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The First Two Cacti with Larger Multiplicative Eccentricity Resistance-Distance

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Abstract. For a connected graph G, the multiplicative eccentricity resistance-distance $\xi_R^*(G)$ is defined as $\xi_R^*(G) = \sum_{\{x,y\}\subseteq V(G)} \varepsilon(x) \cdot \varepsilon(y) R_G(x, y)$, where $\varepsilon(\cdot)$ is the eccentricity of the corresponding vertex and $R_G(x, y)$ is the effective resistance between vertices x and y. A cactus is a connected graph in which any two simple cycles have at most one vertex in common. Let $Cat(n; t)$ be the set of cacti possessing n vertices and t cycles, where $0 \le t \le \frac{n-1}{2}$. In this paper, we first introduce some edge-grafting transformations which will increase $\xi_R^*(G)$. As their applications, the extremal graphs with maximum and second-maximum $\xi_R^*(G)$-value in $Cat(n; t)$ are characterized, respectively.

1. Introduction

All graphs considered in this paper are simple and connected. Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G. For graph-theoretical terms that are not defined here, we refer to Bollobás's book [1].

Let P_n, C_n and S_n be the path, the cycle and the star on n vertices, respectively. A *cactus* is a connected graph in which any two simple cycles have at most one vertex in common. Equivalently, every edge in such a graph belongs to at most one cycle. Denote by $Cat(n; t)$ the set of cacti possessing n vertices and t cycles, where $0 \le t \le \frac{n-1}{2}$. A cycle C of G is said to be an end cycle at v if v is the unique vertex in C which is adjacent to a vertex in $V_G \setminus V_C$. Let G be a graph in $Cat(n; t)$, this unique vertex v in C is called the anchor of C. Let $d_G(v)$ (for simplicity, $d(v)$) be the degree of v in G. For a path $P_k = v_1 v_2 \dots v_k (k \ge 2)$ with $d(v_1) \ge 3$ in G, it is called a pendent path if $d(v_k) = 1$, and a internal path if $d(v_k) \ge 3$. If $E_0 \subset E_G$, we denote by $G - E_0$ the subgraph of G obtained by deleting the edges in E_0. If E_1 is the subset of the edge set of the complement of G, $G + E_1$ denotes the graph obtained from G by adding the edges in E_1. Similarly, if $W \subset V_G$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them. For simplicity, we write $G - xy$, $G + xy$ and $G - x$ instead of $G - \{xy\}$, $G + \{xy\}$ and $G - \{x\}$.

The *distance* $d_G(x, y)$ between two vertices x and y is defined as the length of a shortest (x, y)-path in G. The *eccentricity* $\varepsilon_G(x)$ (for simplicity, $\varepsilon(x)$) of a vertex x is the distance between x and a furthest vertex from

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x. The resistance distance $r_G(x, y)$ is defined to be equal to the effective resistance between vertices x and y of graph G, with unit resistors taken over any edge of G. Although the distance has great important effect on many problems with respect to graphs, the use of shortest path has some obvious drawbacks. In many cases, shortest paths form a small subset of all paths between two vertices; it follows that paths even slightly longer than the shortest one are not considered at all in the studying of some problems. Furthermore, the distance between the vertices does not consider the actual number of (shortest) paths that lie among the two vertices: two vertices that are separated by a single path have the same distance of two vertices that are separated by many paths of the same length. To overcome these limitations, the resistance distance is obvious an alternative choice. Based on this consideration, there is a family of resistive descriptors $F(G)$ proposed, with the general formula

$$F(G) = \sum_{\{x,y\}\subseteq V_G} f_G(x, y)R_G(x, y) \quad (1)$$

where $R_G(x, y)$ is the effective resistance between vertices x and y, $f_G(x, y)$ is some real function on V_G.

If $f_G(x, y) \equiv 1$, it is the well-known Kirchhoff index $Kf(G)$, proposed by D. klein and M. Randić in [10]. This index has very nice purely mathematical and physical interpretations (for example, see [20]), and has been investigated extensively in mathematical, physical and chemical literatures, for more detail information the readers are referred to recent papers [21] and references therein. In [2], H. Chen and F. Zhang introduced naturally an index $R^*(G)$ named *multiplicative degree-Kirchhoff index* from the relations between resistance distance and the normalized Laplacian spectrum. It is defined exactly by (1.1) when taking $f_G(x, y) = d_G(x) \cdot d_G(y)$. Comparing with this index, I. Gutman, L. Feng and G. Yu [4] proposed the *additive degree-Kirchhoff index* $R^+(G)$ which can also be obtained by letting $f_G(x, y) = d_G(x)+d_G(y)$ in (1.1). S. Li and W. Wei [11] defined the *eccentricity resistance-distance sum* $\xi^R(G)$ from (1.1) by taking $f_G(x, y) = \varepsilon(x) + \varepsilon(y)$. Some mathematical properties and extremal problems on $\xi^R(G)$ are considered. Some interested properties and relations among these Kirchhoffian indices are obtained, see [8, 9, 13, 14] and references therein. Motivated by these works above, we defined a new index $\xi^*_R(G)$ [7] from (1.1) by taking $f_G(x, y) = \varepsilon(x) \cdot \varepsilon(y)$, name it as *multiplicative eccentricity resistance-distance*, and some mathematical properties on $\xi^*_R(G)$ were studied, as an application, the extremal graphs with minimum and second minimum $\xi^*_R(G)$-value in $Cat(n; t)$ were characterized. In this paper, we will further study some mathematical properties of $\xi^*_R(G)$ and their applications.

The following results are useful for our main results. For convenience, let $Kf_v(G) = \sum_{u\in V_G} R_G(u, v)$.

Lemma 1.1. *[10] Let C_k be a cycle with length k and $v \in V_{C_k}$. Then $Kf(C_k) = \frac{k^3-k}{12}$, $Kf_v(C_k) = \frac{k^2-1}{6}$.*

Lemma 1.2. *[17] Let G be a connected graph with a cut-vertex v such that G_1 and G_2 are two connected subgraphs of G having v as the only common vertex and $V_{G_1} \cup V_{G_2} = V_G$. Let $n_1 = |V_{G_1}| - 1$, $n_2 = |_{G_2}| - 1$. Then $Kf(G) = Kf(G_1) + Kf(G_2) + n_1Kf_v(G_1) + n_2Kf_v(G_2)$.*

2. Some edge-grafting transformations increased $\xi^*_R(G)$

In this section, we introduce some edge-grafting transformations which are increased $\xi^*_R(G)$. For convenience, for any two vertices x, y of G (resp. G', G''), let $\varepsilon(x) = \varepsilon_G(x)$ (resp. $\varepsilon'(x) = \varepsilon_{G'}(x)$, $\varepsilon''(x) = \varepsilon_{G''}(x)$) and $R_{xy} = R_G(x, y)$ (resp. $R'_{xy} = R_{G'}(x, y)$, $R''_{xy} = R_{G''}(x, y)$).

Lemma 2.1. *Given a connected graph G with a cut vertex u and $d_G(u) \geq 3$. Let $P_1 = uu_1u_2 \cdots u_k$ and $P_2 = uv_1v_2 \cdots v_t (k \geq t)$ be two pendent paths attaching at u, and set $G' = G - v_{t-1}v_t + u_kv_t$ (as shown in Figure 1). Then $\xi^*_R(G) < \xi^*_R(G')$.*

Proof. Let $H = G - (V_{P_1} \cup V_{P_2} - u)$, $A = \{u_1, \cdots , u_k\}$, $B = \{v_1, v_2, \cdots , v_t\}$, $C = V_H$. For simplicity, let $d = \varepsilon_H(u)$.

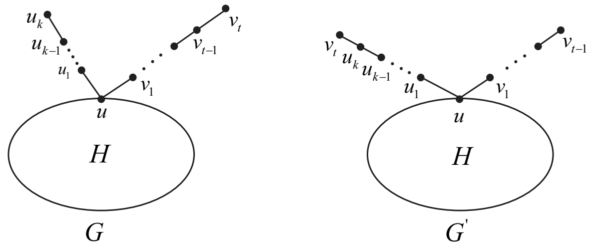

Figure 1: The graphs G and G' in Lemma 2.1.

Case 1. $d \geq t$. Note that $k \geq t$, one can get that

$$\begin{aligned}
\varepsilon'(v_t) &> \varepsilon(v_t),\ \ \varepsilon'(x) \geq \varepsilon(x), x \in V_G \setminus \{v_t\};\\
R'_{xy} &= R_{xy}\ \ x, y \in V_G \setminus \{v_t\},\ \ R'_{u_iv_t} = k - i + 1,\ \ R_{u_iv_t} = i + t\ (i \in \{1, 2, \cdots, k\});
\end{aligned}$$

$$\begin{aligned}
R'_{v_jv_t} &= k + j + 1,\ \ R_{v_jv_t} = t - j\ \ (j \in \{1, 2, \cdots, t-1\});\\
R'_{xv_t} &= R_{xu} + k + 1,\ \ R_{xv_t} = R_{xu} + t,\ \ x \in C.
\end{aligned}$$

It follows that

$$\xi_1 = \sum_{\{x,y\} \subseteq V_G \setminus \{v_t\}} [\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] \geq 0.$$

Note that $\varepsilon'(x)\varepsilon'(v_t) > \varepsilon(x)\varepsilon(v_t)$ for $x \in V_G \setminus \{v_t\}$. Hence

$$\begin{aligned}
\xi_2 &= (\sum_{x\in A} + \sum_{x \in B\setminus\{v_t\}} + \sum_{x\in C})[\varepsilon'(x)\varepsilon'(v_t)R_{xv_t} - \varepsilon(x)\varepsilon'(v_t)R'_{xv_t}]\\
&> (\sum_{x\in A} + \sum_{x \in B\setminus\{v_t\}} + \sum_{x\in C})[\varepsilon(x)\varepsilon(v_t)(R'_{xv_t} - R_{xv_t})]\\
&> d^2[(\sum_{x\in A} + \sum_{x \in B\setminus\{v_t\}} + \sum_{x\in C})(R'_{xv_t} - R_{xv_t})]\\
&= d^2[\sum_{i=1}^{k}(k - i + 1 - (i - t)) + \sum_{j=1}^{t-1}(k + j + 1 - (t - j)) + \sum_{x\in C}(k + 1 - t)]\\
&= d^2[(t - k - 1) + |C|(k - t + 1)] = d^2(|C| - 1)(k - t + 1) > 0.
\end{aligned}$$

Therefore, by the definition of $\xi_R^*(G)$. We get $\xi_R^*(G') - \xi_R^*(G) = \xi_1 + \xi_2 > 0$.

