $x \prec y$. The measure of quality of any realized task is a probability measure, as we explained in further text. Hence, it is a function that corresponds one real number to each realized task. Note that $\mathcal{Z}$ is partially ordered, meaning that every possible project realization is partially ordered too.

As it is much more easier to deal with sequences of realized options instead with partially ordered sets, we present $\mathcal{Z}$ as a sequence of different elements. Being finite, the set $\mathcal{Z}$ may be presented in different ways in the form of a sequence. For the process of the quality improvement of $\mathcal{P}$, the order of tasks defined with the relation $\prec$ in the sequence construction is of no importance.

Theorem 2.1. *The set $\mathcal{Z}$ may be presented as a sequence of different elements, indexed by some initial portion of the set N.*

In the proof of this theorem we choose one of the possible constructions which preserves best the dependency of tasks. See Appendix.

In each subset in $\mathcal{Z}$ there is a minimal and a maximal element. For each task, there is a predecessor, except for the minimal one and a successor, except for the maximal one.

Remark 2.2. *Some partially ordering properties of $\mathcal{Z}$ are being examined in literature, but the real question is how to identify characteristic properties of some partially ordered set in order to be considered as a project. Here we point out one property of a partial ordered set that is not included in classical project approach.*

Specifically, let $\mathcal{P}$ be a project and let $x \in \mathcal{Z}$. Also, let $ext(x) = \{z \mid z \prec x\}$ be the set of all previous tasks of x, and $int(x) = \{z \mid x \prec z\}$ the set of all tasks followers of x. Usually, for $x \neq y$ can not be $ext(x) = ext(y)$ and $int(x) = int(y)$ in same time, so that different tasks x and y can not have the same position in $\mathcal{Z}$, which is not the case in our approach. Consequently, we can define $p-$equality of tasks:

Definition 2.3. *x and y are $p-$equal tasks of $\mathcal{P}$, denoted by $x =_p y$, if $ext(x) = ext(y) \leftrightarrow int(x) = int(y)$.*

Obviously $=_p$ is the equivalence relation such that each $=_p$ class can be considered as a task with a set of options as its elements.

Hence, according to the above, we generalize in natural way the classical notion of project.

Let us denote the number of all possible options of a task x with n_x and assume that $1 \leq n_x \leq 100$. All options of the given task could be compared in quality and every option of the task x could be identified by an index of an option quality j_x, satisfying $1 \leq j_x \leq m_x$, where m_x is the highest quality index for the given task x and $1 \leq m_x \leq 100$. It is assumed that different options have different quality indexes heaving in mind that the smaller index, the better option. So, for the given task x the option satisfying $j_x = 1$ (if such option exists) is the best quality option, while the option satisfying $j_x = m_x$ is the worst quality option. For the given task $x \in \mathcal{Z}$ the i_x-th option, $1 \leq i_x \leq n_x$ with the quality index $j_x \in \{1, 2, ..., m_x\}$ is denoted by $x^{i_x j_x}$.

See that the quality scale of options differs from task to task. The definition of each quality scale is an assignment for different profile experts who must cooperate, since the tasks depend on each other. Often it can occur that the quality units of neighboring tasks cannot be compared, but for the project improvement success it is not crucial. What is important is that the selection of neighboring task options be accordant with the relation $\prec$. That means, if $x, y \in \mathcal{Z}$ and $x \prec y$, then for each option $x^{i_x j_x}$ of x there is at least one option $y^{i_y j_y}$ of y with the possibility of realization only if $x^{i_x j_x}$ is previously defined and closed.

The project is finalized when all tasks are accomplished, which means that exactly one of the options of each task is realized. Thus, when the project is closed we may identify each realized task $x \in \mathcal{Z}$ with the chosen option. We conclude that the project is terminated, if the sequence of all project tasks is realized. The sequence of realized options in the completed project we call the *optional sequence*. Once we determine the deficiencies of a final project, it is crucial to revise them in order to improve the quality of the project. The choice of different combination of options that could be applied wouldn't guarantee better result and experimenting with various optional sequences would be too expensive.

Values of quality indexes are approximately determined, except for the case of special options when accuracy is possible. Comparison within quality indexes of different tasks options is also approximate, since the accuracy in defining these relations is unlikely possible even for experts. If accuracy were possible and if we chose the options with equal quality indexes, the quality of a project realization would only depend on the chosen quality index value. This way, we would obtain success improvement of a project simply by lowering quality indexes of all tasks. So, big differences among indexes of realized options obviously influence on the project success and therefore it is necessary to find the procedure to overcome them. We need to identify indicators of those differences in order to lower them and that way to obtain more equable indexes of realized options. Thus, we define the *difference indicator of one option in relation to other options.* In what follows, we shell call it simply the *difference indicator*.

Definition 2.4. *Let $x^{i_x j_x}$ be a realized option in project $\mathcal{P}$. Difference indicator of an option $x^{i_x j_x}$ is a real number:*

$$p_x = \frac{j_x}{\sum_{z \in \mathcal{Z}} j_z}, \tag{1}$$

where j_z, with $z \in \mathcal{Z}$ are quality indexes of all realized options $z^{i_z j_z}$ in $\mathcal{P}$.

The *difference indicator* shows us how much the realization option of some task z affects the overall success of the realized project.

