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New Error Estimation Based on Midpoint Iterative Method for Solving Nonlinear Fuzzy Fredholm Integral Equations

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Abstract. In this paper, first, we apply the successive approximations method in terms of midpoint quadrature formula to solve nonlinear fuzzy Fredholm integral equations of the second kind (NFFIE-2). Considering some assumptions, we acquire a new error estimation. Moreover, we prove the convergence of the proposed method. Then, we study the numerical stability of the proposed method with respect to the first iteration choice. Eventually, to demonstrate the accuracy of the suggested method, we present two numerical examples.

1. Introduction

The fuzzy integral equations (FIEs) are used to solve many problems in several applied sciences such as mathematical economics, electrical engineering, medicine, biology and optimal control theory. Since these equations usually can not be solved explicitly, it is required to obtain the approximate solutions. Many authors have used Banach fixed point theorem to prove the existence and uniqueness of solutions of these equations [4, 6, 13]. Numerical methods for solving fuzzy Fredholm integral equations of the second kind, based on the successive approximation methods and some other techniques, have been investigated in [5–7, 15, 18–23]. Ezzati and Ziari [8] suggested an iterative procedure via the trapezoidal rule to solve fuzzy Fredholm integral equations, and also proved the convergence of the proposed method. Baghmisheh and Ezzati investigated the error estimation and numerical solution of nonlinear fuzzy Fredholm integral equations of the second kind using triangular functions [17]. In [16], authors presented a numerical method based on the iterative method and midpoint quadrature formula for solving linear fuzzy Fredholm integral equations of the second kind. In this paper, we apply the proposed method introduced in [16] to solve nonlinear fuzzy Fredholm integral equation. Also, by considering some assumptions, we acquire a new error estimation.
Due to the error bound of the midpoint rule in comparing Trapezoidal and Simpson rules, and also according to the fact that Trapezoidal and Simpson rules can not be used to approximate integrals which are not defined in the first and the end points of the integration, in this paper we use the midpoint rule to approximate the solution of fuzzy integral equations with singular kernel.

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Here, first, we present a numerical method based on the iterative procedure and the midpoint formula to approximate the solution of NFFIE-2

$$u(x) = f(x) \oplus (FR)\int_a^b K(x,s) \odot G(u(s))ds, \tag{1}$$

where $x \in [a,b]$, $K(x,s)$ is a positive crisp kernel defined in $[a,b] \times [a,b]$, and $f : [a,b] \to R_F$, $u(x)$ is a fuzzy function and $G : R_F \to R_F$ is continuous. Then, we present the error estimation of this method. We suppose that $K : [a,b] \times [a,b] \to \mathbb{R}$ is continuous. So, K is uniformly continuous. Therefore, there exists $M_K > 0$ such that

$$M_K = \max_{a \le x,s \le b} |K(x,s)|.$$

The structure of the paper is as follows: In Section 2, basic concepts of fuzzy set theory, fuzzy Reiman integrable function and modulus of continuity are reviewed. In Section 3, the midpoint quadrature formula for solving NFFIE-2 is introduced. The error estimate, convergence and numerical stability analysis of the suggested method are presented in Section 4. We present two numerical examples in Section 5 to show the efficiency of the method. Finally, the conclusions of the paper are presented in section 6.

2. Basic concepts of fuzzy logic

Definition 1 (See [2]). A fuzzy number is a function $u : R \to [0, 1]$ having the properties:

(1) u is normal, that is $\exists\, x_0 \in R$ such that $u(x_0) = 1$,

(2) u is fuzzy convex set, i.e. $u(\lambda x + (1-\lambda)y) \ge \min\{u(x), u(y)\}, \quad \forall x, y \in R, \lambda \in [0,1]$,

(3) u is upper semi-continuous on R,

(4) the $\overline{\{\, x \in R : u(x) > 0\}}$ is compact set.

The set of all fuzzy numbers is denoted by R_F. An alternative definition which yields the same R_F is given by [11].

Definition 2 (See [10, 12]). An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \le r \le 1$, which satisfy the following requirements:

(1) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1],

(2) $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0,1],

(3) $\underline{u}(r) \le \overline{u}(r) \quad , \quad 0 \le r \le 1$.