Case 2. $d < t$. Let $E = \{u_k, \cdots, u_1, u, v_1, \cdots, v_t\}, F = V_G \setminus E$, we have

$$\varepsilon'(x) = d(x, u) + k + 1, \varepsilon(x) = d(x, u) + k\ \ x \in F;\ \ R'_{xy} = R_{xy}\ \ x, y \in F\ \ or\ \ x \in F, y \in E \setminus \{v_t\}.$$

Then

$$\xi_3 = (\sum_{\{x,y\}\subseteq V_F} + \sum_{x\in F, y=v_t})[\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] > 0.$$

Let $P_3 = u_k \cdots u_1uv_1 \cdots v_t$ and $P_4 = v_tu_k \cdots u_1uv_1 \cdots v_{t-1}$. Obviously, $P_3 = P_4$. So we get

$$\xi_4 = \sum_{\{x,y\}\subseteq E} [\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] = \xi_R^*(P_4) - \xi_R^*(P_3) = 0.$$

Further, for $y \in E$, it is easy to see that $\varepsilon'(y) + 1 \geq \varepsilon(y)$. Then

$$\begin{aligned}
\xi_5 &= \sum_{x\in F, y\in E\setminus\{v_t\}} [\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] \\
&= \sum_{x\in F, y\in E\setminus\{v_t\}} [(d(x,u)+k+1)\varepsilon'(y) - (d(x,u)+k)\varepsilon(y)]R_{xy} \\
&= \sum_{x\in F, y\in E\setminus\{v_t\}} [(d(x,u)+k)(\varepsilon'(y)-\varepsilon(y)) + \varepsilon'(y)]R_{xy} \\
&= \sum_{x\in F, y\in A\setminus\{v_t\}} [\varepsilon(y) - (d(x,u)+k+1)]R_{xy} + \sum_{x\in F, y\in B\setminus\{v_t\}} [\varepsilon(y) + (d(x,u)+k+1)]R_{xy} \\
&\geq \sum_{x\in F, y\in A\setminus\{v_t\}} \sum_{i=1}^{k} [(t-1)+i-(d(x,u)+k+1)]R_{xy} \\
&\quad + \sum_{x\in F, y\in B\setminus\{v_t\}} \sum_{j=1}^{t-1} [(k+1)+j+(d(x,u)+k+1)]R_{xy} \\
&= \sum_{x\in F, y\in A\setminus\{v_t\}} [k(t-k-2) - d(x,u)k + \frac{(k+1)k}{2}]R_{xy} \\
&\quad + \sum_{x\in F, y\in B\setminus\{v_t\}} [(t-1)d(x,u) + 2(t-1)(k+1) + \frac{(t-1)t}{2}]R_{xy} \\
&= \sum_{x\in F, y\in E\setminus\{v_t\}} [(t-k-1)d(x,u) + 2(t-1)(k+1) - k(t-k-2) + \frac{(t-1)t+(k+1)k}{2}]R_{xy} \\
&> \sum_{x\in F, y\in E\setminus\{v_t\}} [(t-k-1)t + 2(t-1)(k+1) - k(t-k-2) + \frac{(t-1)t+(k+1)k}{2}]R_{xy} \\
&= \sum_{x\in F, y\in E\setminus\{v_t\}} [(t-k-1)t + 2(t-1)(k+1) - k(t-k-2) + \frac{(t-1)t+(k+1)k}{2}]R_{xy} \\
&= \sum_{x\in F, y\in E\setminus\{v_t\}} [\frac{3t^2+2t-4+3k^2+k}{2}]R_{xy} > \sum_{x\in F, y\in E\setminus\{v_t\}} [\frac{6t^2+3t-4}{2}]R_{xy} \\
&= \sum_{x\in F, y\in E\setminus\{v_t\}} [3t^2 + \frac{3}{2}t - 2]R_{xy} = \sum_{x\in F, y\in E\setminus\{v_t\}} [3(t+\frac{1}{4})^2 - \frac{35}{16}]R_{xy} > 0 \ \ (\textit{since}\ \ t \geq 1).
\end{aligned}$$

Therefore, $\xi^*_R(G') - \xi^*_R(G) = \xi_3 + \xi_4 + \xi_5 > 0$.

Combining Case 1 with Case 2, we have $\xi^*_R(G') > \xi^*_R(G)$. □

Given three disjoint connected graphs G_1, G_2 and a path $P = vv_1 \cdots v_{t-1}v_t$, let $u_1 \in V_{G_1}, u_2 \in V_{G_2}$. Suppose that H is the graph obtained from G_1 and G_2 by identifying u_1 and u_2 to u. We call this procedure an identification operation [18], which denoted by the formula $(H, u) = (G_1, u_1) \oplus (G_2, u_2)$. Let G be a connected graph constructed by identifying v in P and u in H, that is, $(G, u) = (H, u) \oplus (P, v)$. Let G' and G'' be the graphs formed by the identification operation as follows:

$$(G_3, v_t) = (G_1, u_1) \oplus (P, v_t), \ \ (G', v) = (G_3, v) \oplus (G_2, u_2);$$
$$(G_4, v) = (G_1, u_1) \oplus (P, v), \ \ (G'', v_t) = (G_4, v_t) \oplus (G_2, u_2).$$

The graphs G, G' and G'' are depicted in Figure 2.

Lemma 2.2. *Let G, G' and G'' be the three graphs defined above. Then $\xi^*_R(G') > \xi^*_R(G)$ or $\xi^*_R(G'') > \xi^*_R(G)$.*

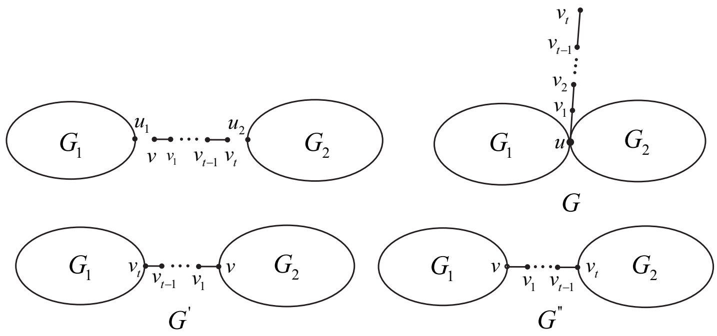

Figure 2: The graphs G, G' and G'' in Lemma 2.2.

Proof. Let $\varepsilon_{G_1}(v) = d_1, \varepsilon_{G_2}(v) = d_2$, and put $V_1 = V_{G_1} \setminus \{u_1\}, V_2 = V_{G_2} \setminus \{u_2\}, V_3 = \{v, v_1, v_2, \cdots, v_t\}$.

Case 1. $d_2 \geq d_1$. In this case, we consider the transformation from G to G'. It is clear that

$$\varepsilon'(x) = \varepsilon(x) \;\; x \in V_G; \;\; R'_{xy} = R_{xy}, \;\; x, y \in V_1 \;\; or \;\; x, y \in V_2;$$
$$R'_{xy} = R_{xy}, \;\; x, y \in V_3 \;\; or \;\; x \in V_2, y \in V_3; \;\; R'_{xy} \geq R_{xy}, \;\; x \in V_1, y \in V_2;$$
$$R'_{xv_i} = R_{xv} + t - i; \;\; R_{xv_i} = R_{xv} + i, \;\; x \in V_1, v_i \in V_3 (i = 1, 2, \cdots, t-1).$$

Let $U = \{1, 2, \cdots, t-1\}$, we have

$$\begin{aligned}
\xi_6 &= \Big(\sum_{\{x,y\}\subseteq V_1} + \sum_{\{x,y\}\subseteq V_2} + \sum_{\{x,y\}\subseteq V_3} + \sum_{x\in V_1, y\in V_2} + \sum_{x\in V_3, y\in V_2}\Big)[\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] > 0.\\
\xi_7 &= \sum_{x\in V_1, y\in V_3}[\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] \geq \sum_{x\in V_1, y\in V_3}[\varepsilon(x)\varepsilon(y)(R'_{xy} - R_{xy})]\\
&\geq d_2^2\Big(\sum_{x\in V_1, y\in\{v, v_t\}} + \sum_{x\in V_1, y\in V_3\setminus\{v, v_t\}}\Big)(R'_{xy} - R_{xy})\\
&\geq d_2^2 \sum_{x\in V_1, i\in U}(R'_{xv_i} - R_{xv_i}) = d_2^2 \sum_{x\in V_1, i\in U}[(R_{xv} + t - i) - (R_{xv} + i)]\\
&= d_2^2 |V_1| \sum_{i\in U}(t - 2i) \geq 0.
\end{aligned}$$

Therefore, we get $\xi_R^*(G') - \xi_R^*(G) = \xi_6 + \xi_7 > 0$.