Predictability of a project success depends on the differences among indexes of realized options; the smaller differences, the bigger predictability. Any case, we get the highest quality project when all indexes of realized options are close to 1. Really, assume that all realized options are of the same quality indexes, that is to say $j_x = j_y$ for every pair of tasks $x, y \in \mathcal{Z}$ and therefore $p_x = \frac{1}{n_{\mathcal{Z}}}$, where $n_{\mathcal{Z}}$ is the number of tasks in $\mathcal{Z}$. Thus, difference indicators do not depend on separate option indexes but only on $n_{\mathcal{Z}}$. Having in mind that difference indicators differ from each other and depend on the index of each separate option, it is needed to equalize them, as much as it is possible.

Therefore, the sum of all indexes of realized options should be calculated. Since $\mathcal{Z}$ is partially ordered set and the number of tasks in a project can be very big, there is a problem of finding $\sum_{z\in\mathcal{Z}} j_z$. According to Theorem 2.1, it is possible to construct the finite sequence of all realized tasks options, i.e. the optional sequence. Next theorem trivially holds.

Theorem 2.5. *Let $\mathcal{Z}$ be a set of all tasks in a realized project $\mathcal{P}$ and $\beta = \mathbb{P}(\mathcal{Z})$, where $\mathbb{P}(\mathcal{Z})$ is the power set of $\mathcal{Z}$ and $P : \mathcal{Z} \to R$ defined as:*

$$(\forall x)(x \in \mathcal{Z} \to P(x) = p_x).$$

Then $(\mathcal{Z}, \beta, P)$ is a probability space.

Let $\mathcal{A} \subset \mathcal{Z}$ be set of tasks in a project, for example, the set of tasks defining a certain phase in the project, the probability $P(\mathcal{A})$ shows how much the realization of $\mathcal{A}$ affects the quality of the realized project.

Theorem 2.5 shows that each optional sequence generates a probability space, such that the probabilities of the sequence members are the corresponding difference indicators.

Definition 2.6. *Let $\mathcal{P}$ be the realized project and $x^{i_x j_x}$ a realized option of each task $x \in \mathcal{Z}$. We call $\mu = \max_{x\in\mathcal{Z}} p_x$ the measure of quality or failure of the realized project $\mathcal{P}$.*

Since $\mathcal{Z}$ is finite and $j_x \geq 1$ for every task $x \in \mathcal{Z}$, it holds that $p_x > 0$. Also, since every project $\mathcal{P}$ contains more than one task it will be $p_x < 1$. Also, we have that $\mu = \sup\{p_x\}_{x\in\mathcal{Z}} = \max\{p_x\}_{x\in\mathcal{Z}} < 1$.

Remark 2.7. *The more μ value is higher, the more options with a lower quality are realized. Hence, in order to increase the project quality we need to lower the measure of μ. So, next we define the process of the quality improvement by defining the process of lowering μ.*

The assignment of improving the project quality is not trivial. In order to explore this issue we establish next principles.

2.1. Reliability principle.

In order to improve the success of some project $\mathcal{P}$, taking into consideration the experience achieved in the similar projects should not necessarily be in the focus. It is because similar projects need not have the same tasks; even if they do, its realizations don't have to be obtained through the options as in the project $\mathcal{P}$. So, using the conclusions from the previous projects is not quite reliable. The reliability should be accomplished by focusing on the project $\mathcal{P}$, its realized tasks and options. Note that, by using this principle, we are trying avoid the experiences from other similar projects, since they were realized with different resources and in different project environments.

2.2. Revision principle.

Let us suppose that $\mathcal{P}$ is a completed project and that we determined its measure of failure μ. Next we substitute all unsatisfying tasks with new separate projects. Unsatisfying task would be every task whose realization has higher difference indicator than the initially given level. That way, we expect to obtain revised project with smaller μ, meaning a project with less disadvantages. We repeat the previous steps and construct revised projects as long as it takes, in order to obtain the project at satisfying level of μ. The main point is that we construct a sequence of dependent projects, where every project is the revision of

its previous one. This principle assumes that, at certain point we expect a project with failures bellow the initially given level.

Now, assume that for every task $x \in \mathcal{Z}$ there is at least one project $\mathcal{P}_x$ with the set of tasks $\mathcal{Z}_{\mathcal{P}_x}$ such that for every task $y \in \mathcal{Z}_{\mathcal{P}_x}$ and all its options holds:

$$(\forall y)(y \in \mathcal{P}_x \rightarrow j_y < j_x). \tag{2}$$

If we substitute the realized task $x \in \mathcal{Z}$ in $\mathcal{P}$ with a set of such optional sequences in $\mathcal{P}_x$ we get to the revision of realized the project $\mathcal{P}$ generated by x and denoted by $\mathcal{P}^{\{x\}}$. If we denote the set of all tasks in $\mathcal{P}^{\{x\}}$ with $\mathcal{Z}_{\mathcal{P}^{\{x\}}}$ we will have $\mathcal{Z}_{\mathcal{P}^{\{x\}}} = (\mathcal{Z} - \{x\}) \cup \mathcal{Z}_{\mathcal{P}_x}$, where $\mathcal{Z}_{\mathcal{P}_x}$ is the set of all tasks in $\mathcal{P}_x$. The relation $\prec$ in $\mathcal{P}^{\{x\}}$ is the extension of $\prec$ from $\mathcal{P}$. Let $z \in \mathcal{P}$ and $y \in \mathcal{P}_x$. Then if $x \prec z$ then $y \prec z$. Also, if $z \prec x$ then it holds that $z \prec y$.

We presume also that the condition

$$\sum_{y \in \mathcal{Z}_{\mathcal{P}_x}} j_y \geq j_x \tag{3}$$

holds with no loss of generality.

Really, note that if $\sum_{y \in \mathcal{Z}_{\mathcal{P}_x}} j_y < j_x$, we may extend $\mathcal{Z}_{\mathcal{P}_x}$ with certain additional tasks with the same index 1 which will not influence the revision.