The addition and scalar multiplication of fuzzy numbers in R_F are defined as follows:

(1) $(u \oplus v)(r) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,

(2) $(\lambda \odot u)(r) = \begin{cases} (\lambda\underline{u}(r), \lambda\overline{u}(r))\lambda \ge 0, \\ \\ (\lambda\overline{u}(r), \lambda\underline{u}(r))\lambda < 0. \end{cases}$

Definition 3 (See [1]). For two fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ the quantity $D(u,v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$ is the distance between u and v.

The following properties are hold (See [1]):

(1) (R_F, D) is a complete metric space,

(2) $D(u \oplus w, v \oplus w) = D(u, v),\ \forall\ u, v, w \in R_F$,

(3) $D(k \odot u, k \odot v) = |k|\, D(u, v),\ \forall\ u, v \in R_F,\ \forall\ k \in R$,

(4) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e),\ \forall\ u, v, w, e \in R_F$.

(5) $D(a \odot u, b \odot u) \leq |a - b| D(u, 0),\ \forall\ u \in R_F,\ \forall\ a, b \in R,\ ab > 0$.

Theorem 1 (See [2, 3]).

(1) The pair $(R_F, \oplus)$ is a commutative semigroup with $\tilde{0} = \chi_0$ zero element.

(2) For fuzzy numbers which are not crisp, there is no opposite element (that is, $(R_F, \oplus)$ can not be a group).

(3) For any $a, b \in R$ with $a, b \geq 0$ or $a, b \leq 0$ and for any $u \in R_F$, we have
$(a + b) \odot u = a \odot u \oplus b \odot u$.
For arbitrary $a, b \in R$, this property is not fulfilled.

(4) For any $\lambda, \mu \in R$ and $u \in R_F$, we have
$\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot u$.

(5) For any $\lambda \in R$ and $u, v \in R_F$, we have
$\lambda \odot (\mu \odot u) = (\lambda.\mu) \odot u$.

(6) The function $||.||_F : R_F \to R$ defined by $||u||_F = D(u, \tilde{0})$ has the usual properties of the norm, that is,
$||u||_F = 0$ if and only if $u = \tilde{o}$,
$||\lambda \odot u||_F = |\lambda| ||u||_F$, and
$||u \oplus v||_F \leq ||u||_F + ||v||_F$.

(7) $|||u||_F - ||v||_F| \leq D(u, v)$ and $D(u, v) \leq |u||_F + ||v||_F$ for any $u, v \in R_F$.

Definition 4 (See [11]). A fuzzy real number valued function $f : [a, b] \to R_F$ is said to be continuous in $x_0 \in [a, b]$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$, whenever $x \in [a, b]$ and $|x - x_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $x_0 \in [a, b]$, and denote the space of all such functions by $C_F[a, b]$.

Definition 5 (See [3]). If $X = \{f : [a, b] \to R_F |$ f is continuous$\}$, then X together with the metric

$$D^*(f, g) = \sup_{a \leq s \leq b} D(f(s), g(s))$$

is complete metric space.

Definition 6 (See [1]). Let $f : [a, b] \to R_F$, be a bounded mapping, then function
$\omega_{[a,b]}(f, .) : R_+ \cup \{0\} \to R_+$ defined by

$$\omega_{[a,b]}(f, \delta) = \sup \{D(f(x), f(y)) |\ x, y \in [a, b], |x - y| \leq \delta\}, \tag{2}$$

is called the modulus of oscillation of f on $[a, b]$. In addition, if $f \in C_F[a, b]$ (i.e. $f : [a, b] \to R_F$ is continuous on $[a, b]$), then $\omega_{[a,b]}(f, \delta)$ is called the modulus of continuity of f on $[a, b]$. Some properties of the modulus of continuity are given in below.

Theorem 2 (See [8]). The following properties hold:

(1) $D(f(x), f(y)) \leq \omega_{[a,b]}(f, |x - y|)$, for any $x, y \in [a, b]$,

(2) $\omega_{[a,b]}(f,\delta)$ is increasing function of δ,

(3) $\omega_{[a,b]}(f,0) = 0$,

(4) $\omega_{[a,b]}(f,\delta_1+\delta_2) \leq \omega_{[a,b]}(f,\delta_1) + \omega_{[a,b]}(f,\delta_2)$ for any $\delta_1, \delta_2 \geq 0$,

(5) $\omega_{[a,b]}(f,n\delta) \leq n\omega_{[a,b]}(f,\delta)$ for any $\delta \geq 0$, $n \in N$,

(6) $\omega_{[a,b]}(f,\lambda\delta) \leq (\lambda+1)\omega_{[a,b]}(f,\delta)$ for any $\delta, \lambda \geq 0$,

(7) if $[a,b] \subseteq [c,d]$ then $\omega_{[a,b]}(f,\delta) \leq \omega_{[c,d]}(f,\delta)$.