Case 2. $d_1 > d_2$. In this case, we consider the transformation from G to G''. It is easy to see that

$$\varepsilon''(x) = \varepsilon(x) \;\; x \in V_G; \;\; R''_{xy} = R_{xy}, \;\; x, y \in V_1 \;\; or \;\; x, y \in V_2;$$
$$R''_{xy} = R_{xy}, \;\; x, y \in V_3 \;\; or \;\; x \in V_2, y \in V_3; \;\; R''_{xy} \geq R_{xy}, \;\; x \in V_1, y \in V_2;$$
$$R''_{xv_i} = R_{xv} + t - i; \;\; R_{xv_i} = R_{xv} + i, \;\; x \in V_1, v_i \in V_3 (i = 1, 2, \cdots, t-1).$$

In a similar way to case 1, we have

$$\begin{aligned}
\xi_8 &= \Big(\sum_{\{x,y\}\subseteq V_1} + \sum_{\{x,y\}\subseteq V_2} + \sum_{\{x,y\}\subseteq V_3} + \sum_{x\in V_1, y\in V_1} + \sum_{x\in V_3, y\in v_2}\Big)[\varepsilon''(x)\varepsilon''(y)R''_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] > 0.\\
\xi_9 &= \sum_{x\in V_2, y\in V_3}[\varepsilon''(x)\varepsilon''(y)R''_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}]\\
&\geq \sum_{x\in V_2, y\in V_3}[\varepsilon(x)\varepsilon(y)(R'_{xy} - R_{xy})] \geq d_1^2\Big(\sum_{x\in V_1, y\in\{v, v_t\}} + \sum_{x\in V_1, y\in V_3\setminus\{v, v_t\}}\Big)(R'_{xy} - R_{xy}) \geq 0.
\end{aligned}$$

Therefore, we get $\xi_R^*(G') - \xi_R^*(G) = \xi_8 + \xi_9 > 0$.

This completes the proof. □

Lemma 2.3. *Let u and v be two vertices in G such that the distance between u and v is equal to the diameter of G. Let w be a cut vertex of G which is the common vertex of G_1 and G_2. Let G' (resp. G'') be the graph obtained from G_1 and G_2 by identifying w of G_2 with v (resp. u) of G_1, as shown in Figure 3. Then (i) If $Kf_v(G_1) \geq Kf_w(G_1)$, $\xi^*_R(G') > \xi^*_R(G)$; (ii) If $Kf_u(G_1) \geq Kf_w(G_1)$, $\xi^*_R(G'') > \xi^*_R(G)$.*

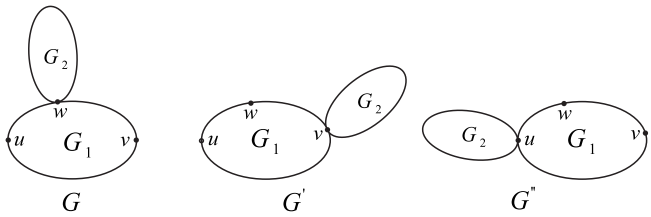

Figure 3: The graphs G, G' and G'' in Lemma 2.3

Proof. First of all, it can be conformed that either $\varepsilon_{G_2}(w) \leq d_G(w,u)$ or $\varepsilon_{G_2}(w) \leq d_G(w,v)$. Otherwise, without loss of generality, let $\varepsilon_{G_2}(w) > d_G(w,v)$. Then $\varepsilon_{G_2}(w) + d_G(w,u) > d_G(w,v) + d_G(w,u) \geq d(u,v)$. It means that there exists a shortest path which length is greater than diameter, this is a contradiction. Therefore, we have $\varepsilon(x) \leq d(u,v) < \varepsilon_{G_2}(w) + d(u,v) = \varepsilon'(x)$ for any $x \in V_{G_2}$.

According to the definitions of G and G', it can be concluded that

$$R_{xy} = R'_{xy},\ \ x,y \in V_{G_1};\ \ R_{xy} = R'_{xy},\ \ x,y \in V_{G_2};$$
$$R_{xy} = R_{xw} + R_{wy},\qquad R'_{xy} = R_{xv} + R_{wy},\ \ x \in V_{G_1}, y \in V_{G_2};$$
$$\varepsilon'(x) \geq \varepsilon(x), x \in V_{G_1};\qquad \varepsilon'(x) > \varepsilon(x),\ \ x \in V_{G_2}.$$

Hence we have

$$\begin{aligned}
\xi_{10} &= \Big(\sum_{\{x,y\}\subseteq V_{G_1}} + \sum_{\{x,y\}\subseteq V_{G_2}}\Big)[\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] > 0.\\
\xi_{11} &= \sum_{x\in V_{G_1}, y\in V_{G_2}} [\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}]\\
&= \sum_{x\in V_{G_1}, y\in V_{G_2}} [\varepsilon'(x)\varepsilon'(y)(R_{xv} + R_{wy}) - \varepsilon(x)\varepsilon(y)(R_{xw} + R_{wy})]\\
&> \sum_{x\in V_{G_1}, y\in V_{G_2}} [\varepsilon'(x)\varepsilon'(y)R_{xv} - \varepsilon(x)\varepsilon(y)R_{xw}] > \sum_{x\in V_{G_1}, y\in VG_2} [\varepsilon(x)\varepsilon(y)(R_{xv} - R_{xw})]\\
&> \sum_{x\in V_{G_1}} (R_{xv} - R_{xw}) = Kf_v(G_1) - Kf_w(G_1).
\end{aligned}$$

Therefore, when $Kf_v(G_1) \geq Kf_w(G_1)$, it follows that

$$\xi^*_R(G') - \xi^*_R(G) = \xi_{10} + \xi_{11} > Kf_v(G_1) - Kf_w(G_1) \geq 0.$$

Similarly, for $Kf_u(G_1) \geq Kf_w(G_1)$, we also have $\xi^*_R(G'') > \xi^*_R(G)$. □

Lemma 2.4. *Let $C_k (k \geq 4)$ be an end cycle at vertex u in G, and $u_{k-3}, u_{k-2}, u_{k-1}$ be three successive vertices lying in C_k, as shown in Figure 4. Let $G' = G - uu_{k-1} + u_{k-3}u_{k-1}$, then $\xi^*_R(G') > \xi^*_R(G)$.*

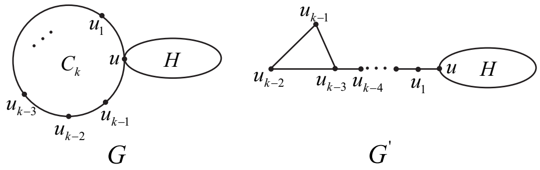

Figure 4: The graphs G and G' in Lemma 2.4.

Proof. Let $H = G - \{u_1, u_2, \cdots, u_{k-1}\}$. For the transformation from G to G', we know that $\varepsilon'(x) \geq \varepsilon(x)$ for any $x \in V_G$. It is easy to see that

$$\begin{aligned}\xi_R^*(G') - \xi_R^*(G) &= \sum_{\{x,y\}\subseteq V_G} [\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}] \geq \sum_{\{x,y\}\subseteq V_G} [\varepsilon(x)\varepsilon(y)(R'_{xy} - R_{xy})] \\ &\geq \sum_{\{x,y\}\subseteq V_G} (R'_{xy} - R_{xy}) = Kf(G') - Kf(G).\end{aligned}$$

By lemmas 1.1 and 1.2, we have

$$Kf(G') = Kf(H) + n_2 Kf_u(H) + \frac{1}{6}(k-2)(7k^2 - 15k + 14) + \frac{4}{3}k + \frac{10}{3},$$
$$Kf(G) = Kf(H) + n_2 Kf_u(H) + \frac{1}{12}(k-1)(3k^2 + k - 2),$$
$$Kf(G') - Kf(G) = \frac{1}{12}(k-1)(11k^3 - 56k^2 + 107k - 18) = f(k).$$

Note that for real number k, $f'(k) = \frac{1}{12}(33k^2 - 108k + 107) = \frac{11}{4}(k - \frac{18}{11})^2 - \frac{81}{11} > 0$ for $k \geq 4$. Then $f(k) \geq f(4) = \frac{109}{6} > 0$, that is, $Kf(G') - Kf(G) > 0$ for $k \geq 4$. So we have $\xi_R^*(G') > \xi_R^*(G)$. □

Lemma 2.5. *Let G_1, G_2 and $C_k (k \geq 4)$ be three disjoint graphs where $v_1 \in V_{G_1}, v_2 \in V_{G_2}, u, u_i \in V_{C_k}$. Let $G = ((G_1, v_1) \oplus (C_k, u_i), u) \oplus (G_2, v_2)$, as shown in Figure 5. Suppose that u_{i-1}, u_i, u_{i+1} are three successive vertices in C_k. Let $G' = G - \{uu_{k-1}, u_{i-1}u_{i-2}\} + \{u_{i-2}u_{k-1}, u_{i-1}u_{i+1}\}$, then $\xi_R^*(G') > \xi_R^*(G)$.*

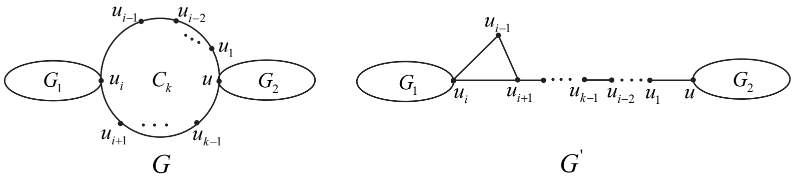

Figure 5: The graphs G and G' in Lemma 2.5.