Theorem 2.8. *Assume that the conditions (2) and (3) holds. Let x be any task in $\mathcal{P}$ and let $\mathcal{P}_x = \{z_i\}_{i \in \{1,2,\ldots,k\}}$ be an optional sequence with indexes $j_{z_1}, j_{z_2}, \ldots, j_{z_k}$. Then $p_{z_i} < p_x$, for $i = 1, 2, \ldots, k$.*

This theorem shows that the revised project has higher quality then the previously realized one. Please see the proof in the Appendix.

Theorem 2.9. *Let $\{x_n\}_{n \in N}$ be any sequence of tasks such that $x_1 = x$ is in the project $\mathcal{P}$, $x_2 \in \mathcal{Z}_{\mathcal{P}_{x_1}}, \ldots, x_{n+1} \in \mathcal{Z}_{\mathcal{P}_{x_n}}, \ldots$. Then for every real number $\epsilon > 0, \epsilon \in R$ there is $n(\epsilon) \in N$ such that for every $n > n(\epsilon)$ holds that $p_{x_n} < \epsilon$.*

The proof of Theorem 2.10 is given in the Appendix.

Remark 2.10. *According to Theorem 2.8., we can conclude that every difference indicator, after certain number of revisions, may be lowered bellow the initially given level ϵ, $\epsilon \in R$. If we replace the given task with a set of new tasks satisfying (2), and (3) in the same time we will by equalizing the quality indexes decrease their values. Note that, by lowering the difference indicator of $x \in \mathcal{P}$ we lower at the same time the difference indicators of non-revised tasks.*

Since difference indicators are probabilities, lowering the probabilities of certain tasks may cause the increase of some other tasks' probabilities. Therefore, naturally appears a question can we obtain, in certain point, a realized project with all difference indicators bellow the expected level ϵ.

In Theorems 3.9 and 3.11 in next Section we obtain positive answer to the above question by starting from some infinite probability space with the atomless probability measure. The optional sequence of a realized project generates certain decomposition of that probability space. The probabilities of such decomposition members are exactly the corresponding difference indicators. The revisions produces finer decompositions of the same space. So, the process of project quality improvement may be presented as a sequence of decompositions, such that every decomposition is finer than its previous. According to Theorems 3.9 and 3.11, it is possible to achieve the decomposition with all member probabilities less than ϵ, meaning that it is possible to obtain a project with the measure of failure μ less than ϵ.

3. Probabilistic analysis of the model

In this section we generalize the notion of the optional sequence by defining *the optional sequence of real numbers*, in order to enable further researches on project success growth.

The set of tasks, as well as any optional sequence in the given project is always finite. During the project revision the number of tasks increases. In the process of quality improvement of a project the number of tasks can be increased infinitely. For this reason, we define random optional sequences which are, in general, infinite. In order to apply these results in further mathematical researches, we may assume that the members of those sequences are real numbers.

Definition 3.1. *Let (Ω, β, P) be a probability space, where Ω is the sample space of the elementary outcomes, β is the σ-field of the events and P is the probability measure defined on β space of events. Let ϕ be a random variable $\phi : \Omega \to N \subset R$ and $a = \{a_n\}_{n\in N}$ a sequence of reals, such that $a : N \to R$, then the random variable $X = a \circ \phi$ is the optional random sequence of real numbers and we denote it by $\{a_n^{\phi}\}_{n\in N}$. For the given member of the optional random sequence $\{a_n^{\phi}\}_{n\in N}$, $a_{n_0}^{\phi}$, we define the probability of its appearance with $P(\phi^{-1}(n_0))$.*

Directly, based on Theorem 2.5., we conclude that every optional sequence in realized project may be identified with the optional sequence of real numbers, where the probability space is the set $\mathcal{Z}$ and the members of the sequence are the quality indexes of the realized options.

Remark 3.2. *In the denotation of a random optional sequence there appears a random variable which defines it. So, it can undoubtedly be clear if it is a random optional sequence or a sequence of reals. For that reason, when there is no confusion we will use the term sequence in both cases.*

It is easy to prove that every composition of the sequence of reals $a : N \to R$ and the random variable $\phi : \Omega \to N$ is a random variable.

The "randomness" of the given optional random sequence $\{a_n^{\phi}\}_{n\in N}$ is in the fact that all the members of the sequence appear with certain probabilities $P(\phi^{-1}(n_0))$. For the given index $n_0 \in \phi(\Omega)$, $\phi^{-1}(n_0)$ is measurable, so there exists its probability $P(\phi^{-1}(n_0))$. In that sense, we say that the probability of appearance of a member of the sequence $\{a_n^{\phi}\}_{n\in N}$, $a_{n_0}^{\phi}$, in the above definition is in fact the probability $P(\phi^{-1}(n_0))$ of index $n_0 \in N$ of the sequence $\{a_n\}_{n\in N}$.

The probability attached to any member of the sequence $\{a_n^{\phi}\}_{n\in N}$ depends on the set $\phi(\Omega) \cap N$, so it depends on Ω and ϕ. If index $n \notin \phi(\Omega)$ then $\phi^{-1}(\{n\}) = \emptyset$, so it is $P(\phi^{-1}(\{n\})) = 0$. We conclude that for all the indexes $n \in N$ in the set $N \backslash \phi(\Omega)$, corresponding members of the sequence $\{a_n^{\phi}\}_{n\in N}$ appear with the probability 0.

