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S-Iteration Process of Halpern-Type for Common Solutions of Nonexpansive Mappings and Monotone Variational Inequalities

D. R. Sahu[a], Ajeet Kumar[a], Ching-Feng Wen[b,c]

[a]*Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India*
[b]*Center for Fundamental Science and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung-80708, Taiwan*
[c]*Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung-80708, Taiwan*

Abstract. This paper is devoted to the strong convergence of the S-iteration process of Halpern-type for approximating a common element of the set of fixed points of a nonexpansive mapping and the set of common solutions of variational inequality problems formed by two inverse strongly monotone mappings in the framework of Hilbert spaces. We also give some numerical examples in support of our main result.

1. Introduction

The theory of variational inequalities, introduced in 1964 by the italian mathematician Stampacchia [1] has emerged as a powerful tool in nonlinear analysis and optimization. During last three decades, this theory has been developed in several directions using novel and innovative techniques; see for example [2–7] and the references therein. Various kinds of iterative algorithms to solve the variational inequalities have been developed by many authors. It is well known that the variational inequality problems are equivalent to the fixed point problems as well as zero point problems; see for example [3, 8]. Due to this equivalence formulation, the solution of variational inequalities can be computed by using the iterative projection techniques; see for example [9–11]. Recently, some authors have computed the common elements of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequalities; see [12–14] as well as solved the system of variational inequalities; see for example [15–17].

We consider the following variational inequality problem:

$$\text{find } x^* \in C \text{ such that } \langle Ax^*, x - x^*\rangle \geq 0 \text{ for all } x \in C, \tag{1}$$

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Corresponding author: Ching-Feng Wen

Email addresses: drsahudr@gmail.com (D. R. Sahu), ajeetbhu09@gmail.com (Ajeet Kumar), cfwen@kmu.edu.tw (Ching-Feng Wen)

where C is a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ is a monotone mapping. The problem (1) is denoted by $VI(C, A)$ and the solution set of the variational inequality problem $VI(C, A)$ is denoted by $\Omega[VI(C, A)]$, i.e.,

$$\Omega[VI(C, A)] = \{x^* \in C : \langle Ax^*, x - x^* \rangle \geq 0 \text{ for all } x \in C\}.$$

In 1967, Browder [18] and Halpern [19] independently proved the strong convergence of the path $\{x_t = tu + (1-t)Sx_t : t \in (0, 1)\}$ as $t \to 0^+$ for nonexpansive mapping S on a bounded subset C in a Hilbert spaces and in 1980, Reich [20] proved the strong convergence in uniformly smooth Banach spaces. In 1967, Halpern [19] introduced an iteration process for approximation of fixed points of a nonexpansive mapping $S : C \to C$ as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n \text{ for all } n \in \mathbb{N}, \tag{2}$$

where $u \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. He proved that the iteration process (2) converges to the fixed point of S under the choice of $\alpha_n = \frac{1}{n^\theta}$, where $\theta \in (0, 1)$. The iteration process (2) is called Halpern iteration process. The strong convergence of the explicit iteration process (2) was further studied by Lions [21], Shioji and Takahashi [22], Wong, Sahu and Yao [23] and many more under certain assumptions on iteration parameter α_n.

For computing an element of $Fix(S) \cap \Omega[VI(C, A)]$, in 2005, Iiduka and Takahashi [24] introduced an iterative scheme:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \text{ for all } n \in \mathbb{N}, \tag{3}$$

where $A : C \to H$ is an α-inverse strongly monotone mapping, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ and proved a strong convergence theorem. Chen, Zhang and Fan [25] considered an iterative scheme by viscosity approximation method as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \text{ for all } n \in \mathbb{N}, \tag{4}$$

where f is a contraction mapping from C into itself. They proved that the sequence $\{x_n\}$ generated by (4) converges strongly to $x^* \in Fix(S) \cap \Omega[VI(C, A)]$, which is also a solution of the following variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0 \text{ for all } x \in Fix(S) \cap \Omega[VI(C, A)]. \tag{5}$$

In 2010, Jung [26] introduced a composite iterative algorithm for finding an element in $Fix(S) \cap \Omega[VI(C, A)]$ as follows:

$$\begin{cases} x_{n+1} = (1 - \beta_n)y_n + \beta_n SP_C(y_n - \lambda_n Ay_n), \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \text{ for all } n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, 2\alpha)$ and $\{\beta_n\} \subset [0, 1)$ are some sequences. He proved the strong convergence of the sequence $\{x_n\}$ generated by (6) to a point $x^* \in Fix(S) \cap \Omega[VI(C, A)]$ satisfying (5). Recently, Cho, Li and Kang [27] introduced a new iterative algorithm to solve the problem of finding a common solution to the zero point problems involving two monotone operators and fixed point problems involving asymptotically strictly pseudo-contractive mappings in Hilbert spaces. Following Jung [26], in 2017, Lin, Sharma, Kumar and Gurudwan [28] introduced a viscosity approximation method for common fixed point problems of a finite family of nonexpansive mappings.

On the other hand, Mann iteration [29] and Ishikawa iteration [30] are well known iteration processes for approximating fixed points of nonexpansive mappings; see [31–33]. In 2007, Agarwal, O'Regan and Sahu [34] introduced the S-iteration process as follows:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Sy_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Sx_n \text{ for all } n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying some suitable conditions. It is remarked that the S-iteration process (7) is independent of the Mann [29] and Ishikawa [30] iteration processes. The S-iteration process (7) is more applicable than the Picard [35], Mann [29] and Ishikawa [30] iteration processes because it converges faster than these iteration processes for contraction mappings and also works for nonexpansive mappings. In recent years the S-iteration process attracted many mathematicians as an alternate iteration process for fixed point problems, common fixed point problems and other allied areas; see [36, 37].

Motivated by Halpern [19] and Agarwal, O'Regan and Sahu [34], in 2011, Sahu [38] introduced the S-iteration process of Halpern-type as follows:

$$\begin{cases} x_{n+1} = \beta_n S x_n + (1-\beta_n) S y_n, \\ y_n = \alpha_n u + (1-\alpha_n) x_n \text{ for all } n \in \mathbb{N}, \end{cases} \tag{8}$$

where $u \in C$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are some sequences in $(0,1)$ satisfying some suitable conditions. He proved the strong convergence of the sequence $\{x_n\}$ generated by (8) to $R_{Fix(S)}(u)$, where $R_{Fix(S)}$ is the sunny nonexpansive retraction from C onto $Fix(S)$ in the framework of uniformly convex Banach space.

Inspired and motivated by the results in [25, 26], the purpose of this paper is to introduce a new iterative algorithm based on S-iteration process of Halpern-type (8) for solving the following variational inequality problem:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0 \text{ for all } x \in Fix(S) \cap \Omega[VI(C,A)] \cap \Omega[VI(C,B)], \tag{9}$$

where $S : C \to C$ is nonexpansive and $A, B : C \to H$ are inverse strongly monotone mappings. It is interesting to note that algorithms (4), (6) and (7) are not applicable for solving variational inequality problem (9). Our algorithm algorithm is applicable for solving variational inequality problem (9) and hence it is an improvement upon algorithms (4), (6) and (7). We also provide some numerical examples to show the implementation of our main result.

2. Preliminaries

This section contains some definitions and lemmas which will be needed in proof of our main result.

Let C be a nonempty subset of a real Hilbert space H with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively. We denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. The symbol $\mathbb{N}$ stands for the set of all natural numbers and I the identity mapping of H. A mapping $T : C \to H$ is called (see [39])

(i) monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0 \text{ for all } x, y \in C,$$

(ii) η-strongly monotone if there exists a positive real number η such that

$$\langle Tx - Ty, x - y \rangle \geq \eta \left\| x - y \right\|^2 \text{ for all } x, y \in C,$$

(iii) α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \left\| Tx - Ty \right\|^2 \text{ for all } x, y \in C,$$

(iv) k-Lipschitzian if there exists a constant $k \in [0, \infty)$ such that

$$\left\| Tx - Ty \right\| \leq k \left\| x - y \right\| \text{ for all } x, y \in C,$$

(v) nonexpansive if

$$\left\| Tx - Ty \right\| \leq \left\| x - y \right\| \text{ for all } x, y \in C.$$

It is easy to see that an α-inverse strongly monotone mapping T is monotone and Lipschitz continuous.

A mapping $T : C \to C$ is called λ-strictly pseudocontractive if there exists a constant λ with $0 \leq \lambda < 1$ such that

$$\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 + \lambda\|(I - T)x - (I - T)y\|^2 \text{ for all } x, y \in C.$$

Put $A = I - T$, where $T : C \to C$ is λ-strictly pseudocontractive mapping. Then A is $\frac{1-\lambda}{2}$-inverse strongly monotone (see [40]).