Proof. For the transformation from G to G', it is easy to see that $\varepsilon'(x) \geq \varepsilon(x)$ for any $x \in V_G$. Therefore

$$\begin{aligned}\xi_R^*(G') - \xi_R^*(G) &= \sum_{\{x,y\}\subseteq V_G} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy} \geq \sum_{\{x,y\}\subseteq V_G} \varepsilon(x)\varepsilon(y)(R'_{xy} - R_{xy}) \\ &\geq \sum_{\{x,y\}\subseteq V_G} (R'_{xy} - R_{xy}).\end{aligned}$$

Let $A = V_{G_1} \setminus \{u_i\}$, $B = V_{G_2} \setminus \{u\}$, $C = \{u, u_1, \cdots, u_{k-1}\}$. We have

$$\xi_{12} = (\sum_{\{x,y\}\subseteq A} + \sum_{\{x,y\}\subseteq B} + \sum_{x\in A, y\in B})(R'_{xy} - R_{xy}) > 0.$$

According to Lemma 1.1 and Lemma 1.2, if $k \geq 4$, we get

$$\xi_{13} = \sum_{\{x,y\}\subseteq C} (R'_{xy} - R_{xy}) = \frac{1}{6}(k^3 - 19k + 50) - \frac{1}{12}(k^3 - k) = \frac{1}{12}(k^3 - 37k + 100) > 0,$$

$$\xi_{14} = \sum_{x\in A, y\in C} (R'_{xy} - R_{xy}) = \sum_{x\in A, y\in C} (R'_{u_i y} - R_{u_i y}) = \frac{1}{6}|A|(2k^2 - 11k + 15) > 0,$$

$$\xi_{15} = \sum_{x\in B, y\in C} (R'_{xy} - R_{xy}) = \sum_{x\in A, y\in C} (R'_{uy} - R_{uy}) = \frac{1}{6}|B|(2k^2 - 3k - 9) > 0.$$

Therefore, it follows that $\xi^*_R(G') - \xi^*_R(G) \geq \xi_{12} + \xi_{13} + \xi_{14} + \xi_{15} > 0$, that is, $\xi^*_R(G') > \xi^*_R(G)$. □

Definition. A chain cactus is a graph G if each block of it has at most two cut vertices and each cut vertex is shared by exactly two blocks. A chain 3-cactus is a chain cactus in which every cycle is a triangle. A path 3-cactus is a chain cactus in which every block is a triangle. A path 3-cactus with $t(t \geq 0)$ triangles is denoted by $C^3(t)$.

Let G_1 be a path 3-cactus and G_2 a chain 3-cactus, and let $\varepsilon_{G_1}(v) = d_1, \varepsilon_{G_2}(v) = d_2$, where $v_1 \in G_1, v_2 \in G_2$ and d_1, d_2 are the diameter of G_1, G_2, respectively. Suppose that $P_u = uu_1u_2\cdots u_{k-1}u_k$ is a path , we construct the graph H by the identification operation

$$(M, v_1) = (G_1, v_1) \oplus (P_u, u), \ \ (H, v_2) = (M, u_k) \oplus (G_2, v_2).$$

Let $G = H + \{vu_{k-1}, vv_2\}$, $G' = H + \{vv_1, vu_1\}$, as shown in Figure 6, we have the following result.

Lemma 2.6. *Suppose that G and G′ are two graphs illustrated in Figure 6. If* $|V_{G_1}| \leq |V_{G_2}|$*, then* $\xi^*_R(G') \geq \xi^*_R(G)$*, the equality holds if and only if* $G_1 \cong G_2$.

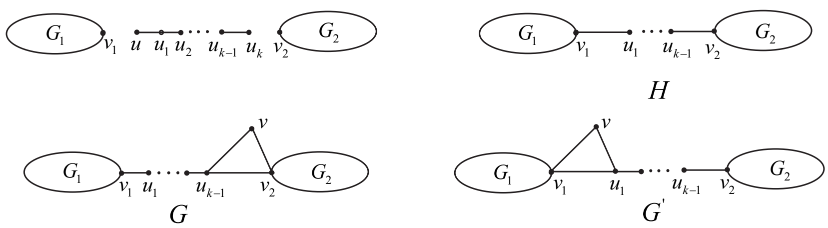

Figure 6: The graphs G and G' in Lemma 2.6.

Proof. From the definition of the path 3-cactus, if $|V_{G_1}| \leq |V_{G_2}|$, we have $d_1 \leq d_2$. Let $U = \{u_1, u_2, \cdots, u_{k-1}\}$, for the transformation from G to G', we have

$$\varepsilon'(x) \geq \varepsilon(x), \ \ x \in V_G; \ \ R'_{xy} = R_{xy}, \ \ x, y \in V_{G_1} \ \ or \ \ x, y \in V_{G_2}$$
$$R'_{xy} = R_{xy}, \ \ x, y \in U; \ \ R'_{xy} = R_{xy}, \ \ x \in V_{G_1}, y \in V_{G_2};$$
$$R'_{xu_i} = R_{xv_1} + \frac{2}{3} + i - 1, \ \ R_{xu_i} = R_{xv_1} + i, \ \ x \in V_{G_1}, u_i \in U;$$
$$R'_{xu_i} = R_{xv_2} + k - i, \ \ R_{xu_i} = R_{xv_2} + \frac{2}{3} + k - 1 - i, \ \ x \in V_{G_2}, u_i \in U;$$
$$R'_{xv} = R_{xv_1} + \frac{2}{3}, \ \ R_{xv} = R_{xv_1} + \frac{2}{3} + k - 1, \ \ x \in V_{G_1};$$
$$R'_{xv} = R_{xv_2} + \frac{2}{3} + k - 1, \ \ R_{xv} = R_{xv_2} + \frac{2}{3}, \ \ x \in V_{G_2}.$$
$$R'_{u_i v} = \frac{2}{3} + i - 1, \ \ R_{u_i v} = \frac{2}{3} + k - 1 - i, \ \ u_i \in U.$$

Hence we have

$$\xi_R^*(G') - \xi_R^*(G) = \sum_{\{x,y\}\subseteq V_G} [\varepsilon'(x)\varepsilon'(y)R'_{xy} - \varepsilon(x)\varepsilon(y)R_{xy}]$$

$$\geq \sum_{\{x,y\}\subseteq V_G} [\varepsilon(x)\varepsilon(y)(R'_{xy} - R_{xy})] \geq \sum_{\{x,y\}\subseteq V_G} (R'_{xy} - R_{xy}).$$

Note that $V(G) = V_{G_1} \cup V_{G_2} \cup U \cup \{v\}$, we get

$$\xi_{16} = (\sum_{\{x,y\}\subseteq V_{G_1}} + \sum_{\{x,y\}\subseteq V_{G_2}} + \sum_{\{x,y\}\subseteq U} + \sum_{x\in V_{G_1}, y\in V_{G_2}})(R'_{xy} - R_{xy}) = 0,$$

$$\xi_{17} = (\sum_{x\in V_{G_1}, u_i\in U} + \sum_{x\in V_{G_2}, u_i\in U})(R'_{xy} - R_{xy}) = \frac{1}{3}(k-1)(|G_2| - |G_1|),$$

$$\xi_{18} = (\sum_{x\in V_{G_1}} + \sum_{x\in V_{G_2}})(R'_{xy} - R_{xy}) = (k-1)(|G_2| - |G_1|),$$

$$\xi_{19} = \sum_{u_i\in U}(R'_{u_iv} - R_{u_iv}) = \sum_{i=1}^{k-1}(2i-k) = 0.$$

Therefore, we get $\xi_R^*(G') - \xi_R^*(G) \geq \xi_{16} + \xi_{17} + \xi_{18} + \xi_{19} = \frac{4}{3}(k-1)(|V_{G_2}| - |V_{G_1}|)$. This implies that $\xi_R^*(G') \geq \xi_R^*(G)$ when $|V_{G_2}| \geq |V_{G_1}|$, and the equality holds if and only if $G_1 \cong G_2$. □

3. Applications of the increasing transformations

In this section, we will determine the graphs in $Cat(n;t)$ with the maximum and second-maximum multiplicative eccentricity resistance-distance in $cat(n;t)$. Assume that $C_{n,t} \in cat(n;t)$ is a chain 3-cactus consisting of two path 3-cacti $C^3(k)$, $C^3(t-k)$ and an internal path P, where $k = [\frac{t}{2}]$, as shown in Figure 7.