Definition 7 (See [14]). Let $f : [a,b] \to R_F$, for $\Delta x : a = x_0 < x_1 < \dots < x_n = b$, partition of the intervals $[a,b]$. Let us consider the intermediate points $\xi_i \in [x_{i-1}, x_i]$, $i = 1, \dots, n$, and $\delta : [a,b] \to R_+$. The division $P_x = ([x_{i-1}, x_i]; \xi_i); i = 1, \dots, n$, denoted shortly by $P_x = (\Delta^n, \xi)$ is said to be δ-fine if $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$. The function f is called Henstock integrable to $I \in R_F$, if for any $\varepsilon > 0$, there is function $\delta : [a,b] \to R_+$ such that for any δ-fine division we have $D(\sum_{i=0}^{n}(x_i - x_{i-1}) \odot f(\xi_i, \eta_j), I)$, where Σ denotes the fuzzy summation. Then I is called the Henstock integral of f and denoted by $I(f) = (FH)\int_a^b f(s)ds$. If the above δ is a constant function, then one recaptures the concept of Riemann integral. In this case $I \in R_F$ is called integral of f on $[a,b]$ and is be denoted by $(FR)\int_a^b f(s)ds$.

In [8], it is proved that if $f \in C_F[a,b]$, then its definite integral exists, and,

$$\underline{(FR)\int_a^b f(s;r)ds} = \int_a^b \underline{f}(s;r)ds, \quad \overline{(FR)\int_a^b f(s;r)ds} = \int_a^b \overline{f}(s;r)ds.$$

Lemma 1 (See [9]). If $f, g : [a,b] \to R_F$ are fuzzy continuous functions, then the function $F : [a,b] \to R_+$ by $F(s) = D(f(s), g(s))$ is continuous on $A = [a,b]$, and

$$D\left((FR)\int_a^b f(s)ds, (FR)\int_a^b g(s)ds\right) \leq \int_a^b D\left(f(s), g(s)\right)ds.$$

Lemma 2 (See [5]). Let $f : [a,b] \to R_F$ be a L-Lipschitz function. Then for $\Delta x : a = x_0 < x_1 < \dots < x_n = b$, we have:

$$D\left((FR)\int_a^b f(s)ds, \quad (b-a)\odot f(x)\right) \leq L\left(\frac{(b-a)^2}{4} + (x - \frac{a+b}{2})^2\right), \tag{3}$$

for any $x \in [a,b]$.

Lemma 3 (See [3]). Let $f : [a,b] \to R_F$ be a L-Lipschitz function. Then for $\Delta x : a = x_0 < x_1 < \dots < x_n = b$ and $\eta_i \in [x_{i-1}, x_i]$, $i = 1, 2, \dots, n$, we have:

$$D\left((FR)\int_a^b f(s)ds, \sum_{i=1}^{n}(x_i - x_{i-1}) \odot f(\eta_i)\right) \leq \frac{L(b-a)^2}{2n}. \tag{4}$$

3. Introducing the quadrature rule

In this section, we consider the NFFIE-2 (1), where $K(x,s)$ is a continuous positive crisp kernel defined on $[a,b]\times[a,b]$ and $G : R_F \to R_F$ is continuous function. We assume that K is continuous and therefore uniformly continuous with respect to x, s. This property implies that there exists $M_K > 0$ such that

$$M_K = \max_{a \leq x, s \leq b} |K(x,s)|.$$

3.1. Existence of a unique solution

Theorem 3 (See [8]). Let the function $K(x,s)$ be continuous and positive defined on $[a,b]\times[a,b]$ and $f:[a,b]\to R_F$ is a fuzzy continuous function. Moreover, assume that there exists $L_1>0$ such that

$$D\Big(G(u_1(x), G(u_2(y)\Big) \leq L_1.D(u_1(x), u_2(y)).$$

If $B = L_1M_K(b-a) < 1$, then the fuzzy integral equation (1) has a unique solution $u^* \in X$ and it can be obtained by the following successive approximation method

$$u_0(x) = f(x), \quad u_m(x) = f(x) \oplus (FR)\int_a^b K(x,s)\odot G(u_{m-1})(s)ds. \tag{5}$$