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point $P_C(x)$ of C such that

$$\|x - P_C(x)\| \leq \|x - y\| \text{ for all } y \in C.$$

The mapping P_C is called the metric projection [41] from H onto C. It is remarkable that the metric projection mapping P_C is nonexpansive from H onto C (see Agarwal O'Regan and Sahu [32] for other properties of projection mappings).

Lemma 2.1. ([42]) *For the metric projection mapping P_C, the following properties hold:*

(i) $P_C(x) \in C$ *for all* $x \in H$;
(ii) $\langle x - P_C(x), P_C(x) - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$;
(iii) $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$ *for all* $x \in H$ *and* $y \in C$;
(iv) $\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2$ *for all* $x, y \in H$.

Lemma 2.2. ([43]) *In a real Hilbert space H, there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \text{ for all } x, y \in H.$$

Lemma 2.3. ([42]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ a nonexpansive mapping. Then the mapping $(I - T)$ is demiclosed on C, i.e., $x_n \rightharpoonup x$ in H and $(I - T)x_n \to y$ imply that $x \in C$ and $(I - T)x = y$.*

Lemma 2.4. ([44]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \beta_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following conditions:

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \dfrac{\beta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} \beta_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main result

Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be two α-inverse strongly monotone and β-inverse strongly monotone mappings, respectively. Assume that $f : C \to C$ is a k-contraction mapping and $S : C \to C$ is a nonexpansive mapping such that $Fix(S) \cap \Omega[VI(C, A)] \cap \Omega[VI(C, B)] \neq \emptyset$. We introduce our iterative algorithm for solving the variational inequality problem (9) as follows:

Algorithm 3.1. (1) *Initialization: Select $x_1 \in C$ arbitrarily.*
(2) *Iterative step: Select $\{\alpha_n\}, \{\beta_n\}, \{s_n\}$ and $\{t_n\}$ as iteration parameters and compute the $(n + 1)^{th}$ iteration as follows:*

$$\begin{cases} x_{n+1} = \beta_n SP_C(x_n - s_n Ax_n) + (1 - \beta_n)SP_C(y_n - t_n By_n), \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n \text{ for all } n \in \mathbb{N}, \end{cases} \tag{10}$$

where $\{\alpha_n\}, \{\beta_n\}, \{s_n\}$ and $\{t_n\}$ are real sequences satisfying

(SH1) $\{\alpha_n\} \subset (0, 1)$ *and* $\{\beta_n\} \subset (0, 1)$ *with* $0 < a \leq \beta_n \leq b < 1$ *for all* $n \in \mathbb{N}$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(SH2) $\{s_n\} \subset (0, 2\alpha)$ *and* $\{t_n\} \subset (0, 2\beta)$ *with* $0 < c \leq s_n \leq d < 2\alpha$, $0 < l \leq t_n \leq m < 2\beta$;

(SH3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$; $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$.

Remark 3.2. *If $A = 0$ and $B = 0$, then our algorithm* (10) *reduces to the iteration process S-iteration process of Halpern-type defined by* (8).

We now establish a strong convergence theorem which shows that the element of $Fix(S) \cap \Omega[VI(C, A)] \cap \Omega[VI(C, B)]$ can be approximated by Algorithm 3.1.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone mappings, respectively. Assume that $f : C \to C$ is a k-contraction mapping and $S : C \to C$ is a nonexpansive such that $Fix(S) \cap \Omega[VI(C, A)] \cap \Omega[VI(C, B)]$ is nonempty. Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then the sequence $\{x_n\}$ converges strongly to $x^* \in Fix(S) \cap \Omega[VI(C, A)] \cap \Omega[VI(C, B)]$, which is the unique solution of variational inequality problem* (9).

Proof. Set $z_n := P_C(x_n - s_n Ax_n)$ and $v_n := P_C(y_n - t_n By_n)$ for all $n \in \mathbb{N}$. Let $q \in Fix(S) \cap \Omega[VI(C, A)] \cap \Omega[VI(C, B)]$. Then $q = P_C(q - s_n Aq) = P_C(q - t_n Bq)$. Since $(I - s_n A)$ and $(I - t_n B)$ are nonexpansive, we have

$$\|z_n - q\| = \|P_C(x_n - s_n Ax_n) - P_C(q - s_n Aq)\| \leq \|x_n - q\| \text{ for all } n \in \mathbb{N}$$

and

$$\|v_n - q\| \leq \|y_n - q\| \text{ for all } n \in \mathbb{N}.$$

We proceed with the following steps.

Step 1. *$\{x_n\}$ is bounded.*

From (10), we have

$$\begin{aligned} \|y_n - q\| &\leq \alpha_n \|f(x_n) - q\| + (1 - \alpha_n)\|x_n - q\| \\ &\leq \alpha_n(\|f(x_n) - f(q)\| + \|f(q) - q\|) + (1 - \alpha_n)\|x_n - q\| \\ &\leq \alpha_n(k\|x_n - q\| + \|f(q) - q\|) + (1 - \alpha_n)\|x_n - q\| \\ &= (1 - \alpha_n(1 - k))\|x_n - q\| + \alpha_n\|f(q) - q\|. \end{aligned} \tag{11}$$

From (10) and (11), we have

$$\begin{aligned} \|x_{n+1} - q\| &\leq \beta_n\|Sz_n - q\| + (1 - \beta_n)\|Sv_n - q\| \\ &\leq \beta_n\|z_n - q\| + (1 - \beta_n)\|v_n - q\| \\ &\leq \beta_n\|x_n - q\| + (1 - \beta_n)\|y_n - q\| \\ &\leq \beta_n\|x_n - q\| + (1 - \beta_n)\Big((1 - \alpha_n(1 - k))\|x_n - q\| + \alpha_n\|f(q) - q\|\Big) \\ &= \Big(1 - \alpha_n(1 - \beta_n)(1 - k)\Big)\|x_n - q\| + \alpha_n(1 - \beta_n)\|f(q) - q\| \\ &\leq \max\left\{\|x_n - q\|, \frac{1}{1 - k}\|f(q) - q\|\right\} \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Hence $\{x_n\}$ is bounded. Observe that the sequences $\{y_n\}, \{z_n\}, \{v_n\}, \{Ax_n\}, \{By_n\}, \{Sz_n\}$ and $\{Sv_n\}$ are bounded.

Step 2. $\|x_{n+1} - x_n\| \to 0$ *as* $n \to \infty$.

Note that

$$\begin{aligned} \|z_n - z_{n-1}\| &\leq \|(x_n - s_n Ax_n) - (x_{n-1} - s_{n-1}Ax_{n-1})\| \\ &= \|(x_n - s_n Ax_n) - (x_{n-1} - s_n Ax_{n-1}) + (x_{n-1} - s_n Ax_{n-1}) - (x_{n-1} - s_{n-1}Ax_{n-1})\| \\ &\leq \|x_n - x_{n-1}\| + |s_{n-1} - s_n|\|Ax_{n-1}\|. \end{aligned}$$

Similarly, we have

$$\|v_n - v_{n-1}\| \leq \|y_n - y_{n-1}\| + |t_{n-1} - t_n|\|By_{n-1}\|.$$

From (10), we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - \alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})x_{n-1}\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) + (1-\alpha_n)x_n - (1-\alpha_n)x_{n-1} + (1-\alpha_n)x_{n-1} \\
&\quad -\alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})x_{n-1}\| \\
&= \|\alpha_n(f(x_n) - f(x_{n-1})) + (1-\alpha_n)(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})(f(x_{n-1}) - x_{n-1})\| \\
&\leq \alpha_n k\|x_n - x_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - x_{n-1}\| \\
&= (1 - \alpha_n(1-k))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - x_{n-1}\|. \qquad (12)
\end{aligned}$$