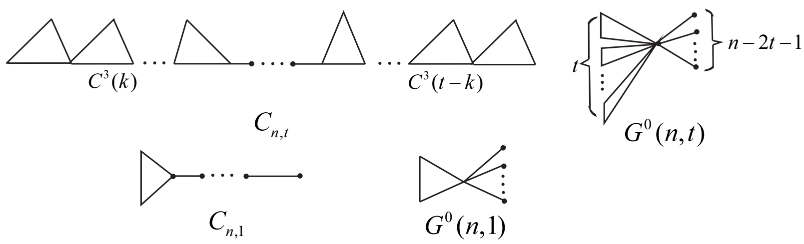

Figure 7: The graphs $C_{n,t}, G^0(n,t), C_{n,1}$ and $G^0(n,1)$

3.1. *The extremal cacti with maximum $\xi_R^*(G)$-value*

Theorem 3.1. *Let $G \in Cat(n;t)$ with $n \geq 6$, then $\xi_R^*(G) \leq \xi_R^*(C_{n,t})$, the equality holds if and only if $G \cong C_{n,t}$.*

Proof. Suppose that $G \ncong C_{n,t}$. By Lemma 2.1 and Lemma 2.2, we can convert the paths incident with vertices of cycles of G into pendent paths or internal paths, and the new graph obtained from G is denoted by H. Obviously, we have $\xi_R^*(G) < \xi_R^*(H)$.

If H is not isomorphic to chain cactus, then a chain cactus H_1 is created by repeated applications of Lemma 2.3. Obviously, all the cut vertices of H_1 are in the path which length is equal to the diameter of H_1. By Lemma 2.3, it follows that $\xi_R^*(H) < \xi_R^*(H_1)$.

Assume that H_1 is not isomorphic to chain 3-cactus, we can form a 3-cactus H_2 according to Lemma 2.4 and Lemma 2.5. It is easy to see that $\xi_R^*(H_1) < \xi_R^*(H_2)$.

Finally, let $H_2 \ncong C_{n,t}$, we use the transformation defined in Lemma 2.6 repeatedly, then we have the chain 3-cactus $C_{n,t}$. Further, we get $\xi_R^*(H_2) < \xi_R^*(C_{n,t})$. Therefore, we have $\xi_R^*(G) \le \xi_R^*(C_{n,t})$, the equality holds if and only if $G \cong C_{n,t}$. □

Let $G^0(n,t)$ be the graph as shown in Figure 7. Combining the result in [7], we have the following corollary.

Corollary 3.2. *For $G \in Cat(n;t)$, $\xi_R^*(G^0(n,t)) \le \xi_R^*(G) \le \xi_R^*(C_{n,t})$, with equality on the left-hand side holds if and only if $G \cong G^0(n,t)$, with equality on the right-hand side holds if and only if $G \cong C_{n,t}$.*

For $Cat(n;t)$, if $t = 0,1$, $Cat(n;0)$ and $Cat(n;1)$ are the class of trees and unicyclic graphs, respectively. Further, we have the following results by the discussion above.

Corollary 3.3. *Let G be a tree of order n different from S_n and P_n, then $\xi_R^*(S_n) < \xi_R^*(G) < \xi_R^*(P_n)$.*

Corollary 3.4. *Let G be a unicyclic graph of order n and $G \ncong G^0(n,1)$, the graph $C_{n,1}$ is as shown in Figure 7, then $\xi_R^*(G^0(n,1)) < \xi_R^*(G) < \xi_R^*(C_{n,1})$.*

3.2. The extremal cacti with second maximum $\xi_R^*(G)$-value

Lemma 3.5. *Let $C_{n,t}^1, C_{n,t}^2$ be graphs as shown in Figure 8. (i) If n is odd, $\xi_R^*(C_{n,t}^1) > \xi_R^*(C_{n,t}^2)$; (ii) If n is even, $\xi_R^*(C_{n,t}^1) \ge \xi_R^*(C_{n,t}^2)$, the equality holds if and only if $n = 4k+2$, where $k = \lfloor \frac{t}{2} \rfloor$, that is, k is the maximum integer which does not exceed $\frac{t}{2}$.*

Proof. For simplicity, let $G = C_{n,t}^1, G' = C_{n,t}^2$. Let $A = V_{C^3(k-1)}$, $B = V_{C^3(t-k-1)}$, $C = V_G \backslash (A\backslash\{u\}) \cup (B\backslash\{u'\})$, $C' = V_{G'} \backslash (A\backslash\{v\}) \cup (B\backslash\{v'\})$. Then

$$\varepsilon(x) = \varepsilon'(x),\ \ \varepsilon(y) = \varepsilon'(y),\ \ R_{xy} = R'_{xy},\ \ x, y \in A \ \ or\ \ B;$$
$$\varepsilon(x) = \varepsilon'(x),\ \ \varepsilon(y) = \varepsilon'(y),\ \ R_{xy} = R'_{xy},\ \ x \in A, x \in B.$$

Further we have

$$\begin{aligned}
& \xi_R^*(G) - \xi_R^*(G') \\
= & \ (\sum_{\{x,y\}\subseteq A} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{\{x,y\}\subseteq A} \varepsilon'(x)\varepsilon'(y)R'_{xy}) + (\sum_{\{x,y\}\subseteq B} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{\{x,y\}\subseteq B} \varepsilon'(x)\varepsilon'(y)R'_{xy}) \\
& + (\sum_{x\in A, y\in B} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{x\in A, y\in B} \varepsilon'(x)\varepsilon'(y)R'_{xy}) + (\sum_{\{x,y\}\subseteq C} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{\{x,y\}\subseteq C'} \varepsilon'(x)\varepsilon'(y)R'_{xy}) \\
& + (\sum_{\{x\in A\backslash\{u\}, y\in C} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{x\in A\backslash\{v\}, y\in C'} \varepsilon'(x)\varepsilon'(y)R'_{xy}) \\
& + (\sum_{x\in B\backslash\{u'\}, y\in C} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{x\in B\backslash\{v'\}, y\in C'} \varepsilon'(x)\varepsilon'(y)R'_{xy}) \\
= & \ (\sum_{\{x,y\}\subseteq C} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{\{x,y\}\subseteq C'} \varepsilon'(x)\varepsilon'(y)R'_{xy}) \\
& + (\sum_{x\in A\backslash\{u\}, y\in C} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{x\in A\backslash\{v\}, y\in C'} \varepsilon'(x)\varepsilon'(y)R'_{xy}) \\
& + (\sum_{x\in B\backslash\{u'\}, y\in C} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{x\in B\backslash\{v'\}, y\in C'} \varepsilon'(x)\varepsilon'(y)R'_{xy})
\end{aligned}$$

Case 1. If t is even, let $t = 2k$. We can distinguish two cases as the following.

Subcase 1.1. If n is odd, we have

$$\begin{aligned}
\eta_1 &= \sum_{\{x,y\}\subseteq C} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{\{x,y\}\subseteq C'} \varepsilon'(x)\varepsilon'(y)R'_{xy} \\
&= \{\frac{2}{3}(n-3k)^2 + 2(n-3k)[\frac{2}{3}(n-3k-1) + \frac{5}{3}(n-3k-2) + \cdots + \\
&\quad \frac{n-2k-1}{2}(\frac{n-4k-1}{2} + \frac{2}{3}) + \cdots + (n-3k-2)(n-4k-2+\frac{2}{3}) + \\
&\quad 2(n-3k-1)(n-4k-2+\frac{4}{3}) + (n-3k)(n-4k-1+\frac{4}{3})] \\
&\quad +(n-3k)[(n-3k-1) + \frac{5}{3}(n-3k-1) + \frac{5}{3}(n-3k-2) + \frac{8}{3}(n-3k-3) + \cdots \\
&\quad +\frac{n-2k-1}{2}(\frac{n-4k-1}{2} + \frac{2}{3}) + \cdots + (n-3k-1)(\frac{n-4k-1}{2} + \frac{2}{3})]\} \\
&\quad -\{(n-3k)[(n-3k-1) + 2(n-3k-2) + \cdots + \frac{n-2k-1}{2}(\frac{n-4k-1}{2} + 1) + \cdots \\
&\quad +(n-3k-2)(n-4k-1) + 2(n-3k-1)(n-4k-1+\frac{2}{3}) + 2(n-3k)(n-4k-1+\frac{4}{3})] \\
&\quad +2(n-3k)[\frac{2}{3}(n-3k-1) + \frac{4}{3}(n-3k-1) + \frac{4}{3}(n-3k-2) + \frac{5}{3}(n-3k-3) + \cdots \\
&\quad +\frac{n-2k-1}{2}(\frac{n-4k-1}{2} + \frac{1}{3}) + \cdots + (n-3k-1)(n-4k-2+\frac{4}{3})] + \frac{2}{3}(n-3k)^2\} \\
&= (n-3k)\{\frac{1}{3}[(n-3k-2) + \cdots + \frac{n-2k-1}{2} + \frac{n-2k+1}{2} + \cdots + (n-3k-2)] \\
&\quad +(n-3k)(n-3k-1)(n-4k-1) - (n-3k)(n-3k-1)\} \\
&= \frac{1}{3}(n-3k)[\frac{(3n-8k-5)(n-4k-1)}{4} - \frac{n-2k-1}{2}] + (n-3k)(n-3k-1)(n-4k-2) \\
&= \frac{n-3k}{12}[15n^2 - (104k+46)n + (176k^2+152k+31)], \\
\eta_2 &= \sum_{x\in A\setminus\{u\}, y\in C} \varepsilon(x)\varepsilon(y)R_{xy} - \sum_{x\in A\setminus\{v\}, y\in C'} \varepsilon'(x)\varepsilon'(y)R'_{xy} \\
&= -\frac{1}{3}(n-3k-1)[(n-3k) + 2(n-3k+1) + 2(n-3k+2) + \cdots + 2(n-2k-1)] \\
&\quad -\frac{1}{3}(n-3k-2)[(n-3k) + 2(n-3k+1) + 2(n-3k+2) + \cdots + 2(n-2k-1)] - \cdots \\
&\quad -\frac{1}{3}\frac{n-2k-1}{2}[(n-3k) + 2(n-3k+1) + 2(n-3k+2) + \cdots + 2(n-2k-1)] - \cdots \\
&\quad -\frac{1}{3}(n-3k-2)[(n-3k) + 2(n-3k+1) + 2(n-3k+2) + \cdots + 2(n-2k-1)] \\
&\quad -\frac{2}{3}(n-3k-1)[(n-3k) + 2(n-3k+1) + 2(n-3k+2) + \cdots + 2(n-2k-1)] \\
&\quad -(n-3k)[(n-3k)(n-4k-1+\frac{4}{3}) + 2(n-3k+1)(n-4k-1+2) + 2(n-3k+2) \\
&\quad + \cdots + 2(n-2k-1)(n-4k-1+\frac{2}{3}(k+1))] \\
&\quad +(n-3k)[\frac{2}{3}(n-3k) + 2(n-3k+1)\times\frac{4}{3} + \cdots + 2(n-2k-1)\times\frac{2}{3}k] \\
&= [(n-3k)(n-4k-\frac{1}{3}) - \frac{(3n-8k-1)(n-4k-1)}{12}][(2n-5k+2)k-n],
\end{aligned}$$