Moreover, the sequence of successive approximation $(u_m)_{m\geq 1}$ converges to the solution u^*, furthermore the following error estimation holds

$$D(u^*(x), u_m(x)) \leq \frac{\Gamma_0 B^{m+1}}{L_1(1-B)}, \quad \text{where} \quad \Gamma_0 = \sup_{x\in[a,b]} \parallel G(f(x)) \parallel. \tag{6}$$

3.2. Presentation of the numerical method

Now, we introduce the numerical method to find the approximate solution of the NFFIE-2 (1). For this aim, we consider the uniform partition of $[a,b]$ as $\Delta_x: a = s_0 < s_1 < \dots < s_{n-1} < s_n = b$, with $s_j = s_0 + jh$, where $h = s_j - s_{j-1}$. The following process gives the approximate solution of 1 at point x

$$F_0(x) = f(x), \quad F_m(x) = f(x) \oplus \sum_{j=0}^{n-1} h \odot K(x, s_j + \frac{h}{2}) \odot G(F_{m-1}(s_j + \frac{h}{2})). \tag{7}$$

4. Convergence and stability analysis

Here we obtain an error estimate for the proposed method. First we prove the following lemma.
Lemma 4. In the approach of integral equation(1), let the following conditions hold:

(1) $f \in C([a,b], R_F)$, $G \in C([a,b]\times R_F, R_F)$, $K \in C([a,b]\times[a,b], R)$, $K(x,s) \geq 0$, $\forall x,s \in [a,b]$,

(2) there exists $L_1 \geq 0$ such that $D(G(u(s)), G(v(s))) \leq L_1 D(u(s), v(s))$, $\forall u,v \in R_F$,

(3) $L_1 M_K(b-a) < 1$, where $M_K \geq 0$ is such that $|K(x,s)| \leq M_K$, $\forall x,s \in [a,b]$,

(4) there exists $\delta \geq 0$ such that $D(f(x), f(y)) \leq \delta|x-y|$, $\forall x,y \in [a,b]$,

(5) there exists $\eta \geq 0$ such that $|K(x,s) - K(y,s)| \leq \eta|x-y|$, $\forall x,y \in [a,b]$,

(6) there exists $\gamma \geq 0$ such that $|K(x,s) - K(x,t)| \leq \gamma|s-t|$, $\forall x,s,t \in [a,b]$,

then the function $K(x,s)\odot G(u(s))$ is $L-$ Lipschitz, where $L = [M_K L_1(\delta + \eta M(b-a)) + M\gamma]$.

Proof. For arbitrary fixed $x \in [a,b]$ we can write

$$\begin{aligned} &D\Big(K(x,s_1)\odot G(u_m(s_1)), K(x,s_2)\odot G(u_m(s_2))\Big) \leq D\Big(K(x,s_1)\odot G(u_m(s_1)), K(x,s_1)\odot G(u_m(s_2))\Big) \\ &+ D\Big(K(x,s_1)\odot G(u_m(s_2)), K(x,s_2)\odot G(u_m(s_2))\Big) \leq |K(x,s_1)| D\Big(G(u_m(s_1)), G(u_m(s_2))\Big) \\ &+ |K(x,s_1) - K(x,s_2)| D\left(G(u_m(s_2)), \tilde{0}\right) \leq M_K L_1 D\Big(u_m(s_1), u_m(s_2)\Big) + D\Big(G(u_m(s_2)), \tilde{0}\Big)\gamma|s_1 - s_2|, \end{aligned} \tag{8}$$

but

$$D\Big(G(u_m(s_2),\tilde{0})\Big) \leq D\Big(G(u_m(s_2)),G(u_0(s_2))\Big) + D\Big(G(u_0(s_2)),\tilde{0})\Big)$$
$$\leq L_1 D(u_m(s),u_0(s)) + \Gamma_0 \leq \frac{L_1\Gamma_0 M_K(b-a)}{1-L_1M_K(b-a)} + \Gamma_0 = M, \tag{9}$$