$\mathcal{D} = \{\phi^{-1}(n)\}_{n\in\phi(\Omega)}$ is the decomposition of the Ω into mutually disjoint sets. The members of the decomposition are measurable. According to $\sigma-$ additivity we have:

$$P(\bigcup_{n\in N} \phi^{-1}(n)) = \sum_{n\in N} P(\phi^{-1}(n)) = P(\Omega) = 1. \tag{4}$$

Example 3.3. *If the random variable ϕ is a constant, for example $\phi(\omega) = n_0 \in N$, for every $\omega \in \Omega$, then the member $a_{n_0}^{\phi}$ of the sequence $\{a_n^{\phi}\}_{n\in N}$ appears with the probability $P(a_{n_0}) = 1$ and all the rest members are observed with the probability 0.*

Example 3.4. *Let $\sum_{n=1}^{+\infty} b_n = B$ be a series of reals with positive members. If we put $\frac{b_n}{B} = p_n$, then for any set $A \subset N$, we define $P(A) = \sum_{n_k\in A} p_{n_k}$. Obviously, $P(A)$ exists for every $A \subset N$, since it is the limit of monotony increasing sequence upper bounded with number 1. This way, we obtain the probability space $(N, P(N), P)$.*

Remark 3.5. *Identical mapping $Id_N : N \to N$ is a random variable, so the optional random sequence $\{a_n^{id_N}\}_{n\in N}$ is in the same time an ordinary sequence of reals $a = \{a_n\}_{n\in N}$, where $a : (N, P(N), P) \to R$ and $a = X$.*

Let P be the probability measure on Ω. The set $A \subset \Omega$ is an *atom of P* if $P(A) > 0$ and if for every $X \subset A$ we have either $P(X) = 0$ or $P(X) = P(A)$.

A measure P is *atomless* if it has no atoms. If measure P is atomless, then every set X of positive measure can be split into two disjoint sets of positive measure (Jech [6]).

Example 3.6. *Lebesgue measure on the interval $\Omega = [0, 1]$ is atomless probability measure (see Halmos [4]).*

Next proposition (see Appendix for the proof) presents an interesting example of the optional random sequence.

Proposition 3.7. *Let us suppose that P is two-valued measure on the set Ω and $\mathcal{U}$ corresponding $\sigma-$complete ultrafilter. Then for every optional random sequence $\{a_n^{\phi}\}_{n\in N}$ there exists only one index $n_0 \in N$ such that $P(a_{n_0}^{\phi}) = 1$.*

Assume that (Ω, β, P) is the atomless probability space and $\mathcal{P}$ a project with the set of tasks $\mathcal{Z}$. Since $\mathcal{Z}$ is finite, for every realized option $x^{i_x j_x}$, there is the member Γ_x of the decomposition $\mathcal{D}$ such that $P(\Gamma_x)$ equals p_x.

Similarly, note that the sequence of project revisions generates the corresponding sequence of decompositions of the given atomless probability space (Ω, β, P). The results of following theorems reduce the problem of project success into existence of special sequence of decompositions. Next theorems generalize both the Theorem 2.9. and the Remark 2.10.

Theorem 3.8. *Let (Ω, β, P) be the probability space, where Ω is the infinite set, P atomless probability measure on Ω and $\{a_n\}_{n\in N}$ any sequence of reals. Then for every real number $\epsilon > 0, \epsilon \in R$ and some fixed member a_{n_0} of the given sequence $\{a_n\}_{n\in N}$, there exists a random variable $\psi : \Omega \to N$ such that $P(a_{n_0}^{\psi}) < \epsilon$.*

Theorem 3.9. *Let (Ω, β, P) be the probability space, where Ω is the infinite set and P atomless probability measure on Ω. For every real number $\epsilon > 0, \epsilon \in R$ there exists some decomposition of the set Ω into the disjoint subsets $\mathcal{D}_{\mathcal{M}}$ such that $\cup\mathcal{D}_{\mathcal{M}} = \Omega$ and $P(\Gamma) < \epsilon$, for every $\Gamma \in \mathcal{D}_{\mathcal{M}}$.*

Theorem 3.10. *Let us suppose that (Ω, β, P) is a probability space, P atomless measure and $\epsilon > 0, \epsilon \in R$. Then, every decomposition $\mathcal{D} = \{\Gamma_\lambda\}_{\lambda<\tau}$ of a set Ω with $0 < P(\Gamma_\lambda) < \epsilon$, for $\lambda < \tau$, has cardinality $|\mathcal{D}| \leq 2^{\aleph_0}$, where $2^{\aleph_0}$ is cardinality of real line.*

Theorem 3.11. *Assume that every decomposition of the set Ω into the disjoint sets of positive measure is at most countable, in some probability space (Ω, β, P), where P is atomless measure on Ω and $\{a_n\}_{n\in N}$ any sequence of real numbers. Then, for every real number $\epsilon > 0$ exists some random variable $\psi : \Omega \to N$, such that for any member a_n^{ψ} of the optional random sequence $\{a_n^{\psi}\}_{n\in N}$, $P(a_n^{\psi}) < \epsilon$.*

The proofs of the above theorems are in the Appendix.

Definition 3.12. *Let $\{a_n^{\psi}\}_{n\in N}$ be any optional random sequence. The member $a_{n_0}^{\psi}$ is a missing member of order $\epsilon > 0, \epsilon \in R$ if $P(a_{n_0}^{\psi}) < \epsilon$. For a member $a_{n_0}^{\psi}$ we simply say that it is missing member of the sequence $\{a_n^{\psi}\}_{n\in N}$, if $P(a_{n_0}^{\psi}) = 0$.*

Theorem 3.13. *Missing members of the sequence $\{a_n^{\psi}\}_{n\in N}$ of order $\epsilon, \epsilon \in R$ form a subsequence of the given optional random sequence.*

The proof of Theorem 3.13 is straightforward, see the Appendix.