From (10) and (12), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\beta_n Sz_n + (1-\beta_n)Sv_n - \beta_{n-1}Sz_{n-1} - (1-\beta_{n-1})Sv_{n-1}\| \\
&= \|\beta_n Sz_n - \beta_n Sz_{n-1} + \beta_n Sz_{n-1} + (1-\beta_n)Sv_n - (1-\beta_n)Sv_{n-1} + (1-\beta_n)Sv_{n-1} \\
&\quad -\beta_{n-1}Sz_{n-1} - (1-\beta_{n-1})Sv_{n-1}\| \\
&= \|\beta_n(Sz_n - Sz_{n-1}) + (1-\beta_n)(Sv_n - Sv_{n-1}) + (\beta_n - \beta_{n-1})(Sz_{n-1} - Sv_{n-1})\| \\
&\leq \beta_n\|z_n - z_{n-1}\| + (1-\beta_n)\|v_n - v_{n-1}\| + |\beta_n - \beta_{n-1}|\|z_{n-1} - v_{n-1}\| \\
&\leq \beta_n\{\|x_n - x_{n-1}\| + |s_{n-1} - s_n|\|Ax_{n-1}\|\} + (1-\beta_n)\{\|y_n - y_{n-1}\| \\
&\quad +|t_{n-1} - t_n|\|By_{n-1}\|\} + |\beta_n - \beta_{n-1}|\|z_{n-1} - v_{n-1}\| \\
&\leq \beta_n\{\|x_n - x_{n-1}\| + |s_{n-1} - s_n|\|Ax_{n-1}\|\} + (1-\beta_n)\{(1 - \alpha_n(1-k))\|x_n - x_{n-1}\| \\
&\quad +|\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - x_{n-1}\| + |t_{n-1} - t_n|\|By_{n-1}\|\} + |\beta_n - \beta_{n-1}|L_1 \\
&\leq \Big(1 - \alpha_n(1-\beta_n)(1-k)\Big)\|x_n - x_{n-1}\| + |s_{n-1} - s_n|\|Ax_{n-1}\| + |t_{n-1} - t_n|\|By_{n-1}\| \\
&\quad +|\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - x_{n-1}\| + |\beta_n - \beta_{n-1}|L_1 \\
&\leq \Big(1 - \alpha_n(1-\beta_n)(1-k)\Big)\|x_n - x_{n-1}\| + L_2|s_{n-1} - s_n| + L_3|t_{n-1} - t_n| \\
&\quad +L_4|\alpha_n - \alpha_{n-1}| + L_1|\beta_n - \beta_{n-1}|, \qquad (13)
\end{aligned}$$

where $L_1 = \sup_{n\in\mathbb{N}}\{\|z_n - v_n\|\}$, $L_2 = \sup_{n\in\mathbb{N}}\{\|Ax_n\|\}$, $L_3 = \sup_{n\in\mathbb{N}}\{\|By_n\|\}$ and $L_4 = \sup_{n\in\mathbb{N}}\{\|f(x_n) - x_n\|\}$. Note $\sum_{n=1}^{\infty}\alpha_n = \infty$. Therefore, from (13) and Lemma 2.4, we obtain that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Moreover, from (12), we have $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.

Step 3. *$\|By_n - Bq\| \to 0$ and $\|Ax_n - Aq\| \to 0$ as $n \to \infty$.*

Note that $v_n = P_C(I - t_nB)y_n$. Hence, we have

$$\begin{aligned}
\|v_n - q\|^2 &= \|P_C(I - t_nB)y_n - P_C(I - t_nB)q\|^2 \\
&\leq \|(y_n - t_nBy_n) - (q - t_nBq)\|^2 \\
&= \|(y_n - q) - t_n(By_n - Bq)\|^2 \\
&= \|y_n - q\|^2 + t_n^2\|By_n - Bq\|^2 - 2t_n\langle y_n - q, By_n - Bq\rangle \\
&\leq \|y_n - q\|^2 - t_n(2\beta - t_n)\|By_n - Bq\|^2 \\
&\leq \|y_n - q\|^2 - l(2\beta - m)\|By_n - Bq\|^2.
\end{aligned}$$

Similarly,

$$\|z_n - q\|^2 \leq \|x_n - q\|^2 - c(2\alpha - d)\|Ax_n - Aq\|^2.$$

From (10), we have

$$\begin{aligned}
& \|x_{n+1}-q\|^2 \\
\leq\ & \beta_n\|z_n-q\|^2+(1-\beta_n)\|v_n-q\|^2 \\
\leq\ & \beta_n\|x_n-q\|^2+(1-\beta_n)\Big(\|y_n-q\|^2-l(2\beta-m)\|By_n-Bq\|^2\Big) \\
\leq\ & \beta_n\|x_n-q\|^2+(1-\beta_n)\Big(\alpha_n\|f(x_n)-q\|^2+(1-\alpha_n)\|x_n-q\|^2-l(2\beta-m)\|By_n-Bq\|^2\Big) \\
=\ & [\beta_n+(1-\beta_n)(1-\alpha_n)]\|x_n-q\|^2+(1-\beta_n)\alpha_n\|f(x_n)-q\|^2-(1-\beta_n)l(2\beta-m)\|By_n-Bq\|^2 \\
=\ & (1-\alpha_n(1-\beta_n))\|x_n-q\|^2+(1-\beta_n)\alpha_n\|f(x_n)-q\|^2-(1-\beta_n)l(2\beta-m)\|By_n-Bq\|^2 \\
\leq\ & \|x_n-q\|^2+\alpha_n\|f(x_n)-q\|^2-(1-\beta_n)l(2\beta-m)\|By_n-Bq\|^2,
\end{aligned}$$

which immediately gives that

$$\begin{aligned}
(1-\beta_n)l(2\beta-m)\|By_n-Bq\|^2 &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+\alpha_n\|f(x_n)-q\|^2 \\
&\leq (\|x_n-q\|+\|x_{n+1}-q\|)\|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-q\|^2.
\end{aligned}$$

Since $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$, we have $\|By_n-Bq\|\to 0$ as $n\to\infty$. Proceeding in the same manner as above and using (10), we have

$$\begin{aligned}
& \|x_{n+1}-q\|^2 \\
\leq\ & \beta_n\|z_n-q\|^2+(1-\beta_n)\|v_n-q\|^2 \\
\leq\ & \beta_n\Big(\|x_n-q\|^2-s_n(2\alpha-s_n)\|Ax_n-Aq\|^2\Big)+(1-\beta_n)\|y_n-q\|^2 \\
\leq\ & \beta_n\Big(\|x_n-q\|^2-s_n(2\alpha-s_n)\|Ax_n-Aq\|^2\Big)+(1-\beta_n)\Big[\alpha_n\|f(x_n)-q\|^2+(1-\alpha_n)\|x_n-q\|^2\Big] \\
=\ & [\beta_n+(1-\beta_n)(1-\alpha_n)]\|x_n-q\|^2-\beta_n s_n(2\alpha-s_n)\|Ax_n-Aq\|^2+(1-\beta_n)\alpha_n\|f(x_n)-q\|^2 \\
=\ & (1-\alpha_n(1-\beta_n))\|x_n-q\|^2-\beta_n s_n(2\alpha-s_n)\|Ax_n-Aq\|^2+(1-\beta_n)\alpha_n\|f(x_n)-q\|^2 \\
\leq\ & \|x_n-q\|^2+\alpha_n\|f(x_n)-q\|^2-\beta_n c(2\alpha-d)\|Ax_n-Aq\|^2,
\end{aligned}$$

which immediately gives that

$$\begin{aligned}
\beta_n c(2\alpha-d)\|Ax_n-Aq\|^2 &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+\alpha_n\|f(x_n)-q\|^2 \\
&\leq (\|x_n-q\|+\|x_{n+1}-q\|)\|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-q\|^2.
\end{aligned}$$

Since $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$, we have $\|Ax_n-Aq\|\to 0$ as $n\to\infty$.
Step 4. $\|y_n-v_n\|\to 0$ *and* $\|x_n-z_n\|\to 0$ *as* $n\to\infty$.