and

$$D\left(u_m(s_1),u_m(s_2)\right) \leq D\left(f(s_1),f(s_2)\right) + D\Big((FR)\int_a^b K(s_1,s)\odot G(u_m(s))ds,(FR)\int_a^b K(s_2,s)\odot G(u_m(s))ds\Big)$$
$$\leq \delta|s_1-s_2| + \int_a^b |K(s_1,s)-K(s_2,s)|D\left(G(u_m(s)),\tilde{0}\right)ds \leq \delta|s_1-s_2| + \int_a^b \eta|s_1-s_2|Mds$$
$$\leq \Big(\delta+\eta M(b-a)\Big)|s_1-s_2|, \tag{10}$$

using (8), (9) and (10) we get

$$D\Big(K(x,s_1)\odot G(u(s_1)),K(x,s_2)\odot G(u(s_2))\Big) \leq L|s_1-s_2|$$

where

$$L = [M_K L_1\left(\delta+\eta M(b-a)\right) + M\gamma].$$

□

Theorem 4. Assume (7) satisfies the conditions of lemma (4), then the iterative procedure (7), converges to the unique solution of E.q. (1), u^*, and its error estimate is as follows

$$D^*(u^*,F_m) \leq \frac{1}{1-B}\left(\frac{\Gamma_0 B^{m+1}}{L_1} + \frac{L(b-a)^2}{4n}\right). \tag{11}$$

Proof. Since

$$u_1(x) = f(x)\oplus(FR)\int_a^b K(x,s)\odot G(u_0(s))ds,$$

we have

$$D(u_1(x),F_1(x)) = D\Big((FR)\int_a^b K(x,s)\odot G(u_0(s))ds, \sum_{j=0}^{n-1} h\odot K(x,s_j+\frac{h}{2})\odot G(F_0(s_j+\frac{h}{2}))\Big)$$
$$= D\Big((FR)\int_a^b K(x,s)\odot G(f(s))ds, \sum_{j=0}^{n-1} h\odot K(x,s_j+\frac{h}{2})\odot G(f(s_j+\frac{h}{2}))\Big)$$
$$= D\Big(\sum_{j=0}^{n-1}\int_{s_j}^{s_{j+1}} K(x,s)\odot G(f(s))ds, \sum_{j=0}^{n-1} h\odot K(x,s_j+\frac{h}{2})\odot G(f(s_j+\frac{h}{2}))\Big)$$
$$\leq \sum_{j=0}^{n-1} D\Big(\int_{s_j}^{s_{j+1}} K(x,s)\odot G(f(s))ds, (s_{j+1}-sj)\odot K(x,s_j+\frac{h}{2})\odot G(f(s_j+\frac{h}{2}))\Big),$$

using lemma (2) and lemma (4), we conclude that

$$D(u_1(x),F_1(x)) \leq \sum_{j=0}^{n-1} L\times\left(\frac{(s_{j+1}-s_j)^2}{4} + (s_j+\frac{h}{2}-\frac{s_j+s_{j+1}}{2})^2\right) \leq L\frac{(b-a)^2}{4n}.$$

Clearly

$$u_2(x) = f(x) \oplus (FR)\int_a^b K(x,s) \odot G(u_1(s))ds,$$

so

$$\begin{aligned}
D(u_2(x), F_2(x)) =& D\Big((FR)\int_a^b K(x,s)\odot G(u_1(s))ds, \sum_{j=0}^{n-1} h\odot K(x,s_j+\frac{h}{2})\odot G(F_1(s_j+\frac{h}{2}))\Big)\\
&\leq D\Big((FR)\int_a^b K(x,s)\odot G(u_1(s))ds, \sum_{j=0}^{n-1} h\odot K(x,s_j+\frac{h}{2})\odot G(u_1(s_j+\frac{h}{2}))\Big)\\
&+ D\Big(\sum_{j=0}^{n-1} h\odot K(x,s_j+\frac{h}{2})\odot G(u_1(s_j+\frac{h}{2})), \sum_{j=0}^{n-1} h\odot K(x,s_j+\frac{h}{2})\odot G(F_1(s_j+\frac{h}{2}))\Big)\\
&\leq \sum_{j=0}^{n-1} D\Big((FR)\int_{s_j}^{s_{j+1}} K(x,s)\odot G(u_1(s))ds, h\odot K(x,s_j+\frac{h}{2})\odot G(u_1(s_j+\frac{h}{2}))\Big)\\
&+ h\sum_{j=0}^{n-1} D\Big(K(x,s_j+\frac{h}{2})\odot G(u_1(s_j+\frac{h}{2})), K(x,s_j+\frac{h}{2})\odot G(F_1(s_j+\frac{h}{2}))\Big)\\
&\leq \frac{L(b-a)^2}{4n} + h\sum_{j=0}^{n-1} |K(x,s_j+\frac{h}{2})| \times D\left(G(u_1(s_j+\frac{h}{2})), G(F_1(s_j+\frac{h}{2}))\right) \leq \frac{L(b-a)^2}{4n}\\
&+ hM_K L_1 \sum_{j=0}^{n-1} D\left(u_1(s_j+\frac{h}{2}), F_1(s_j+\frac{h}{2})\right) \leq \frac{L(b-a)^2}{4n} + M_K L_1(b-a)\frac{L(b-a)^2}{4n}\\
&= \frac{L(b-a)^2}{4n}(1+B).
\end{aligned}$$