$$\begin{aligned}
\eta_3 &= \sum_{x\in B\setminus\{u'\},y\in C}\varepsilon(x)\varepsilon(y)R_{xy}-\sum_{x\in B\setminus\{v'\},y\in C'}\varepsilon'(x)\varepsilon'(y)R'_{xy}\\
&= -(n-3k)[\frac{2}{3}(n-3k)+2(n-3k+1)\times\frac{4}{3}+\cdots+2(n-2k-1)\times\frac{2}{3}k]\\
&\quad+\frac{1}{3}(n-3k-1)[(n-3k)+2(n-3k+1)+2(n-3k+2)+\cdots+2(n-2k-1)]\\
&\quad+\frac{1}{3}(n-3k-2)[(n-3k)+2(n-3k+1)+2(n-3k+2)+\cdots+2(n-2k-1)]-\cdots\\
&\quad+\frac{1}{3}\frac{n-2k-1}{2}[(n-3k)+2(n-3k+1)+2(n-3k+2)+\cdots+2(n-2k-1)]-\cdots\\
&\quad+\frac{1}{3}(n-3k-2)[(n-3k)+2(n-3k+1)+2(n-3k+2)+\cdots+2(n-2k-1)]\\
&\quad+\frac{2}{3}(n-3k-1)[(n-3k)+2(n-3k+1)+2(n-3k+2)+\cdots+2(n-2k-1)]\\
&\quad+(n-3k)[(n-3k)(n-4k-1+\frac{4}{3})+2(n-3k+1)(n-4k-1+2)+2(n-3k+2)\\
&\quad+\cdots+2(n-2k-1)(n-4k-1+\frac{2}{3}(k+1))]\\
&= [\frac{(3n-8k-1)(n-4k-1)}{12}-(n-3k)(n-4k-\frac{1}{3})][(2n-5k+2)k-n].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\xi_R^*(G)-\xi_R^*(G') &= \eta_1+\eta_2+\eta_3\\
&= \frac{n-3k}{12}[15n^2-(104k+46)n+(176k^2+152k+31)].
\end{aligned}$$

Let $f(x)=15x^2-(104k+46)x+(176k^2+152k+31)$ $(x>4k+1,\ k\geq 1)$, then $f'(x)=30x-(104k+46)>30(4k+1)-(104k+46)=16(k-1)\geq 0$, that is, $f'(x)>0$. So $f(x)$ is increasing. Note that n is odd, hence we get $f(x)\geq f(4k+3)=16k+28>0$. It is easy to see that $\xi_R^*(G)-\xi_R^*(G')=\frac{n-3k}{12}f(n)\geq\frac{n-3k}{12}f(4k+3)\geq\frac{1}{3}(k+3)(4k+7)>0$. So we get

$$\xi_R^*(G)>\xi_R^*(G')\ \ (n>4k+1,\ \ k\geq 1).$$

Subcase 1.2. If n is even, in a similar way to Subcase 1.1, by direct calculation, we have

$$\begin{aligned}
\xi_R^*(G)-\xi_R^*(G') &= \frac{1}{12}(n-3k)(n-4k-2)(15n-44k-16)\\
&\geq 0\ \ (\textit{Since}\ \ n\geq 4k+2,\ \ k\geq 1).
\end{aligned}$$

Hence we have

$$\xi_R^*(G)\geq\xi_R^*(G')\ \ (n\geq 4k+2,\ \ k\geq 1).$$

Case 2. If t is odd, then $t=2k+1$. In a similar discussion as in Case 1, we have two cases as follow.
Subcase 2.1. If n is odd, we have $\xi_R^*(C_{n,t}^1)>\xi_R^*(C_{n,t}^2)$.
Subcase 2.2. If n is even, we have (i) When $n=4k+4$, $C_{n,t}^1\cong C_{n,t}$, $\xi_R^*(C_{n,t}^1)>\xi_R^*(C_{n,t}^2)$; (ii) When $n\neq 4k+4$, $\xi_R^*(C_{n,t}^1)>\xi_R^*(C_{n,t}^2)$.

Combining Case 1 with Case 2, we get (i) If n is odd, $\xi_R^*(C_{n,t}^1)>\xi_R^*(C_{n,t}^2)$; (ii) If n is even, $\xi_R^*(C_{n,t}^1)\geq\xi_R^*(C_{n,t}^2)$, the equality holds if and only if $n=4k+2$. □

Lemma 3.6. *Let $C_{n,t}^2, C_{n,t}^3$ be graphs as shown in Figure 8. If $k\geq 2$, then $\xi_R^*(C_{n,t}^2)>\xi_R^*(C_{n,t}^3)$.*

Proof. Let $G' = C^2_{n,t}, G'' = C^3_{n,t}, A' = V_{C^3(k-1)}, B' = V_{C^3(t-k)}, D = V_{G'}\backslash(A'\backslash\{v\}) \cup (B'\backslash\{v''\}), D' = V_{G''}\backslash(A\backslash\{w\}) \cup (B\backslash\{w''\})$. Then

$$\varepsilon'(x) = \varepsilon''(x),\ \ \varepsilon'(y) = \varepsilon''(y),\ \ R'_{xy} = R''_{xy},\ \ x, y \in A'\ \ or\ \ B';$$
$$\varepsilon'(x) = \varepsilon''(x),\ \ \varepsilon'(y) = \varepsilon''(y),\ \ R'_{xy} = R''_{xy},\ \ x \in A', x \in B'.$$

Hence we get

$$\begin{aligned}
\xi^*_R(G') - \xi^*_R(G'') &= (\sum_{\{x,y\}\subseteq A'} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{\{x,y\}\subseteq A'} \varepsilon''(x)\varepsilon''(y)R''_{xy}) \\
&\quad +(\sum_{\{x,y\}\subseteq B'} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{\{x,y\}\subseteq B'} \varepsilon''(x)\varepsilon''(y)R''_{xy}) \\
&\quad +(\sum_{x\in A',y\in B'} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{x\in A',y\in B'} \varepsilon''(x)\varepsilon''(y)R''_{xy}) \\
&\quad +(\sum_{\{x,y\}\subseteq D} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{\{x,y\}\subseteq D'} \varepsilon''(x)\varepsilon''(y)R''_{xy}) \\
&\quad +(\sum_{\{x\in A'\backslash\{v\},y\in D} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{x\in A'\backslash\{w\},y\in D'} \varepsilon''(x)\varepsilon''(y)R''_{xy}) \\
&\quad +(\sum_{x\in B'\backslash\{v''\},y\in D} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{x\in B\backslash\{w''\},y\in D'} \varepsilon''(x)\varepsilon''(y)R''_{xy}) \\
&= (\sum_{\{x,y\}\subseteq D} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{\{x,y\}\subseteq D'} \varepsilon'(x)\varepsilon'(y)R'_{xy}) \\
&\quad +(\sum_{x\in A'\backslash\{v\},y\in D} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{x\in A'\backslash\{w\},y\in D'} \varepsilon''(x)\varepsilon''(y)R'_{xy}) \\
&\quad +(\sum_{x\in B'\backslash\{v''\},y\in D} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{x\in B'\backslash\{w''\},y\in D'} \varepsilon''(x)\varepsilon''(y)R''_{xy})
\end{aligned}$$