Note that $v_n=P_C(I-t_nB)y_n$. Therefore, we have

$$\begin{aligned}
& \|v_n-q\|^2 \\
=\ & \|P_C(I-t_nB)y_n-P_C(I-t_nB)q\|^2 \\
\leq\ & \langle (I-t_nB)y_n-(I-t_nB)q, v_n-q\rangle \\
=\ & \frac{1}{2}\Big(\|(I-t_nB)y_n-(I-t_nB)q\|^2+\|v_n-q\|^2-\|(I-t_nB)y_n-(I-t_nB)q-(v_n-q)\|^2\Big) \\
\leq\ & \frac{1}{2}\Big(\|y_n-q\|^2+\|v_n-q\|^2-\|(y_n-v_n)-t_n(By_n-Bq)\|^2\Big) \\
=\ & \frac{1}{2}\Big(\|y_n-q\|^2+\|v_n-q\|^2-\|y_n-v_n\|^2+2t_n\langle y_n-v_n, By_n-Bq\rangle-t_n^2\|By_n-Bq\|^2\Big),
\end{aligned}$$

which implies that

$$\|v_n-q\|^2 \leq \|y_n-q\|^2-\|y_n-v_n\|^2+2t_n\langle y_n-v_n, By_n-Bq\rangle-t_n^2\|By_n-Bq\|^2.$$

Similarly, we have

$$\|z_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - z_n\|^2 + 2s_n \langle x_n - z_n, Ax_n - Aq\rangle - s_n^2\|Ax_n - Aq\|^2.$$

Therefore, from (10), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
\leq\ & \beta_n\|z_n - q\|^2 + (1-\beta_n)\|v_n - q\|^2 \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)\|v_n - q\|^2 \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)\Big(\|y_n - q\|^2 - \|y_n - v_n\|^2 + 2t_n \langle y_n - v_n, By_n - Bq\rangle - t_n^2\|By_n - Bq\|^2\Big) \\
\leq\ & \beta_n\|x_n - q\|^2 + (1-\beta_n)\Big(\alpha_n\|f(x_n) - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \|y_n - v_n\|^2 \\
& +2t_n \langle y_n - v_n, By_n - Bq\rangle - t_n^2\|By_n - Bq\|^2\Big) \\
\leq\ & \Big(\beta_n + (1-\beta_n)(1-\alpha_n)\Big)\|x_n - q\|^2 + \alpha_n\|f(x_n) - q\|^2 - (1-\beta_n)\|y_n - v_n\|^2 \\
& +2t_n \langle y_n - v_n, By_n - Bq\rangle - t_n^2\|By_n - Bq\|^2 \\
=\ & (1 - \alpha_n(1-\beta_n))\|x_n - q\|^2 + \alpha_n\|f(x_n) - q\|^2 - (1-\beta_n)\|y_n - v_n\|^2 + 2t_n \langle y_n - v_n, By_n - Bq\rangle \\
& -t_n^2\|By_n - Bq\|^2 \\
\leq\ & \|x_n - q\|^2 + \alpha_n\|f(x_n) - q\|^2 - (1-\beta_n)\|y_n - v_n\|^2 + 2t_n \langle y_n - v_n, By_n - Bq\rangle - t_n^2\|By_n - Bq\|^2,
\end{aligned}$$

which immediately gives that

$$\begin{aligned}
(1-\beta_n)\|y_n - v_n\|^2 \leq\ & \alpha_n\|f(x_n) - q\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2t_n\langle y_n - v_n, By_n - Bq\rangle \\
& -t_n^2\|By_n - Bq\|^2 \\
\leq\ & \alpha_n\|f(x_n) - q\|^2 + (\|x_n - q\| + \|x_{n+1} - q\|)\|x_{n+1} - x_n\| \\
& +2t_n\|y_n - v_n\|\|By_n - Bq\| - (1-\beta_n)t_n^2\|By_n - Bq\|^2.
\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|By_n - Bq\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{14}$$

Proceeding in the same manner as above and using (10), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
=\ & \|\beta_n Sz_n + (1-\beta_n)Sv_n - q\|^2 \\
\leq\ & \beta_n\|z_n - q\|^2 + (1-\beta_n)\|v_n - q\|^2 \\
\leq\ & \beta_n(\|x_n - q\|^2 - \|x_n - z_n\|^2 + 2s_n \langle x_n - z_n, Ax_n - Aq\rangle - s_n^2\|Ax_n - Aq\|^2) + (1-\beta_n)\|y_n - q\|^2 \\
\leq\ & \beta_n(\|x_n - q\|^2 - \|x_n - z_n\|^2 + 2s_n \langle x_n - z_n, Ax_n - Aq\rangle - s_n^2\|Ax_n - Aq\|^2) \\
& +(1-\beta_n)[\alpha_n\|f(x_n) - q\|^2 + (1-\alpha_n)\|x_n - q\|^2] \\
=\ & (\beta_n + (1-\beta_n)(1-\alpha_n))\|x_n - q\|^2 - \beta_n\|x_n - z_n\|^2 + 2\beta_n s_n \langle x_n - z_n, Ax_n - Aq\rangle \\
& -\beta_n s_n^2\|Ax_n - Aq\|^2 + (1-\beta_n)\alpha_n\|f(x_n) - q\|^2 \\
=\ & (1 - \alpha_n(1-\beta_n))\|x_n - q\|^2 - \beta_n\|x_n - z_n\|^2 + 2\beta_n s_n \langle x_n - z_n, Ax_n - Aq\rangle - \beta_n s_n^2\|Ax_n - Aq\|^2 \\
& +(1-\beta_n)\alpha_n\|f(x_n) - q\|^2 \\
\leq\ & \|x_n - q\|^2 - \beta_n\|x_n - z_n\|^2 + 2s_n\|x_n - z_n\|\|Ax_n - Aq\| - \beta_n s_n^2\|Ax_n - Aq\|^2 + \alpha_n\|f(x_n) - q\|^2,
\end{aligned}$$

which immediately gives that

$$\begin{aligned}
\beta_n\|x_n - z_n\|^2 \leq\ & (\|x_n - q\| + \|x_{n+1} - q\|)\|x_{n+1} - x_n\| + 2s_n\|x_n - z_n\|\|Ax_n - Aq\| \\
& -\beta_n s_n^2\|Ax_n - Aq\|^2 + \alpha_n\|f(x_n) - q\|^2.
\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Ax_n - Aq\| \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{15}$$

Step 5. *$\|v_n - Sv_n\| \to 0$ and $\|z_n - Sz_n\| \to 0$ as $n \to \infty$.*

Since $y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n$, we have

$$\|y_n - x_n\| = \alpha_n \|f(x_n) - x_n\| \to 0 \text{ as } n \to \infty.$$

From (14) and (15), we have

$$\|z_n - v_n\| \le \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - v_n\| \to 0 \text{ as } n \to \infty.$$

From (10), we have

$$\|x_{n+1} - Sv_n\| = \beta_n \|Sz_n - Sv_n\| \le b\|z_n - v_n\| \to 0 \text{ as } n \to \infty.$$

It follows from Step 2, $\|x_n - Sv_n\| \to 0$ as $n \to \infty$. Since $\|x_n - Sv_n\| \to 0$ and $\|y_n - x_n\| \to 0$ as $n \to \infty$, we have

$$\|y_n - Sv_n\| \le \|y_n - x_n\| + \|x_n - Sv_n\| \to 0 \text{ as } n \to \infty. \tag{16}$$

Note that

$$\|x_{n+1} - Sz_n\| = (1 - \beta_n)\|Sz_n - Sv_n\| \le \|z_n - v_n\| \to 0 \text{ as } n \to \infty.$$

Therefore by Step 2, we have

$$\|x_n - Sz_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Sz_n\| \to 0 \text{ as } n \to \infty. \tag{17}$$

From (15) and (17), we have

$$\|Sz_n - z_n\| \le \|Sz_n - x_n\| + \|x_n - z_n\| \to 0 \text{ as } n \to \infty.$$

Similarly, from (14) and (16), we have

$$\|Sv_n - v_n\| \le \|Sv_n - y_n\| + \|y_n - v_n\| \to 0 \text{ as } n \to \infty.$$

Step 6. *$\limsup_{n\to\infty} \langle f(x^*) - x^*, y_n - x^* \rangle \le 0$ for $x^* \in \mathcal{F}$.*

Let us take a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty} \langle f(q) - q, Sv_n - q \rangle = \lim_{k\to\infty} \langle f(q) - q, Sv_{n_k} - q \rangle .$$

Since $\{v_{n_k}\}$ is bounded, there exists a subsequence $\{v_{n_{k_l}}\}$ of $\{v_{n_k}\}$ such that $v_{n_{k_l}} \rightharpoonup z \in H$. Without loss of generality, we may assume that $v_{n_k} \rightharpoonup z$. Since

$$\|Sv_{n_k} - v_{n_k}\| \le \|Sv_{n_k} - y_{n_k}\| + \|y_{n_k} - v_{n_k}\| \to 0,$$

we have $Sv_{n_k} \rightharpoonup z$. Now, we show that $z \in \mathcal{F}$. First we show that $z \in VI(C, B)$. Let

$$Tv = \begin{cases} Bv + N_C(v), & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega[VI(C, B)]$. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Bv + N_C(v)$ and hence $w - Bv \in N_C(v)$. Thus, we have $\langle v - u, w - Bv \rangle \ge 0$ for all $u \in C$. Since $v_n \in C$, therefore $\langle v - v_n, w - Bv \rangle \ge 0$.