By induction, for $m \geq 3$, we have:

$$D(u_m(x), F_m(x)) \leq \frac{L(b-a)^2}{4n}(1+B+...+B^{m-1}) \leq \frac{1-B^m}{1-B}\frac{L(b-a)^2}{4n},$$

$\forall x \in [a,b]$.

Since $B < 1$, according to $\frac{1-B^m}{1-B} \leq \frac{1}{1-B}$ for each $m \in N$, we get:

$$D^*(u_m, F_m) \leq \frac{L(b-a)^2}{4n(1-B)}. \tag{12}$$

Considering inequalities (6) and (12), we have:

$$D^*(u^*, F_m) \leq D^*(u^*, u_m) + D^*(u_m, F_m) \leq \frac{1}{1-B}\left(\frac{\Gamma_0 B^{m+1}}{L_1} + \frac{L(b-a)^2}{4n}\right), \tag{13}$$

which completes the proof. □

Remark 1. Since $B < 1$, it is easy to show that

$$\lim_{m,n\to\infty} D^*(u^*, F_m) = 0. \tag{14}$$

Thus, the proposed method is convergent.

4.1. Numerical stability analysis

In order to investigate the numerical stability of (7) with respect to small perturbation in the starting approximation, we consider another starting approximation $v_0 \in C_F[a,b]$ such that there exists $\varepsilon > 0$ for which $D(u_0(x), v_0(x)) < \varepsilon, \forall x \in [a,b]$. The following sequence of successive approximations is obtained as follows:

$$v_0(x) = f(x), \quad v_m(x) = f(x) \oplus (FR)\int_a^b K(x,s) \odot G(v_{m-1}(s))ds. \tag{15}$$

Thus,

$$\overline{v_m(x)} = f(x) \oplus \sum_{j=0}^{n-1} K(x, s_j + \frac{h}{2})G(\overline{v_{m-1}}(s_j + \frac{h}{2})). \tag{16}$$

As in [6], given the following definition, we prove the numerical stability of the method.

Definition 8 (See [6]). The algorithm used to solve the integral equation (1) is numerically stable with respect to the choice of the first iteration if and only if there exists two constants $K_1, K_2 > 0$, which are independent of h such that,

$$D(F_m(x), \overline{v_m}(x)) < K_1\varepsilon + K_2 h \quad \text{where} \quad x \in [a,b]. \tag{17}$$

Theorem 5. With assumptions of Theorem 4 and $B = L_1 M_K(b-a) < 1$, the suggested method (7) is numerically stable.

Proof. Similarly as Theorem 4, it follows that

$$D^*(v_m, \overline{v_m}) \leq \frac{\bar{L}(b-a)^2}{4n(1-B)}.$$

From Definition 3, we obtain

$$\begin{aligned} D(F_m(x), \overline{v_m}(x)) &\leq D(F_m(x), u_m(x)) + D(u_m(x), v_m(x)) + D(v_m(x), \overline{v_m}(x)) \\ &\leq \frac{L(b-a)^2}{4n(1-B} + \frac{\bar{L}(b-a)^2}{4n(1-B} + D(u_m(x), v_m(x)). \end{aligned}$$

Since $D(u_0(x), v_0(x)) < \varepsilon, \quad \forall x \in [a,b]$, we can write

$$D\left((u_1(x), v_1(x))\right) \leq D\Big((FR)\int_a^b K(x,s) \odot G(u_0(s))ds, (FR)\int_a^b K(x,s) \odot G(v_0(s))ds\Big) \leq M_k L_1(b-a)\varepsilon < \varepsilon.$$