Remark 3.14. *In terms of the Definition 3.12 and Theorem 3.13, we related the notion of an optional random sequence with the notion of a missing data sequence, which may be applied for the incomplete sample problems.*

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4. APPENDIX

Proof. [Proof of Theorem 2.1] Here we choose the construction of a sequence, best preserving the dependency of the tasks. Firstly, we construct the function $f : \mathcal{Z} \to \{1, 2, ..., n_{\mathcal{Z}}\} \subset N$, where $n_{\mathcal{Z}}$ is the number of all tasks in the project $\mathcal{P}$. This can be accomplished in various ways. For example, since partially ordered set $\mathcal{Z}$ is finite, there is some maximal chain l_i. This chain is linearly ordered with minimal element x_m and for each contained element there is its follower. Therefore, there is a strictly monotone function $f_i : l_i \to N$ such that $f_i(x_m) = 1$. For $f_i(x) = k$ and y is a follower of x in l_i in relation $\preceq$ we define $f_i(y) = k + 1$. This way, we identify l_i with the initial portion $\{1, 2, ..., n_i\} \subset N$, where n_i is the number of tasks in the chain l_i. In the set $\mathcal{Z} - l_i$ which is also partially ordered with the relation $\preceq$ there is some maximal chain l_j. There is a strictly monotone function $f_j : l_j \to \{n_i + 1, n_i + 2, ..., n_j\} \subset N$. We choose f_j such that $f_i(l_i) \cap f_j(l_j) = \emptyset$. Next we continue with the procedure until we exploit the whole set $\mathcal{Z}$. Since $\mathcal{Z} = \sum_{i \in I} l_i$, there is a function $f : \mathcal{Z} \to N_{\mathcal{Z}} = \{1, 2, ..., n_{\mathcal{Z}}\} \subset N$ such that $f_i = f \mid_{l_i}$. f is a bijection, so there is an inverse function which represents a finite sequence of all elements in $\mathcal{Z}$. □

Proof. [Proof of Theorem 2.8] Really, let x be a task in a project $\mathcal{P}$. For every $z_i \in \mathcal{P}_x$, $i = 1, 2, ..., k$ we have:

$$p_{z_i} = \frac{j_{z_i}}{j_{z_1} + j_{z_2} + ... + j_{z_k} + \sum_{z \neq x} j_z} = \frac{j_{z_i}}{j_{z_1} + j_{z_2} + ... + j_{z_k} + \sum_{z \in \mathcal{Z}} j_z - j_x}.$$

According to the assumptions of the theorem, it holds that:

$$j_{z_1} + j_{z_2} + ... + j_{z_k} + \sum_{z \in \mathcal{Z}} j_z - j_x > \sum_{z \in \mathcal{Z}} j_z,$$

that is to say:

$$\frac{j_{z_i}}{j_{z_1} + j_{z_2} + ... + j_{z_k} + \sum_{z \in \mathcal{Z}} j_z - j_x} < \frac{j_{z_i}}{\sum_{z \in \mathcal{Z}} j_z} < p_x,$$

because of the fact that $j_{z_i} < j_x$ for $i = 1, 2, ..., k$. □

Proof. [Proof of Theorem 2.9] According to Theorem 2.4, $\{p_{x_n}\}_{n \in N}$ is a decreasing sequence of reals. Let us prove that this is a zero sequence. Really, since (2) holds we may conclude that:

$$\sum_{z \in \mathcal{Z}_{\mathcal{P}_x}} j_z \geq j_x + 1 \tag{5}$$

and therefore:

$$\begin{aligned} p_{x_2} &= \frac{j_{x_2}}{\sum_{z \in \mathcal{Z}_{\mathcal{P}_x}} j_z + \sum_{z \in \mathcal{Z}, z \neq x} j_z} \\ &= \frac{j_{x_2}}{\sum_{z \in \mathcal{Z}_{\mathcal{P}_x}} j_z + \sum_{z \in \mathcal{Z}} j_z - j_x} \\ &\leq \frac{j_{x_2}}{\sum_{z \in \mathcal{Z}} j_z + 1}. \end{aligned}$$

Similarly,

$$p_{x_3} = \frac{j_{x_3}}{\sum_{z \in \mathcal{Z}_{\mathcal{P}_{x_2}}} j_z + \sum_{z \in \mathcal{Z}_{\mathcal{P}_x}, z \neq x_2} j_z + \sum_{z \in \mathcal{Z}, z \neq x} j_z}.$$

Since $\sum_{z \in \mathcal{Z}_{\mathcal{P}_{x_2}}} j_z \geq 1 + j_{x_2}$ and (4) hold, we conclude that:

$$p_{x_3} \leq \frac{j_{x_3}}{2 + \sum_{z \in \mathcal{Z}} j_z}. \tag{6}$$

Following previous steps, we obtain that for every $n \in N$

$$p_{x_n} \leq \frac{j_{x_n}}{(n-1)+\sum_{z\in\mathcal{Z}} j_z} < \frac{100}{(n-1)+\sum_{z\in\mathcal{Z}} j_z}, \tag{7}$$

which obviously completes the proof. □

Proof. [Proof of Proposition 3.7] The collection $\{\phi^{-1}(n)\}_{n\in N}$ is a countable decomposition of the set Ω. Also, the ultrafilter $\mathcal{U}$ is $\sigma-$complete, so there must be at least one member $\phi^{-1}(n_0)$ of the decomposition that belongs to the ultrafilter $\mathcal{U}$. Then it holds:

$$P(\phi^{-1}(n_0)) = 1,$$

that is, $P(a^{\phi}_{n_0}) = 1$. For every index $n \neq n_0$ must be $P(\phi^{-1}(n)) = 0$. Really, all the members of the decomposition $\{\phi^{-1}(n)\}_{n\neq n_0}$ are measurable in the probability space $(\Omega, P(\Omega), P)$, because neither of them doesn't belong to the ultrafilter $\mathcal{U}$. Therefore, it holds $P(\phi^{-1}(n)) = 0$, i.e. $P(a^{\phi}_{n}) = 0$ for all the indexes $n \neq n_0$. □