On the other hand, from $v_n = P_C(y_n - t_nBy_n)$, we have $\langle v - v_n, v_n - (y_n - t_nBy_n)\rangle \geq 0$, and hence $\left\langle v - v_n, \frac{v_n - y_n}{t_n} + By_n\right\rangle \geq 0$. Therefore, from $w - Bv \in N_C(v)$ and $v_{n_k} \in C$, we have

$$\begin{aligned}
\langle v - v_{n_k}, w\rangle &\geq \langle v - v_{n_k}, Bv\rangle \\
&\geq \langle v - v_{n_k}, Bv\rangle - \left\langle v - v_{n_k}, \frac{v_{n_k} - y_{n_k}}{t_{n_k}} + By_{n_k}\right\rangle \\
&= \left\langle v - v_{n_k}, Bv - By_{n_k} - \frac{v_{n_k} - y_{n_k}}{t_{n_k}}\right\rangle \\
&= \langle v - v_{n_k}, Bv - Bv_{n_k}\rangle + \langle v - v_{n_k}, Bv_{n_k} - By_{n_k}\rangle - \left\langle v - v_{n_k}, \frac{v_{n_k} - y_{n_k}}{t_{n_k}}\right\rangle \\
&\geq \langle v - v_{n_k}, Bv_{n_k} - By_{n_k}\rangle - \left\langle v - v_{n_k}, \frac{v_{n_k} - y_{n_k}}{t_{n_k}}\right\rangle.
\end{aligned}$$

Hence, letting $n_k \to \infty$ we obtain $\langle v - z, w\rangle \geq 0$. Since T is maximal monotone, we have, $z \in T^{-1}0$ and hence $z \in \Omega[VI(C, B)]$.
Similarly, we can show that $z \in \Omega[VI(C, A)]$. On the other hand, by (16) and (14), we have

$$\|v_n - Sv_n\| \leq \|v_n - y_n\| + \|y_n - Sv_n\| \to 0 \text{ as } n \to \infty.$$

So by Lemma 2.3, we obtain that $z \in Fix(S)$ and hence $z \in Fix(S) \cap \Omega[VI(C, A)] \cap \Omega[VI(C, B)]$. Note that

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(q) - q, Sv_n - q\rangle &= \lim_{k\to\infty} \langle f(q) - q, Sv_{n_k} - q\rangle = \langle f(q) - q, Sz - q\rangle \\
&= \langle f(q) - q, z - q\rangle = \langle (I - f)(q), q - z\rangle \leq 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(q) - q, y_n - q\rangle &\leq \limsup_{n\to\infty} \langle f(q) - q, y_n - Sv_n\rangle + \limsup_{n\to\infty} \langle f(q) - q, Sv_n - q\rangle \\
&\leq \limsup_{n\to\infty} \|f(q) - q\|\|y_n - Sv_n\| + \limsup_{n\to\infty} \langle f(q) - q, Sv_n - q\rangle \\
&\leq 0.
\end{aligned}$$

Step 7. $x_n \to q$ *as* $n \to \infty$.

Set $A_n = \|x_n - q\|, \gamma_n = 2\alpha_n(1 - \beta_n)(1 - k)$ and $\delta_n = \frac{kM}{1-k}\|y_n - x_n\| + \frac{1}{1-k}\langle f(q) - q, y_n - q\rangle + \frac{\alpha_n M^2}{2(1-k)}$ for all

$n \in \mathbb{N}$. Then from (10), we have

$$\begin{aligned}
A_{n+1}^2 &= \|x_{n+1} - q\|^2 \\
&\leq \beta_n\|z_n - q\|^2 + (1-\beta_n)\|v_n - q\|^2 \\
&\leq \beta_n\|x_n - q\|^2 + (1-\beta_n)\|y_n - q\|^2 \\
&= \beta_n\|x_n - q\|^2 + (1-\beta_n)\|\alpha_n(f(x_n) - q) + (1-\alpha_n)(x_n - q)\|^2 \\
&\leq \beta_n\|x_n - q\|^2 + (1-\beta_n)\Big((1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\left\langle f(x_n) - q, y_n - q\right\rangle\Big) \\
&= \beta_n\|x_n - q\|^2 + (1-\beta_n)\Big((1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\left\langle f(x_n) - f(q), y_n - q\right\rangle \\
&\quad +2\alpha_n\left\langle f(q) - q, y_n - q\right\rangle\Big) \\
&\leq \beta_n\|x_n - q\|^2 + (1-\beta_n)\Big((1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n k\|x_n - q\|\|y_n - q\| \\
&\quad +2\alpha_n\left\langle f(q) - q, y_n - q\right\rangle\Big) \\
&\leq \beta_n\|x_n - q\|^2 + (1-\beta_n)\Big((1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n k\|x_n - q\|(\|y_n - x_n\| + \|x_n - q\|) \\
&\quad +2\alpha_n\left\langle f(q) - q, y_n - q\right\rangle\Big) \\
&= \Big(\beta_n + (1-\beta_n)(1-\alpha_n)^2 + 2\alpha_n(1-\beta_n)k\Big)\|x_n - q\|^2 + 2\alpha_n k(1-\beta_n)\|x_n - q\|\|y_n - x_n\| \\
&\quad +2\alpha_n(1-\beta_n)\left\langle f(q) - q, y_n - q\right\rangle \\
&= (1 - 2\alpha_n(1-\beta_n)(1-k))\|x_n - q\|^2 + 2\alpha_n(1-\beta_n)k\|x_n - q\|\|y_n - x_n\| \\
&\quad +2\alpha_n(1-\beta_n)\left\langle f(q) - q, y_n - q\right\rangle + \alpha_n^2(1-\beta_n)\|x_n - q\|^2 \\
&= (1-\gamma_n)A_n^2 + \gamma_n\left\{\frac{k}{1-k}\|x_n - q\|\|y_n - x_n\| + \frac{1}{1-k}\left\langle f(q) - q, y_n - q\right\rangle + \frac{\alpha_n}{2(1-k)}\|x_n - q\|^2\right\} \\
&\leq (1-\gamma_n)A_n^2 + \gamma_n\left\{\frac{kM}{1-k}\|y_n - x_n\| + \frac{1}{1-k}\left\langle f(q) - q, y_n - q\right\rangle + \frac{\alpha_n}{2(1-k)}M^2\right\} \\
&= (1-\gamma_n)A_n^2 + \gamma_n\delta_n,
\end{aligned}$$

where $M = \sup_{n\in\mathbb{N}}\{\|x_n - q\|\}$. Note that $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. Thus by Lemma 2.4, we obtain that $x_n \to q$. □

Now we provide some numerical examples in support of our main result.

Example 3.4. *Let $H = \mathbb{R}$ with usual norm and $C = [0,1]$. Let $S : C \to C$ be a mapping defined by*

$$S(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}]; \\ x & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Define $A, B : C \to H$ by

$$A(x) = \begin{cases} x - \frac{1}{3} & \text{if } x \in [0, \frac{1}{3}]; \\ 0 & \text{if } x \in (\frac{1}{3}, 1] \end{cases} \quad \text{and } B(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{3}{4}]; \\ x - \frac{3}{4} & \text{if } x \in (\frac{3}{4}, 1]. \end{cases}$$

It is easy to see that S is a nonexpansive mapping with $Fix(S) = [\frac{1}{2}, 1]$. We first show that the mapping A is α-inverse strongly monotone mapping with $\alpha = \frac{1}{3}$.

If $x, y \in [0, \frac{1}{3}]$, then

$$\begin{aligned}
\langle Ax - Ay, x - y\rangle &= \langle (x - 1/3) - (y - 1/3), x - y\rangle = |x - y|^2 \\
&= |(x - 1/3) - (y - 1/3)|^2 \geq 1/3|(x - 1/3) - (y - 1/3)|^2 \\
&= 1/3|Ax - Ay|^2.
\end{aligned}$$

If $x \in [0, \frac{1}{3}]$ *and* $y \in (\frac{1}{3}, 1]$, *then*

$$\begin{aligned}\langle Ax - Ay, x - y\rangle &= (1/3 - x)(y - x) \geq (1/3 - x)(1/3 - x) \\ &= |(x - 1/3)|^2 \geq 1/3|(x - 1/3)|^2 = 1/3|Ax - Ay|^2.\end{aligned}$$

If $x, y \in (\frac{1}{3}, 1]$, *then*

$$\langle Ax - Ay, x - y\rangle \geq 1/3|Ax - Ay|^2.$$

Thus,

$$\langle Ax - Ay, x - y\rangle \geq 1/3|Ax - Ay|^2 \text{ for all } x, y \in C.$$

Hence, A is α*-inverse strongly monotone mapping with* $\alpha = \frac{1}{3}$ *and* $\Omega[VI(C, A)] = [\frac{1}{3}, 1]$.
Next, we show that the mapping B is β*-inverse strongly monotone mapping with* $\beta = \frac{1}{2}$.