By induction, we have

$$D(u_m(x), v_m(x)) < B^m\varepsilon < \varepsilon,$$

thus

$$D(F_m(x), \overline{v_m}(x)) \leq \frac{(L+\bar{L})(b-a)^2}{4n(1-B)} + \varepsilon = K_1\varepsilon + K_2 h,$$

with $K_1 = 1, \quad K_2 = \frac{(L+\bar{L})(b-a)}{4(1-B)}$. □

So, the numerical stability is proved.

5. Numerical examples

To demonstrate the accuracy of the method in the previous section, we present two examples.

Example 1. Consider the following nonlinear fuzzy Fredholm integral equation as:

$$u(x) = f(x) \oplus (FR) \int_0^1 K(x,s) \odot (u(s))^2 ds,$$

where

$$f(x,r) = (\underline{f}(x,r), \overline{f}(x,r)) = \left(rx - \frac{64}{15} r^2 x^{14}, (2-r)x - \frac{64}{15}(2-r)^2 x^{14}\right),$$

and

$$K(x,s) = \frac{4x^{14}}{(1-s)^{1/2}},$$

with the exact solution

$$u(x,r) = \left(\underline{u}(x,r), \overline{u}(x,r)\right) = (rx, (2-r)x).$$

The approximate and exact solutions have been compared in Table 1 .

Table 1. The accuracy of the solution to Example 1 in $x = 0.5$

	n=100, m=3		n=200, m=5	
r-level	$\lvert\underline{u} - \underline{z}_m\rvert$	$\lvert\overline{u} - \overline{z}_m\rvert$	$\lvert\underline{u} - \underline{z}_m\rvert$	$\lvert\overline{u} - \overline{z}_m\rvert$
0.0	0	5.68e-4	0	5.64e-4
0.2	5.68e-6	4.60e-4	5.63e-6	4.57e-4
0.4	2.27e-5	3.63e-4	2.25e-5	3.61e-4
0.6	5.11e-5	2.78e-4	5.07e-5	2.76e-4
0.8	9.09e-5	2.04e-4	9.02e-5	2.03e-4
1.0	1.42e-4	1.42e-4	1.41e-4	1.41e-4

Example 2. Consider the following nonlinear fuzzy Fredholm integral equation :

$$u(x) = f(x) \oplus (FR) \int_0^1 K(x,s) \odot (u(s))^2 ds,$$

where

$$f(x,r) = \left(\underline{f}(x,r), \overline{f}(x,r)\right) = \left(x^3(r^2+5) - \frac{9}{80}(r^2+5)^2(1-\frac{3}{2}x)^7, x^3(6-3r^2) - \frac{9}{80}(6-3r^2)^2(1-\frac{3}{2}x)^7\right),$$

and

$$K(x,s) = \frac{3(1-\frac{3}{2}x)^7}{4s^{1/3}}.$$

The exact solution of this example is given by

$$u(x,r) = \left(\underline{u}(x,r), \overline{u}(x,r)\right) = \left(x^3(r^5+2), x^3(6-3r^2)\right).$$

The approximate and exact solutions have been compared in Table 2.

Table 2. The accuracy of the solution to Example 2 in $x = 0.5$

	n=20 , m=5		n=50 , m=10	
r-level	$\lvert\underline{u} - \underline{z}_m\rvert$	$\lvert\overline{u} - \overline{z}_m\rvert$	$\lvert\underline{u} - \underline{z}_m\rvert$	$\lvert\overline{u} - \overline{z}_m\rvert$
0.0	6.73857e-7	9.69868e-7	1.08012e-7	1.55515e-7
0.2	6.84670e-7	9.31522e-7	1.09746e-7	1.49359e-7
0.4	7.17621e-7	8.21103e-7	1.15033e-7	1.31637e-7
0.6	7.74254e-7	6.52490e-7	1.24119e-7	1.04584e-7
0.8	8.57139e-7	4.48864e-7	1.37420e-7	7.19285e-8
1.0	9.69868e-7	2.42769e-7	1.55515e-7	3.88930e-8

6. Conclusions

In this paper, we applied the proposed method in [16] to solve NFFIE-2. Also, in Lemma 4, we proved that the product of a bounded function and a Lipschitz function is a Lipschitz function. Using this lemma, we acquired a new error estimation for NFFIE-2. The numerical results also show the accuracy of the proposed method.