Proof. [Proof of Theorem 3.8] Really, let $\phi_1 : \Omega \to N$ be some random variable, $\{a^{\phi_1}_n\}_{n\in N}$ a corresponding optional random sequence, $\epsilon > 0$ and $n_0 \in N$ some fixed index.

It can be $P(a^{\phi_1}_{n_0}) < \epsilon$, so the theorem holds. If $P(a^{\phi_1}_{n_0}) \geq \epsilon$, note that $\phi_1^{-1}(n_0) = D_{n_0} \subset \Omega$. That set is the member of decomposition $\mathcal{D} = \{D_n\}_{n\in N}$ of the set Ω, where $\phi_1^{-1}(n) = D_n$ for every index $n \in \phi_1(\Omega)$.

Since D_{n_0} is a set of positive measure and the measure P is atomless, it can be decomposed into the disjoint union of two sets of positive measure, $D_{n_0} = A^1_{n_0} \cup B^1_{n_0}$, where $P(A^1_{n_0}) \leq P(B^1_{n_0})$.

Without loss of generality we can assume that the decomposition $\mathcal{D}$ has more than one member, so we can choose some $D_\lambda \in \mathcal{D}$, for $\lambda \neq n_0$.

Starting from decomposition $\mathcal{D}$ we can construct new decomposition $\mathcal{D}^1$ of the set Ω, where we can take the set $A^1_{n_0} = D^1_{n_0}$ instead of D_{n_0} and $D_\lambda \cup B^1_{n_0} = D^1_\lambda$ instead of D_λ. The rest of the members of the family $\mathcal{D}$ stay the same and we shall rename them into $D_n = D^1_n$, for indexes $n \neq \lambda, n_0$. Then it can be either $P(A^1_{n_0}) < \epsilon$ or $P(A^1_{n_0}) \geq \epsilon$.

If $P(A^1_{n_0}) < \epsilon$ the function $\phi_2 : \Omega \to N$ can be constructed such that it holds:

$$\phi_2|D_n = \phi_1|D_n, \tag{8}$$

for every index $n \neq \lambda, n_0$. For every element $\omega \in D_\lambda \cup B^1_{n_0}$ we may put $\phi_2(\omega) = \lambda$ and for every element $\omega \in A^1_{n_0}$ we may put $\phi_2(\omega) = n_0$. Also holds that $\phi_1(\Omega) = \phi_2(\Omega)$. Obviously ϕ_2 is a random variable so the theorem is proved.

If $P(A^1_{n_0}) \geq \epsilon$, then let us note the set $\phi_2^{-1}(n_0) = D^1_{n_0} \subset \Omega$. This set is a member of the decomposition $\mathcal{D}^1 = \{D^1_n\}_{n\in N}$ of the set Ω, where $\phi_2^{-1}(n) = D^1_n$ for every $n \in N$.

As a set of positive measure, $D^1_{n_0}$ can be decomposed into the disjoint union of two sets $D^1_{n_0} = A^2_{n_0} \cup B^2_{n_0}$ of positive measure, such that $P(A^2_{n_0}) \leq P(B^2_{n_0})$, because the measure P is atomless. So, starting from decomposition $\mathcal{D}^1$ we may construct new decomposition $\mathcal{D}^2$ of the set Ω, indexed as decomposition $\mathcal{D}^1$ with $\phi^1(\Omega)$. Instead of the set $D^1_{n_0}$ we take the set $A^2_{n_0} = D^2_{n_0}$ and instead of the set D^1_λ set $D^1_\lambda \cup B^2_{n_0} = D^2_\lambda$. The rest of the members of the family $\mathcal{D}^1$ remain the same and we rename them with $D^1_n = D^2_n$, for indexes $n \neq \lambda, n_0$.

Then it could be either $P(A^2_{n_0}) < \epsilon$ or $P(A^2_{n_0}) \geq \epsilon$. If $P(A^2_{n_0}) < \epsilon$ we may construct the function $\phi_3 : \Omega \to N$ such that:

$$\phi_3|D_n = \phi_2|D_n, \tag{9}$$

for every $n \neq \lambda, n_0$. For every element $\omega \in D^1_\lambda \cup B^2_{n_0}$ we put $\phi_3(\omega) = \lambda$ and for every element $\omega \in A^2_{n_0}$ we put $\phi_3(\omega) = n_0$. Obviously ϕ_3 is a random variable so we have the proof.