Case 1. If t is even, then $t = 2k$, we have the following two cases.
Subase 1.1. If n is odd, we have

$$\begin{aligned}
\eta_4 &= \sum_{\{x,y\}\subseteq D} \varepsilon'(x)\varepsilon'(y)R'_{xy} - \sum_{\{x,y\}\subseteq D'} \varepsilon''(x)\varepsilon''(y)R''_{xy} \\
&= \{(n-3k)[(n-3k-1) + 2(n-3k-2) + \cdots + \frac{n-2k-1}{2} \times \frac{n-4k-1}{2} + \cdots \\
&\quad +(n-3k-2)(n-4k-1) + 2(n-3k-1)(n-4k-1+\frac{2}{3})] + (n-3k-1)[(n-3k-2) \\
&\quad +2(n-3k-3) + (n-3k-2) + \cdots + \frac{n-2k-1}{2} \times \frac{n-4k-1}{2} + \cdots \\
&\quad +(n-3k-2)(n-4k-2) + 2(n-3k-1)(n-4k-2+\frac{2}{3})] + 2(n-3k-1)[\frac{2}{3}(n-3k-2) \\
&\quad +\frac{5}{3}(n-3k-3) + \cdots + \frac{n-2k-1}{2}(\frac{n-4k-1}{2} - \frac{1}{3}) + \cdots
\end{aligned}$$

$$
\begin{aligned}
& +(n-3k-2)(n-4k-3)+\frac{2}{3})]+\frac{2}{3}(n-3k-1)^2\} \\
& -\{(n-3k)[\frac{3}{2}(n-3k-1)+(n-3k-2)+\cdots+\frac{n-2k-1}{2}\times(\frac{n-4k-1}{2})+\cdots \\
& +(n-3k-2)(n-4k-2)+(n-3k-1)(n-4k-1)]+2(n-3k-1)[\frac{3}{4}(n-3k-2) \\
& +\frac{7}{4}(n-3k-3)+\cdots+\frac{n-2k-1}{2}\times(\frac{n-4k-5}{2}+\frac{3}{4})+\cdots+(n-3k-2)(n-4k-3+\frac{3}{4}) \\
& +(n-3k-1)(n-4k-2+\frac{3}{4})]+(n-3k-1)^2+(n-3k-1)[(n-3k-2)+2(n-3k-3) \\
& +(n-3k-2)+\cdots+\frac{n-2k-1}{2}\times\frac{n-4k-1}{2}+\cdots+(n-3k-2)(n-4k-2)]\} \\
= \; & (n-3k)[-\frac{1}{2}(n-3k-1)+(n-3k-2)+\cdots+\frac{n-2k-1}{2}+\cdots+(n-3k-2) \\
& +(n-3k-1)(n-4k-1+\frac{4}{3})]+2(n-3k-1)^2(n-4k-2+\frac{2}{3}) \\
& -\frac{1}{6}(n-3k-1)[(n-3k-2)+(n-3k-3)\cdots+\frac{n-2k-1}{2}+\cdots+(n-3k-2)] \\
& -[2(n-3k-1)^2(n-4k-1+\frac{3}{4})+\frac{1}{3}(n-3k-1)^2] \\
= \; & \frac{1}{6}(5n-15k+1)[(n-3k-2)+(n-3k-3)\cdots+\frac{n-2k-1}{2}+\cdots+(n-3k-2)] \\
& +(n-3k)(n-3k-1)(n-4k-\frac{1}{6})-\frac{1}{2}(n-3k-1)^2 \\
= \; & \frac{1}{24}(5n-15k+1)[(3n-8k-5)(n-4k-1)-2(n-2k-1)] \\
& +(n-3k)(n-3k-1)(n-4k-\frac{1}{6})-\frac{1}{2}(n-3k-1)^2, \\
\eta_5 = \; & \sum_{x\in A'\setminus\{v\},y\in D}\varepsilon'(x)\varepsilon'(y)R'_{xy}-\sum_{x\in A'\setminus\{w\},y\in D'}\varepsilon''(x)\varepsilon''(y)R'_{xy}) \\
= \; & \frac{1}{2}(n-3k-1)[(n-3k+1)+(n-3k+2)+\cdots+(n-2k-1)]+2(n-3k+2)[(n-3k+1) \\
& +(n-3k+2)+\cdots+(n-2k-1)]+\cdots+(n-2k-1)[(n-3k+1)+(n-3k+2)+\cdots \\
& +(n-2k-1)]+\cdots+2(n-3k-2)[(n-3k+1)+(n-3k+2)+\cdots+(n-2k-1)] \\
& +2(n-3k-1)[(n-3k+1)(n-4k-1+\frac{4}{3})+(n-3k+2)(n-4k-1+\frac{2}{3}\times 3)+\cdots \\
& +(n-2k-1)(n-4k-1+\frac{2}{3}k)]-2(n-3k-1)[(n-3k+1)(\frac{2}{3}+\frac{3}{4}) \\
& +(n-3k+2)(\frac{2}{3}\times 2+\frac{3}{4})+\cdots+(n-2k-1)(\frac{2}{3}(k-1)\times 2+\frac{3}{4})] \\
= \; & [\frac{1}{2}(n-3k-1)+2(n-3k-2)+\cdots+2\times\frac{n-2k-1}{2}+\cdots \\
& +2(n-3k-2)][(n-3k+1)+(n-3k+2)+\cdots+(n-2k-2)+(n-2k-1)] \\
& +2(n-3k-1)(n-4k-\frac{13}{12})[(n-3k+1)+(n-3k+2)+\cdots+(n-2k-2)+(n-2k-1)], \\
\eta_6 = \; & \sum_{x\in B'\setminus\{v''\},y\in D}\varepsilon'(x)\varepsilon'(y)R'_{xy}-\sum_{x\in B'\setminus\{w''\},y\in D'}\varepsilon''(x)\varepsilon''(y)R''_{xy} \\
= \; & \frac{4}{3}(n-3k)[(n-3k)+(n-3k+1)+\cdots+(n-2k-1)] \\
& -\frac{1}{6}(n-3k-1)[(n-3k)+(n-3k+1)+\cdots+(n-2k-1)] \\
& -\frac{2}{3}(n-3k-2)[(n-3k)+(n-3k+1)+\cdots+(n-2k-1)]-\cdots
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{2}{3}\frac{n-2k-1}{2}[(n-3k)+(n-3k+1)+\cdots+(n-2k-1)]-\cdots \\
& -\frac{2}{3}(n-3k-2)[(n-3k)+(n-3k+1)+\cdots+(n-2k-1)] \\
& +2(n-3k-1)[\frac{4}{3}(n-3k)+2(n-3k+1)+\frac{8}{3}(n-3k+2)+\cdots+\frac{2}{3}(k+1)(n-2k-1)] \\
& -2(n-3k-1)[(n-3k)(n-4k-1+\frac{2}{3}+\frac{3}{4})+\cdots+(n-2k-1)(n-4k-2+\frac{2}{3}+\frac{3}{4})] \\
= & \{\frac{4}{3}(n-3k)-\frac{1}{6}(n-3k-1)-\frac{2}{3}[(n-3k-2)+(n-3k-3)+\cdots \\
& +\frac{n-2k-1}{2}+\cdots+(n-3k-2)]\}[(n-3k)+(n-3k+1)+\cdots+(n-2k-1)] \\
& -2(n-3k-1)(n-4k-\frac{23}{12})[(n-3k)+(n-3k+1)+\cdots+(n-2k-2)+(n-2k-1)].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \xi_R^*(G')-\xi_R^*(G'') \\
= & \eta_4+\eta_5+\eta_6 \\
= & \{\frac{4}{3}(n-3k)+\frac{1}{3}(n-3k-1)+\frac{4}{3}[(n-3k-2)+(n-3k-3)+\cdots \\
& +\frac{n-2k-1}{2}+\cdots+(n-2k-2)+(n-2k-1)]\}[n-3k+1)+\cdots+(n-2k-1)] \\
& +\{\frac{4}{3}(n-3k)-\frac{1}{6}(n-3k-1)-\frac{4}{3}[(n-3k-2)+(n-3k-3)+\cdots+\frac{n-2k-1}{2} \\
& +\cdots+(n-2k-2)+(n-2k-1)]\}(n-3k)+\frac{5}{3}(n-3k-1)[n-3k+1) \\
& +\cdots+(n-2k-1)]-2(n-3k-1)(n-4k-\frac{23}{12})(n-3k) \\
& +\frac{1}{24}(5n-15k+1)[(3n-8k-5)(n-4k-1)-2(n-2k-1)] \\
& +(n-3k)(n-3k-1)(n-4k-\frac{1}{6})-\frac{1}{2}(n-3k-1)^2 \\
= & \{\frac{2}{3}(n-3k-1)+\frac{1}{6}[(3n-8k-1)(n-4k+3)+(3n-6k-1)(n-2k-1)]\}+ \\
& \frac{(2n-5k)(k-1)}{2}+\{2(n-3k)-(n-3k-1)(2n-8k-3) \\
& -\frac{1}{12}[(3n-8k-5)(n-4k-1)+(3n-6k-1)(n-2k-1)]\}(n-3k) \\
& +\frac{1}{24}(5n-15k+1)[(3n-8k-5)(n-4k-1)-2(n-2k-1)] \\
& +(n-3k)(n-3k-1)(n-4k-\frac{1}{6})-\frac{1}{2}(n-3k-1)^2 \\
> & \frac{1}{3}(n-3k-1)(2n-5k)(k-1)+\frac{3}{2}(n-3k-1)^2 \\
& +\frac{1}{24}(5n-15k+1)[(3n-8k-5)(n-4k-1)-2(n-2k-1)]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2k-3}{24}[(3n-8k-5)(n-4k-1)+(3n-6k-1)(n-2k-1)](2n-5k)\\
&-(n-3k)(n-3k-1)(n-4k-\frac{17}{6})\\
>\ &\frac{2k-3}{12}[3(n-3k-1)(n-4k-1)+3(n-3k-1)(n-2k-1)](n-3k)\\
&+\frac{5}{8}(n-3k)(n-3k-1)(n-4k-2)-(n-3k)(n-3k-1)(n-4k-2)\\
>\ &\frac{2k-3}{2}(n-3k)(n-3k-1)^2-\frac{3}{8}(n-3k)(n-3k-1)(n-4k-2)\\
>\ &\frac{8k-9}{8}(n-3k)(n-3k-1)(n-4k-2)>0\ \ (Since\ \ n>4k+2, k\geq 2).
\end{aligned}
$$

So we get $\xi^*_R(G') > \xi^*_R(G'')$ $(n > 4k+2,\ \ k \geq 2)$.