If $x, y \in [0, \frac{3}{4}]$, *then*

$$\langle Bx - By, x - y\rangle \geq 1/2|Bx - By|^2.$$

If $x, y \in (\frac{3}{4}, 1]$, *then*

$$\begin{aligned}\langle Bx - By, x - y\rangle &= |(x - 3/4) - (y - 3/4)|^2 \geq 1/2|(x - 3/4) - (y - 3/4)|^2 \\ &= 1/2|Bx - By|^2.\end{aligned}$$

If $x \in [0, \frac{3}{4}]$ *and* $y \in (\frac{3}{4}, 1]$, *then*

$$\begin{aligned}\langle Bx - By, x - y\rangle &= (y - 3/4)(y - x) \geq (y - 3/4)(y - 3/4) \\ &= |(y - 3/4)|^2 \geq 1/2|(y - 3/4)|^2 = 1/2|Bx - By|^2.\end{aligned}$$

Thus

$$\langle Bx - By, x - y\rangle \geq 1/2|Bx - By|^2 \text{ for all } x, y \in C.$$

Hence, B is β*-inverse strongly monotone mapping with* $\beta = \frac{1}{2}$ *and* $\Omega[VI(C, B)] = [0, \frac{3}{4}]$.
Note that $Fix(S) \cap \Omega[VI(C, A)] \cap \Omega[VI(C, B)] = [\frac{1}{2}, \frac{3}{4}]$. *Let* $f : C \to C$ *be defined by* $f(x) = \frac{3x}{5}$ *for all* $x \in C$. *Then* f *is a contraction mapping with contraction coefficient* $k = \frac{3}{5}$. *Taking* $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\beta_n = \frac{1}{3}$, $s_n = \frac{1}{2}$ *and* $t_n = \frac{1}{3}$ *for each* $n \in \mathbb{N}$. *Observe that all the assumptions of Theorem* 3.3 *are satisfied. By Theorem* 3.3, *we conclude that the Algorithm* 3.1 *converges strongly to* $x^* = \frac{1}{2} \in Fix(S) \cap \Omega[VI(C, A)] \cap \Omega[VI(C, B)]$, *which is also the solution of the variational inequality problem* (9).

The numerical values of x_n *for initial points* $x_1 = 0.6$, $x_1 = 0.8$ *and* $x_1 = 1.0$ *are given in Table 1 and the convergence of* $\{x_n\}$ *generated by Algorithm* 3.1 *is shown in Figure 1.*

Table 1: Numerical values of x_n.

n	x_n	x_n	x_n
1	0.600000000000000	0.800000000000000	1.000000000000000
4	0.503703703703704	0.511111111111111	0.518518518518519
8	0.500045724737083	0.500137174211248	0.500228623685414
12	0.500000564502927	0.500001693508781	0.500002822514635
16	0.500000006969172	0.500000020907516	0.500000034845860
20	0.500000000086039	0.500000000258118	0.500000000430196
24	0.500000000001062	0.500000000003187	0.500000000005311
28	0.500000000000013	0.500000000000039	0.500000000000066
32	0.500000000000000	0.500000000000000	0.500000000000000
36	0.500000000000000	0.500000000000000	0.500000000000000

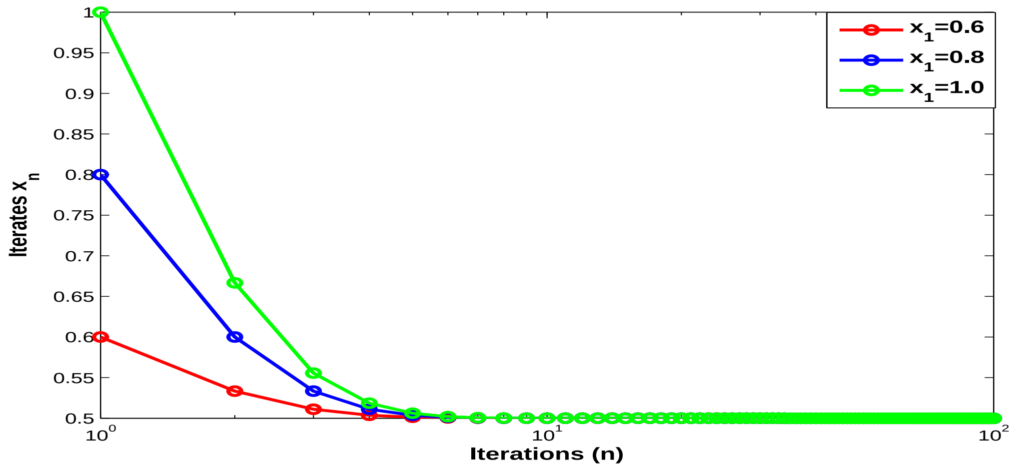

Figure 1: Convergence of Algorithm 3.1.

Example 3.5. *Let $H = \mathbb{R}^2$ with usual norm and $C = [0,1] \times [0,1]$. Let $S : C \to C$ be a mapping defined by*

$$S(x,y) = \left(\frac{x^2}{2}, \frac{y(y+2)}{4}\right) \text{ for all } (x,y) \in C.$$

Define $A, B : C \to H$ by

$$A(x,y) = \left(\frac{x}{3}, 3y\right) \text{ for all } (x,y) \in C \text{ and } B(x,y) = (3x, 2y^2) \text{ for all } (x,y) \in C.$$

It is easy to see that S is a nonexpansive mapping with $Fix(S) = \{(0,0)\}$. We first show that the mapping A is α-inverse strongly monotone with $\alpha = \frac{1}{3}$.

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in C$. Then

$$\begin{aligned}
\langle Ax - Ay, x - y\rangle &= \left\langle \left(\frac{x_1 - y_1}{3}, 3x_2 - 3y_2\right), (x_1 - y_1, x_2 - y_2)\right\rangle \\
&= \frac{(x_1 - y_1)^2}{3} + 3(x_2 - y_2)^2 \geq \frac{(x_1 - y_1)^2}{27} + 3(x_2 - y_2)^2 \\
&= \frac{1}{3}\left[\frac{(x_1 - y_1)^2}{9} + 9(x_2 - y_2)^2\right] = \frac{1}{3}\left\|\left(\frac{x_1 - y_1}{3}, 3x_2 - 3y_2\right)\right\|^2 = \frac{1}{3}\|Ax - Ay\|^2.
\end{aligned}$$

Thus, A is α-inverse strongly monotone mapping with $\alpha = \frac{1}{3}$ and $\Omega[VI(C,A)] = \{(0,0)\}$.

Next we show that the mapping B is β-inverse strongly monotone with $\beta = \frac{1}{4}$. Indeed,

$$\begin{aligned}
\langle Bx - By, x - y\rangle &= \langle (3x_1 - 3y_1, 2x_2^2 - 2y_2^2), (x_1 - y_1, x_2 - y_2)\rangle \\
&= 3(x_1 - y_1)^2 + 2(x_2^2 - y_2^2)(x_2 - y_2) \\
&\geq \frac{9}{4}(x_1 - y_1)^2 + (x_2 + y_2)(x_2^2 - y_2^2)(x_2 - y_2) = \frac{9}{4}(x_1 - y_1)^2 + (x_2^2 - y_2^2)^2 \\
&= \frac{1}{4}[9(x_1 - y_1)^2 + 4(x_2^2 - y_2^2)^2] = \frac{1}{4}\|Bx - By\|^2.
\end{aligned}$$

Thus, B is β-inverse strongly monotone mapping with $\beta = \frac{1}{4}$ and $\Omega[VI(C,B)] = \{(0,0)\}$.

Note that, $Fix(S) \cap \Omega[VI(C,A)] \cap \Omega[VI(C,B)] = \{(0,0)\}$. Let $f : C \to C$ be defined by $f(x,y) = (\frac{x}{2}, \frac{y}{3})$ for all $(x,y) \in C$. Then f is a contraction mapping with contraction coefficient $k = \frac{1}{2}$. Taking $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\beta_n = \frac{1}{3}$, $s_n = \frac{1}{4}$ and $t_n = \frac{1}{6}$ for each $n \in \mathbb{N}$. Observe that all the assumptions of Theorem 3.3 are satisfied. By Theorem 3.3, we conclude that the Algorithm 3.1 converges strongly to $x^* = (0,0) \in Fix(S) \cap \Omega[VI(C,A)] \cap \Omega[VI(C,B)]$, which is also the solution of the variational inequality problem (9).

The numerical values of x_n for initial guess $x_1 = (x_1^1, x_2^1) = (1,1)$ are given in Table 2 and the convergence of $\{x_n\}$ generated by Algorithm 3.1 is shown in Figure 2.