If $P(A^2_{n_0}) \geq \epsilon$ we continue with the procedure. That way, we obtain the sequence of the decompositions:

$$\mathcal{D}, \mathcal{D}^1, \mathcal{D}^2, \mathcal{D}^3, ..., \mathcal{D}^m, ...,$$

decreasing sequence of the sets:

$$D_{n_0} \supset D^1_{n_0} \supset D^2_{n_0} \supset .... \supset D^m_{n_0} \supset ...,$$

increasing sequence of the sets:

$$D_\lambda \subset D^1_\lambda \subset D^2_\lambda \subset .... \subset D^m_\lambda \subset ...,$$

strictly decreasing sequence of numbers:

$$P(D_{n_0}) > P(D^1_{n_0}) > P(D^2_{n_0}) > .... > P(D^m_{n_0}) > ...$$

and strictly increasing sequence of numbers:

$$P(D_\lambda) < P(D^1_\lambda) < P(D^2_\lambda) < .... < P(D^m_\lambda) < ....$$

Strictly decreasing sequence $\{P(D^m_{n_0})\}_{m\in N}$ is convergent because it is lower bounded with 0. Since $P(D^{k+1}_{n_0}) \leq \frac{1}{2}P(D^k_{n_0})$ for every $k \in N$, it holds:

$$\lim_{m\to+\infty} P(D^m_{n_0}) = 0. \tag{10}$$

Therefore, for given $\epsilon > 0$ there exists an index of that sequence $m(\epsilon)$ such that for every index $m > m(\epsilon)$ is $P(D^m_{n_0}) < \epsilon$. If we put $\phi_{m(\epsilon)+1} = \psi$, then $\{a^\psi_n\}_{n\in N}$ is a random sequence, with $P(a^\psi_{n_0}) < \epsilon$. This fact completes the proof. □

Proof. [Proof of Theorem 3.9] Since Ω is a set of positive measure and the measure P is atomless it can be decomposed into the disjoint union of the sets $\Omega = A_1 \cup B_1$ of positive measure, such that $P(A_1) \leq P(B_1)$, that is

$$P(A_1) \leq \frac{1}{2}P(\Omega) = \frac{1}{2}. \tag{11}$$

Since A_1 is a set of of positive measure, it can be decomposed into the disjoint union: $A_1 = A_2 \cup B_2$ of two sets of positive measure such that $P(A_2) \leq P(B_2)$ and

$$P(A_2) \leq \frac{1}{2^2}P(\Omega) = \frac{1}{2^2}.$$

If we continue this procedure, we obtain the sequence:

$$A_1 \supset A_2 \supset A_3 \supset .... \supset A_n \supset ...$$

satisfying:

$$P(A_n) \leq \frac{1}{2^n},$$

for every index $n \in N$, and therefore:

$$\lim_{n\to\infty} P(A_n) = 0.$$

So, for some $\epsilon > 0$ there exists an index of the sequence $m_1(\epsilon)$ such that:

$$P(A_{m_1(\epsilon)}) < \epsilon.$$

Let us put:

$$A_{m_1(\epsilon)} = \Gamma_1.$$

Let us note the set $\Omega - \Gamma_1$ and then apply the above procedure of dividing this set similarly as we did with the set Ω. After finite number $m_2(\epsilon)$ steps we obtain the set Γ_2 satisfying

$$P(\Gamma_2) < \epsilon.$$

Now we apply the procedure on the set $\Omega - (\Gamma_1 \cup \Gamma_2)$ and after $m_3(\epsilon)$ steps we obtain the set Γ_3 holding that

$$P(\Gamma_3) < \epsilon,$$

and so on.

That way, we obtain the collection of disjoint subsets $\{\Gamma_n\}_{n\in N}$ of the set Ω satisfying $P(\Gamma_n) < \epsilon$, for every $n \in N$.

The set Γ_1 is a member of disjoint collections:

$$\{\Gamma_1, \Gamma_2\}, \{\Gamma_1, \Gamma_2, \Gamma_3\}, ...$$

It is easy to see that these disjoint collections obtained by the construction of the starting set Γ_1 are not uniquely determined. Without loss of generality, we will consider one member-collection $\{\Gamma_1\}$ as a trivial collection of disjoint sets.

Let $\mathcal{M}$ be a class of every possible collections of the disjoint sets that may be obtained by the above procedure starting from the set Γ_1.

For two collections $\mathcal{D}_1$ and $\mathcal{D}_2$ we say that $\mathcal{D}_1 \leq \mathcal{D}_2$ if $\mathcal{D}_1 \subset \mathcal{D}_2$. Obviously, $\mathcal{M}$ is partially ordered with the relation $\leq$. Note any chain $\mathcal{L}$ such that $\{\Gamma_1\} \in \mathcal{L}$ and $\mathcal{L} \subset \mathcal{M}$. Let us prove that the chain $\mathcal{L}$ has an upper bound.

If the length of the chain is successor ordinal $\alpha = \beta + 1$, it mins $\mathcal{L} = \{D_\lambda\}_{\lambda\leq\alpha}$, then it's upper bounded with its last element D_α. Otherwise, let $\mathcal{L} = \{D_\lambda\}_{\lambda<\alpha}$ be a chain in the set $\mathcal{M}$, where α is limit ordinal. Now, let us note the set $\bigcup_{\lambda<\alpha} D_\lambda$. This set is the upper bound of the chain $\mathcal{L}$. Really, if $\Gamma \in \bigcup_{\lambda<\alpha} D_\lambda$, there exists an index $\lambda < \alpha$ such that $\Gamma \in D_\lambda$, so we have that $P(\Gamma) < \epsilon$. Also, the set $\bigcup_{\lambda<\alpha} D_\lambda$ is the collection of mutually disjoint subsets of the set Ω. So, if $\Gamma, \Theta \in \bigcup_{\lambda<\alpha} D_\lambda$ there exists some sets D_{λ_1} and D_{λ_2} such that $\Gamma \in D_{\lambda_1}$ and $\Theta \in D_{\lambda_2}$. Since $\mathcal{L}$ is a chain there will be, for example, $D_{\lambda_1} \subset D_{\lambda_2}$, so $\Gamma, \Theta \in D_{\lambda_2}$. Therefore, Γ and Θ are disjoint sets with the probability measures less then ϵ and $\bigcup_{\lambda<\alpha} D_\lambda$ is an upper bound of the chain $\mathcal{L}$. According to Kuratovski-Zorn Lemma (see for example [12]), we conclude that $\mathcal{L}$ is contained in the chain of maximal length having the maximal element in $\mathcal{M}$.