Subcase 1.2 If n is even, in a similar way to Subcase 1.1, by direct calculation, we have

$$
\begin{aligned}
&\xi^*_R(G')-\xi^*_R(G'')\\
=\ &\{\frac{2}{3}(n-3k-1)+\frac{1}{6}[(3n-8k-4)(n-4k-2)+(3n-6k-2)(n-2k)]\}\\
&\frac{(2n-5k)(k-1)}{2}+\{2(n-3k)-(n-3k-1)(2n-8k-3)\\
&-\frac{1}{12}[(3n-8k-4)(n-4k-2)+(3n-6k-2)(n-2k)]\}(n-3k)\\
&+\frac{1}{24}(5n-15k+1)[(3n-8k-4)(n-4k-2)-2(n-2k-1)]\\
&+(n-3k)(n-3k-1)(n-4k-\frac{1}{6})-\frac{1}{2}(n-3k-1)^2\\
>\ &\frac{1}{3}(n-3k-1)(2n-5k)(k-1)+2(n-3k)^2-(n-3k-1)(2n-8k-3)\\
&+\frac{1}{24}(5n-15k+1)(3n-8k-4)(n-4k-2)+(n-3k)(n-3k-1)(n-4k-\frac{1}{6})\\
&+\frac{2k-3}{24}[(3n-8k-4)(n-4k-2)+(3n-6k-2)(n-2k)](2n-5k)-\frac{1}{2}(n-3k-1)^2\\
>\ &\frac{2k-3}{12}[3(n-3k-1)(n-4k-2)+3(n-3k-1)(n-2k)](n-3k)\\
&+\frac{5}{8}(n-3k)(n-3k-1)(n-4k-2)-(n-3k)(n-3k-1)(n-4k-2)\\
&-(n-3k)(n-3k-1)(n-4k-\frac{17}{6})\\
>\ &\frac{2k-3}{2}(n-3k)(n-3k-1)^2-\frac{3}{8}(n-3k)(n-3k-1)(n-4k-2)\\
>\ &\frac{8k-9}{8}(n-3k)(n-3k-1)(n-4k-2)\\
\geq\ &0\ \ (Since\ \ n\geq 4k+2, k\geq 2).
\end{aligned}
$$

Hence $\xi^*_R(G') > \xi^*_R(G'')$ $(n \geq 4k+2,\ \ k \geq 2)$.

Case 2. If t is odd, then $t = 2k+1$. In a similar discussion as in Case 1, when $k \geq 2$, we have $\xi^*_R(C^2_{n,t}) > \xi^*_R(C^3_{n,t})$.

Combining Case 1 and Case 2, when $k \geq 2$, $\xi^*_R(C^2_{n,t}) > \xi^*_R(C^3_{n,t})$. □

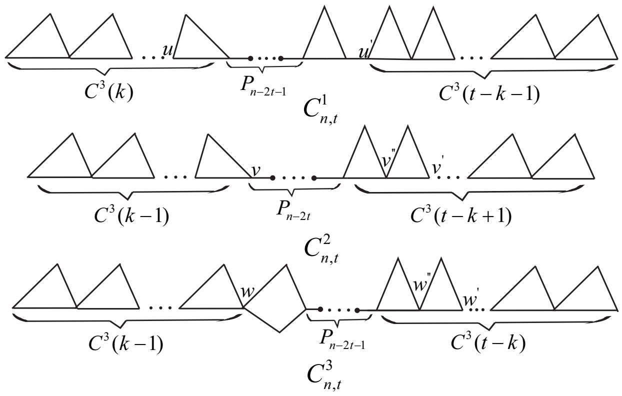

Figure 8: The graphs $C^1_{n,t}, C^2_{n,t}$ and $C^3_{n,t}$.

Theorem 3.7. *Let $G^* \in Cat(n;t)\backslash C_{n,t}$ with $n \geq 6$, the graphs $C^1_{n,t}, C^2_{n,t}$ and $C^3_{n,t}$ are as shown in Figure 8. Then (i) If $t = 2k+1$ and $n = 4k+4$ are not holding at the same time, then $\xi^*_R(G^*) \leq \xi^*_R(C^1_{n,t})$, the equality holds if and only if $G^* \cong C^1_{n,t}$; (ii) If $t = 2k+1, n = 4k+4$, then $\xi^*_R(G^*) \leq \xi^*_R(C^2_{n,t})$, the equality holds if and only if $G^* \cong C^2_{n,t}$.*

Proof. By lemmas 2.2, 2.4, 2.5, 2.6 and Theorem 3.1, one can conclude that G^* has the second multiplicative eccentricity resistance-distance in $Cat(n;t)$, it must be one of the graphs $C^1_{n,t}, C^2_{n,t}, C^3_{n,t}$ which are as shown in Figure 8.

Case 1. When $t = 2k, k \geq 2$, we have the following two cases.

Subcase 1.1 If n is odd, by lemmas 3.5 and 3.6, we have $\xi^*_R(C^1_{n,t}) > \xi^*_R(C^2_{n,t}), \xi^*_R(C^2_{n,t}) > \xi^*_R(C^3_{n,t})(n > 4k+1)$.

Subcase 1.2 If n is even, by lemmas 3.5 and 3.6, we have $\xi^*_R(C^1_{n,t}) \geq \xi^*_R(C^2_{n,t}), \xi^*_R(C^2_{n,t}) > \xi^*_R(C^3_{n,t})(n \geq 4k+2)$.

So we get $\xi^*_R(G^*) \leq \xi^*_R(C^1_{n,t})$, the equality holds if and only if $G^* \cong C^1_{n,t}$.

Case 2. When $t = 2k+1, k \geq 2$, we have the following two cases.

Subcase 2.1 If n is odd, by lemmas 3.5 and 3.6, we have $\xi^*_R(C^1_{n,t}) > \xi^*_R(C^2_{n,t}), \xi^*_R(C^2_{n,t}) > \xi^*_R(C^3_{n,t})(n \geq 4k+5)$.

Subcase 2.2 If n is even, by lemmas 3.5 and 3.6, we have $\xi^*_R(C^1_{n,t}) > \xi^*_R(C^2_{n,t}), \xi^*_R(C^2_{n,t}) > \xi^*_R(C^3_{n,t})(n > 4k+4)$.

Subcase 2.3 If $n = 4k+4$, then $C^1_{n,t} \cong C_{n,t}$, we have $\xi^*_R(C^1_{n,t}) > \xi^*_R(C^2_{n,t}), \xi^*_R(C^2_{n,t}) > \xi^*_R(C^3_{n,t})$. Since $G^* \in Cat(n;t)\backslash C_{n,t}$, hence we have (i) When $n \neq 4k+4$, $\xi^*_R(G^*) \leq \xi^*_R(C^1_{n,t})$; (ii) When $n = 4k+4$, $\xi^*_R(G^*) \leq \xi^*_R(C^2_{n,t})$.

Case 3. When $k = 1, t = 2$, by direct calculation, we have

$$\xi^*_R(G) - \xi^*_R(G'') = \frac{3n-8}{48}[(3n-13)(n-5) + (3n-7)(n-3)] + \frac{1}{6}(5n-13)(n-4) > 0 \ \ (Since \ \ n \geq 6).$$

So we have $\xi^*_R(C^1_{n,t}) > \xi^*_R(C^3_{n,t})(n \geq 6)$.

Similarly, when $k = 1, t = 3$, we have $\xi^*_R(C^1_{n,t}) > \xi^*_R(C^3_{n,t})(n \geq 8)$.

By Lemma 3.5, when $k = 1$, we have $\xi^*_R(C^1_{n,t}) \geq \xi^*_R(C^2_{n,t})$. Therefore, when $k = 1$,we have $\xi^*_R(C^1_{n,t}) \geq \xi^*_R(C^2_{n,t})$, $\xi^*_R(C^1_{n,t}) > \xi^*_R(C^3_{n,t})$.

So we get $\xi^*_R(G^*) \leq \xi^*_R(C^1_{n,t})$.

By case 1, case 2 and case 3, we have

(i) If $t = 2k+1$ and $n = 4k+4$ are not holding at the same time, then $\xi^*_R(G^*) \leq \xi^*_R(C^1_{n,t})$, the equality holds if and only if $G^* \cong C^1_{n,t}$; (ii) If $t = 2k+1, n = 4k+4$, then $\xi^*_R(G^*) \leq \xi^*_R(C^2_{n,t})$, the equality holds if and only if $G^* \cong C^2_{n,t}$. □

Corollary 3.8. *Among all graphs in $Cat(n;t)$, (i) $C_{n,t}$ is the graph with maximum multiplicative eccentricity resistance-distance; (ii) If $t = 2k+1, n = 4k+4$, then $C^2_{n,t}$ is the graph with second-maximum multiplicative eccentricity resistance-distance. Otherwise, $C^1_{n,t}$ is the graph with second-maximum multiplicative eccentricity resistance-distance.*

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