Table 2: Numerical values of $x_n = (x_1^n, x_2^n)$.

n	x_1^n	x_2^n	n	x_1^n	x_2^n
1	1.000000000000000	1.000000000000000	16	0.000000000000000	0.000000008905465
2	0.174870731197417	0.223631493120197	18	0.000000000000000	0.000000000922560
3	0.005571987819672	0.056198981906925	20	0.000000000000000	0.000000000097244
4	0.000005803354507	0.014918912893891	22	0.000000000000000	0.000000000010402
5	0.000000000006408	0.004119061911208	24	0.000000000000000	0.000000000001127
6	0.000000000000000	0.001171544039026	26	0.000000000000000	0.000000000000123
8	0.000000000000000	0.000101081454992	28	0.000000000000000	0.000000000000014
10	0.000000000000000	0.000009269357794	30	0.000000000000000	0.000000000000002
12	0.000000000000000	0.000000887459075	32	0.000000000000000	0.000000000000000
14	0.000000000000000	0.000000087770792	34	0.000000000000000	0.000000000000000

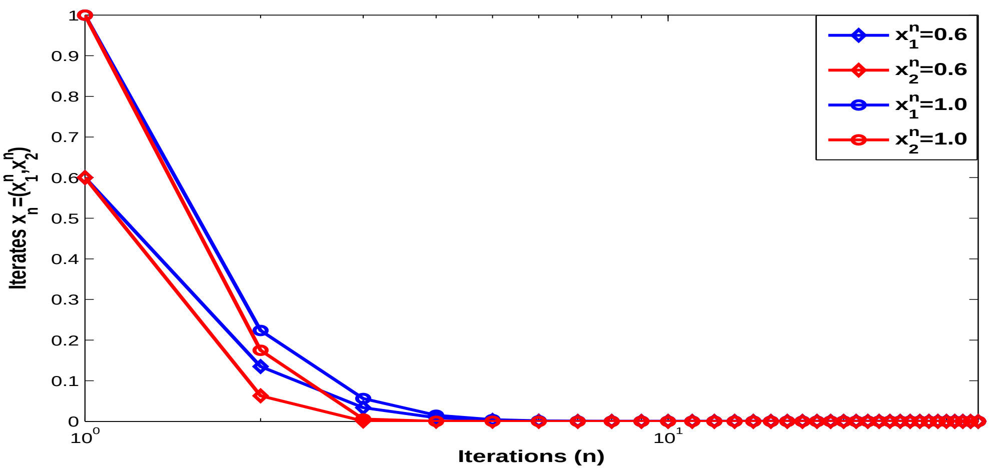

Figure 2: Convergence of Algorithm 3.1.

Note that algorithms (4), (6) and (7) are not applicable for solving variational inequality problem (9) whereas Algorithm 3.1 is applicable for solving variational inequality problem (9). In fact, Theorem 3.3 is more general in nature. So, we can derive some existing and new results in the framework of Hilbert space. In particular, we have the following:

Theorem 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $A : C \to H$ be α-inverse strongly monotone mapping. Assume that $f : C \to C$ is a k-contraction mapping and $S : C \to C$ is a nonexpansive mapping such that $Fix(S) \cap \Omega[VI(C, A)]$ is nonempty. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{s_n\}$ be sequences of real numbers satisfying the conditions* (SH1), (SH4) *and* (SH5), *where*

(SH4) $\{s_n\} \subset (0, 2\alpha)$ *with* $0 < c \leq s_n \leq d < 2\alpha$;

(SH5) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$; $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$.

Then, the sequence $\{x_n\}$ generated by

$$\begin{cases} x_{n+1} = \beta_n SP_C(x_n - s_n A x_n) + (1 - \beta_n) SP_C(y_n - s_n A y_n), \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) x_n \text{ for all } n \in \mathbb{N}, \end{cases} \tag{18}$$

converges strongly to $x^* \in Fix(S) \cap \Omega[VI(C, A)]$, which is the unique solution of variational inequality problem (5).

Proof. The conclusion follows immediately from Theorem 3.3 by putting $B = A$ and $t_n = s_n$ for all $n \in \mathbb{N}$. □

The variational inequality problem (5) is studied by Iiduka and Takahashi [24], Chen, Zhang and Fan [25] and Jung [26] by algorithms (3), (4) and (5), respectively. The algorithm (18) derived from Algorithm 3.1 also deals the problem of computation of solution of (5). The following examples show that the algorithm (18) has better performance than (3), (4) and (6).

Example 3.7. *Consider H, C, $\{\alpha_n\}$, $\{\beta_n\}$, $\{s_n\}$, S, A and f as in Example* 3.4. *Taking $x = 0.2 \in C$ for algorithm* (3). *The comparision of numerical values of x_n for initial point $x_1 = 1.0$ is given in Table* 3 *and the convergence of $\{x_n\}$ generated by algorithms* (3), (4), (6) *and* (18) *is shown in Figure* 3.

Table 3: Numerical values of x_n.

	Algorithm (3)	Algorithm (4)	Algorithm (6)	Algorithm (18)
n	x_n	x_n	x_n	x_n
1	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
5	0.365835921350013	0.394787507655445	0.428347268851679	0.518518518518519
9	0.400000000000000	0.414950518522260	0.448131365981801	0.500076207895138
13	0.416794970566216	0.432770271162393	0.458159632994313	0.500000940838212
17	0.427239312489100	0.442872391524905	0.464053935718083	0.500000011615287
21	0.434534632929202	0.449561509359451	0.468038124989833	0.500000000143399
25	0.440000000000000	0.454397849187426	0.470957435722964	0.500000000001770
29	0.444291398546884	0.458098446921454	0.473212622917328	0.500000000000022
33	0.447776703213291	0.461044710044950	0.475021214455710	0.500000000000000
37	0.450680303808393	0.463460437812950	0.476512721359322	0.500000000000000
.	.	.	.	.
.	.	.	.	.
500	0.486583592135001	0.490808849816997	0.493928429745264	0.500000000000000
1000	0.490513167019495	0.493553060204337	0.495729585464782	0.500000000000000
1500	0.492254033307585	0.494754735388103	0.496521409299662	0.500000000000000

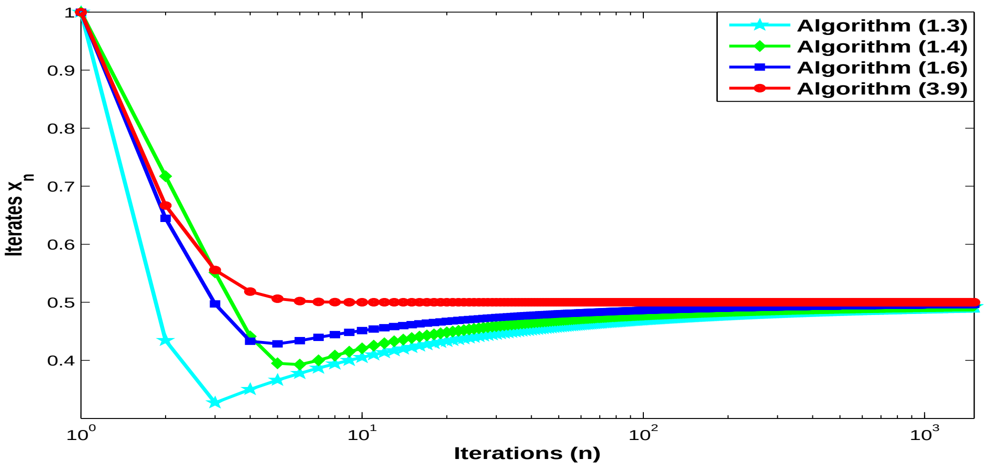

Figure 3: Comparision of algorithms (1.3), (1.4), (1.5) and (3.9).

Example 3.8. *Consider H, C,* $\{\alpha_n\}$, $\{\beta_n\}$, $\{s_n\}$, *S, A and f as in Example* 3.5. *Taking* $x = (0.2, 0.2) \in C$ *for algorithm* (3). *The comparision of numerical values of* x_n *for initial guess* $x_1 = (x_1^1, x_2^1) = (0.2, 1.0)$ *is given in Table* 4 *and the convergence of* $\{x_n\}$ *generated by algorithms* (3), (4), (6) *and* (18) *is shown in Figure* 4.