Let $\mathcal{D}_\mathcal{M}$ be that maximal element. Now put:

$$\mathcal{D}_\mathcal{M} = \{\Gamma_\lambda\}_{\lambda<\tau},$$

where τ is some ordinal. Let us prove that:

$$\bigcup \{\Gamma_\lambda\}_{\lambda<\tau} = \Omega.$$

Suppose the contrary, that

$$\Omega - \bigcup \{\Gamma_\lambda\}_{\lambda<\tau} = M \neq \emptyset.$$

If $P(M) = 0$, then $P(M) < \epsilon$ and $M \cap \Gamma_\lambda = \emptyset$. In this case $\mathcal{D}_\mathcal{M} \cup \{M\}$ is the decomposition satisfying $\mathcal{D}_\mathcal{M} < \mathcal{D}_\mathcal{M} \cup \{M\}$, which is the contradiction with the fact that $\mathcal{D}_\mathcal{M}$ is a maximal element.

If $P(M) \neq 0$ and $P(M) < \epsilon$, we have that $\mathcal{D}_\mathcal{M} \cup \{M\}$ is the decomposition satisfying $\mathcal{D}_\mathcal{M} < \mathcal{D}_\mathcal{M} \cup \{M\}$, which also contradicts with the fact that $\mathcal{D}_\mathcal{M}$ is a maximal element.

Finally, let us suppose that $P(M) \geq \epsilon$. In that case, we may obtain the set $\Gamma^* \subset M$ such that $P(\Gamma^*) < \epsilon$, starting from the set M similarly as we did in the case of the set Ω. We have that:

$$\mathcal{D}_\mathcal{M} < \mathcal{D}_\mathcal{M} \cup \{\Gamma^*\},$$

which also contradicts with the fact that $\mathcal{D}_\mathcal{M}$ is a maximal element. So it holds:

$$\bigcup \{\Gamma_\lambda\}_{\lambda<\tau} = \Omega, \tag{12}$$

which completes the proof. □

Proof. [Proof of Theorem 3.10] Let us put $\mathcal{D} = \{\Gamma_\lambda\}_{\lambda<\tau}$. Starting from the decomposition $\mathcal{D}$ of the set Ω we construct a decomposition $\mathcal{D}^*$ of the set Ω. Denote with $K \subset (0,1]$ the set of real numbers α satisfying that there exists at least one member of the decomposition $\mathcal{D}$ with the probability α. Let $\Gamma^*_\alpha = \cup\{\Gamma_\lambda | P(\Gamma_\lambda) = \alpha\}$. The set Γ^*_α is the union of the most finite many sets from the decomposition $\mathcal{D}$, because the probability measure P is normed and $\sigma-$additive. New decomposition $\mathcal{D}^* = \{\Gamma^*_\alpha\}_{\alpha\in K\subset(0,1]}$ is indexed with the set K with cardinality $|K| \leq 2^{\aleph_0}$. Since every member of the decomposition $\mathcal{D}^*$ is the union of finitely many members of the decomposition $\mathcal{D}$ we have:

$$|\mathcal{D}| \leq \aleph_0 \cdot |K| \leq \aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0},$$

which completes the proof. □

Proof. [Proof of Theorem 3.11] According to the Theorem 3.10, for the probability space (Ω, β, P) there exists its decomposition into the disjoint sets $\mathcal{D} = \{\Gamma_\lambda\}_{\lambda<\tau}$, such that for every index $\lambda < \tau$ holds $P(\Gamma_\lambda) < \epsilon$. According to the assumptions of the theorem, the decomposition $\mathcal{D}$ is at most countable. Therefore, the decomposition $\mathcal{D}$ may be presented as finite or infinite sequence $\mathcal{D} = \{\Gamma_m\}_{m\in K\subset N}$. Every element $\omega \in \Omega$ belongs to exactly one member of the decomposition $\mathcal{D}$. So, there exists the function $\psi : \Omega \to N$ such that for every $\omega \in \Gamma_m \subset \Omega$ is $\psi(\omega) = m \in N$ and $\psi^{-1}(m) \in \mathcal{D}$, for every element $m \in \psi(\Omega) \subset N$. It is easy to prove that the function ψ is a random variable. It is obvious that, for a given sequence $\{a_n\}_{n\in N}$ of real numbers, $\{a^\psi_n\}_{n\in N}$ is a random sequence satisfying $P(a^\psi_n) < \epsilon$ for every $n \in N$, which completes the proof. □

Proof. [Proof of Theorem 3.13] There exists only finite number of the members of optional random sequence $\{a^\psi_n\}_{n\in N}$, satisfying $P(a^\psi_n) \geq \epsilon$. Really, for a given number ϵ holds $m \cdot \epsilon > 1$, if it is $m > [\frac{1}{\epsilon}]$, where $[\frac{1}{\epsilon}]$ is the integral part of $\frac{1}{\epsilon}$. So, the number m of the not missing members of order ϵ can not be higher $[\frac{1}{\epsilon}]$. It means that the rest of members of the optional random sequence form a subsequence. □

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