References

[1] G. A. Anastassiou, Fuzzy mathematics: Approximation theory, Springer-Verlag Berlin Heidelberg, (2010).

[2] G.A. Anastassiou, S.G. Gal, On a fuzzy trigonometric approximation theorem of Weirstrass-type, Jornal of Fuzzy Mathematics, Los Angeles, **9** (2001), 701-708.

[3] A.M. Bica, Error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations, Information Sciences, **178** (2008), 1279-1292.

[4] K. Balachandran, K. Kanagarajan, Existence of solutions of general nonlinear fuzzy Volterra-Fredholm integral equations, J. Appl. Math.Stoch. Anal, **3** (2005), 333–343.

[5] B. Bede, SG. Gal, Quadrature rules for integrals of fuzzy-number-valued functions, Fuzzy Sets Syst, **145**(2004), 359–380.

[6] A.M. Bica, C. Popescu, Approximating the solution of nonlinear Hammerstein fuzzy integral equations, Fuzzy Sets Syst, **95** (2014), 1–17.

[7] A.M. Bica, C. Popescu, Numerical solutions of the nonlinear fuzzy Hammerstein-Volterra delay integral equations, Inf.Sci, **95** (2013), 236–255.

[8] R. Ezzati, S. Ziari, Numerical solution of nonlinear fuzzy Fredholm integral equations using iterative method, Appl. Math. Comput, **225** (2013), 33–42.

[9] SG. Gal, Approximation theory in fuzzy setting, Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton (2000), 617-666.

[10] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems, **18** (1986), 31-43.

[11] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, **24** (1987), 301-317.

[12] M. Friedman, M. Ma, A. Kandel, Numerical solutions of fuzzy differential and integral equations, Fuzzy Sets and Systems **106**(1999), 35-48.

[13] P.V. Subrahmanyam, S.K. Sudarsanam, A note on fuzzy Volterra integral equations, Fuzzy Sets Syst, **95** (1996), 237–240

[14] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions, Fuzzy Sets System, **120** (2001), 523–532.

[15] C. Wu, S. Song, H. Wang, On the basic solutions to the generalized fuzzy integral equation, Fuzzy Sets Syst, **95** (1998), 255–260.

[16] V. Samadpour, R. Ezzati, Numerical solution of linear fuzzy Fredholm integral equations of second kind using iterative method and midpoint quadrature formula, Journal of Intelligent & Fuzzy Systems, **33** (2017), 1293–1302.

[17] M. Baghmisheh, R. Ezzati, Error estimation and numerical solution of nonlinear fuzzy Fredholm integral equations of the second kind using triangular functions, Journal of Intelligent & Fuzzy Systems, **30(2)** (2016), 639–649.

[18] V. Samadpour, R. Ezzati, Numerical solution of two dimensional nonlinear fuzzy Fredholm integral equations of second kind using hybrid of block-pulse functions and Bernstein polynomial, Filomat, **32(14)** (2018), 4923–4935.

[19] A.M. Gholam, R. Ezzati, Solving linear fuzzy Fredholm integral equations of the second kind via iterative method and Simpson quadrature rule: a review, TWMS Journal of pure and applied mathematics, **8(2)** (2018), 121–147.

[20] S. Ziari, R. Ezzati, S. Abbasbandy, Numerical solution of linear fuzzy Fredholm integral equations of the second kind using fuzzy Haar wavelet, Advances in Computational Intelligence, **299** (2012), 79–89.

[21] F. Mokhtarnejad, R. Ezzati, The numerical solution of nonlinear Hammerstein fuzzy integral equations by using fuzzy wavelet like operator, Journal of Intelligent & Fuzzy Systems, **28** (2015), 1617–1626.

[22] M. Baghmisheh, R. Ezzati, Numerical solution of nonlinear fuzzy Fredholm integral equations of the second kind using hybrid of block-pulse functions and Taylor series, Advances in Difference Equations, 5 (2015) 22-34.

[23] F. Mokhtarnejad, R. Ezzati, Existence and uniqueness of the solution of fuzzy-valued integral equations if mixed type, Iranian journal of fuzzy systems, 12(2)(2015), 87–94