Table 4: Numerical values of $x_n = (x_1^n, x_2^n)$.

n	Algorithm (3)		Algorithm (4)	
	x_1^n	x_2^n	x_1^n	x_2^n
1	0.200000000000000	1.000000000000000	0.200000000000000	1.000000000000000
2	0.146343589497924	0.182609465132951	0.075632911379270	0.276890369291158
3	0.119273006202942	0.125337749147151	0.022849110249241	0.068422341209616
4	0.102988458421964	0.107956340191707	0.005821951004222	0.015716694991604
5	0.091906076069271	0.097002982626770	0.001309699861365	0.003431037008519
6	0.083749665624756	0.088911870290293	0.000267767822418	0.000720804095126
7	0.077425944283068	0.082583021586976	0.000050622100014	0.000146863549130
.	.	.	.	.
.	.	.	.	.
16	0.050870601578035	0.055397131141985	0.000000000001535	0.000000000039371
17	0.049330672864302	0.053788615087180	0.000000000000186	0.000000000006911
18	0.047921881100105	0.052313817910224	0.000000000000022	0.000000000001203
19	0.046626646365907	0.050955123730570	0.000000000000003	0.000000000000208
20	0.045430517484575	0.049698008611091	0.000000000000000	0.000000000000036
21	0.044321491071669	0.048530375384853	0.000000000000000	0.000000000000006
22	0.043289503316867	0.047442055525472	0.000000000000000	0.000000000000001
23	0.042326045006761	0.046424430624345	0.000000000000000	0.000000000000000
24	0.041423865967635	0.045470140936651	0.000000000000000	0.000000000000000

n	Algorithm (6)		Algorithm (18)	
	x_1^n	x_2^n	x_1^n	x_2^n
1	0.200000000000000	1.000000000000000	0.200000000000000	1.000000000000000
2	0.051223052969662	0.196529992188561	0.010283803655225	0.093835177232416
3	0.010201070696216	0.034338207996757	0.000029798873416	0.008800193772739
4	0.001715678449267	0.005580845617547	0.000000000264259	0.000856336279976
5	0.000256234682173	0.000862643901349	0.000000000000000	0.000085773771238
6	0.000034880389219	0.000128354932410	0.000000000000000	0.000008776411684
7	0.000004394734035	0.000018524002272	0.000000000000000	0.000000912764987
.	.	.	.	.
.	.	.	.	.
16	0.000000000000003	0.000000000000223	0.000000000000000	0.000000000000002
17	0.000000000000000	0.000000000000028	0.000000000000000	0.000000000000000
18	0.000000000000000	0.000000000000003	0.000000000000000	0.000000000000000

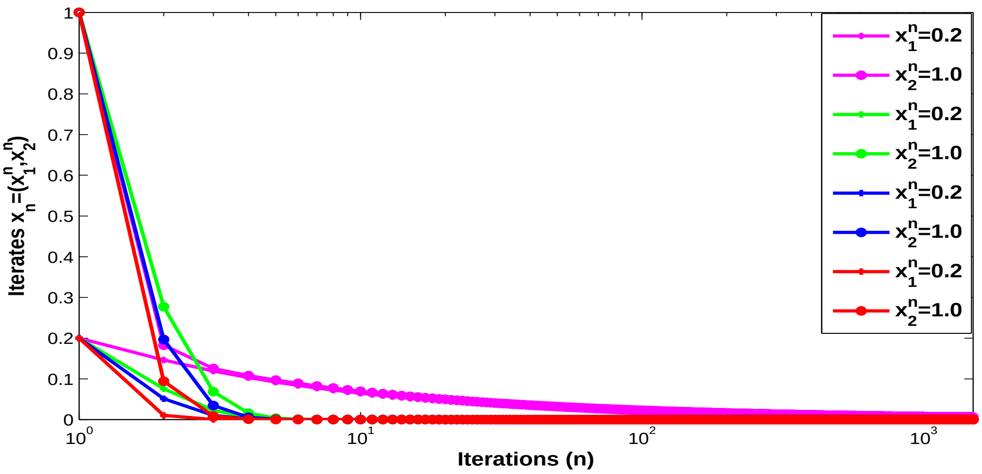

Figure 4: Comparision of algorithms (1.3), (1.4), (1.5) and (3.9).

4. Consequences

The following theorem can be derived from Theorem 5.4 of [38] in real Hilbert space setting.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping such that $Fix(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in $(0, 1)$ satisfying the following condition:*

(SH6) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n(1 - \beta_n) = \infty,\ \lim_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n+1}} = 1,\ \lim_{n\to\infty} \dfrac{\beta_n}{\beta_{n+1}} = 1.$

For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by (8). Then $\{x_n\}$ converges strongly to $P_{Fix(S)}(u)$.

As the consequence of Theorem 3.3, we have the following strong convergence results for solving different types of variational inequality problems.

Corollary 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping such that $Fix(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in $(0,1)$ satisfying the following condition*

(SH7) $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by (8). *Then $\{x_n\}$ converges strongly to $P_{Fix(S)}(u)$.*

Proof. The conclusion follows immediately from Theorem 3.3 by putting $A = 0$, $B = 0$ and $f(x) = u$, a constant mapping. □

Remark 4.3. *Theorem* 4.1 *is more general than Corollary* 4.2 *due to condition* (SH6).

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $f : C \to C$ is a k-contraction mapping and $S : C \to C$ is a nonexpansive mapping. Let $T_1, T_2 : C \to C$ be λ_1-strict pseudocontractive and λ_2-strict pseudocontractive mappings, respectively such that $Fix(S) \cap Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{s_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying* (SH1), (SH3) *and* (SH8), *where*

(SH8) $\{s_n\} \subset (0, 1-\lambda_1)$ *and* $\{t_n\} \subset (0, 1-\lambda_2)$ *with* $0 < c \leq s_n \leq d < 1-\lambda_1$, $0 < l \leq t_n \leq m < 1-\lambda_2$.

Then, the sequence $\{x_n\}$ generated by

$$\begin{cases} x_{n+1} = \beta_n S((1-s_n)x_n + s_n T_1 x_n) + (1-\beta_n) S((1-t_n)y_n + t_n T_2 y_n), \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n \text{ for all } n \in \mathbb{N}, \end{cases}$$

converges strongly to $x^* \in Fix(S) \cap Fix(T_1) \cap Fix(T_2)$, which is the unique solution of variational inequality problem:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0 \text{ for all } x \in Fix(S) \cap Fix(T_1) \cap Fix(T_2).$$

Proof. Put $A = I - T_1$ and $B = I - T_2$. Then A is $\frac{1-\lambda_1}{2}$-inverse strongly monotone and B is $\frac{1-\lambda_2}{2}$-inverse strongly monotone mapping. We also have $P_C(x_n - s_n A x_n) = (1-s_n)x_n + s_n T_1 x_n$ and $P_C(y_n - t_n B y_n) = (1-t_n)y_n + t_n T_2 y_n$ with $\Omega[VI(C,A)] = Fix(T_1)$ and $\Omega[VI(C,B)] = Fix(T_2)$. Therefore, the conclusion follows immediately from Theorem 3.3. □

Theorem 4.5. *Let H be a real Hilbert space. Assume that f is a k-contraction mapping, S is a nonexpansive mapping, A is an α-inverse strongly monotone mapping and B is a β-inverse strongly monotone mapping on H into itself such that $Fix(S) \cap A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{s_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying* (SH1), (SH2) *and* (SH3). *Then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_{n+1} = \beta_n S(x_n - s_n A x_n) + (1-\beta_n) S(y_n - t_n B y_n), \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n \text{ for all } n \in \mathbb{N}, \end{cases}$$

converges strongly to $x^* \in Fix(S) \cap A^{-1}(0) \cap B^{-1}(0)$, which is the unique solution of variational inequality problem:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0 \text{ for all } x \in Fix(S) \cap A^{-1}(0) \cap B^{-1}(0).$$

Proof. Put $C = H$. Then $P_C = I$, $\Omega[VI(C,A)] = A^{-1}(0)$ and $\Omega[VI(C,B)] = B^{-1}(0)$. Therefore, by Theorem 3.3, we get the desired result. □

5. Conclusion

In this paper, we have coupled S-iteration process with the Halpern iteration process and proposed an algorithm to approximate a common element of the set of fixed points of a nonexpansive mapping and the set of common solutions of variational inequality problems formed by two inverse strongly monotone mappings in the framework of Hilbert spaces. We observed that the sequence generated by our proposed Algorithm 3.1 converges strongly to $x^* \in Fix(S) \cap \Omega[VI(C,A)] \cap \Omega[VI(C,B)]$, which is the unique solution of variational inequality problem (9). It has been shown that a particular case of our proposed algorithm, that is, the algorithm (18) has better performance than (3), (4) and (6); see Example 3.7. Some numerical examples are given in support of our main results.

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