Filomat 33:6 (2019), 1619–1639
https://doi.org/10.2298/FIL1906619A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Higher-Order Duality Results for a New Class of Nonconvex Nonsmooth Multiobjective Programming Problems

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Abstract. In this paper, a nonconvex nonsmooth multiobjective programming problem is considered and two its higher-order duals are defined. Further, several duality results are established between the considered nonsmooth vector optimization problem and its dual models under assumptions that the involved functions are higher-order (Φ, ρ)-type I functions.

1. Introduction

In the paper, we consider the following nondifferentiable multiobjective programming problem:

$$\begin{array}{c} \varphi(x) := \left(f_1(x) + \left(x^T B_1 x\right)^{1/2}, \ldots, f_q(x) + \left(x^T B_q x\right)^{1/2}\right) \to \min \\ g_j(x) \leqq 0, \quad j = 1, \ldots, m, \\ x \in X, \end{array} \tag{VP}$$

where $f_i : X \to R$, $i \in I = \{1, \ldots, q\}$, $g_j : X \to R$, $j \in J = \{1, \ldots, m\}$, are differentiable functions on a nonempty open convex set $X \subset R^n$ and, moreover, each B_i, $i \in I$, is an $n \times n$ positive semidefinite symmetric matrix.

Let D be the set of all feasible solutions in the considered vector optimization problem (VP), that is,

$$D = \left\{x \in X : g_j(x) \leqq 0, \quad j = 1, \ldots, m\right\},$$

and, moreover, we define by $J(\overline{x})$ the set of all active inequality constraints at point $\overline{x} \in D$, that is,

$$J(\overline{x}) = \left\{j \in J : g_j(\overline{x}) = 0\right\}.$$

Remark 1.1. *Note that the considered multiobjective programming problem reduces to a usual vector optimization problem if $B_i \equiv 0$ for all $i \in I$.*

2010 *Mathematics Subject Classification*. Primary 90C29; Secondary 90C46, 90C30, 90C26, 26B25
Keywords. Nonsmooth multiobjective programming, Higher-order (Φ, ρ)-type I functions, Higher order Mangasarian duality, Higher order mixed duality.
Received: 30 November 2017; Revised: 30 August 2018; Accepted: 06 February 2019
Communicated by Predrag Stanimirović
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In most real-life problems, decisions are made taking into account several conflicting criteria, rather than by optimizing a single objective. Such an optimization problem is called multiobjective programming problem or vector optimization problem. Multiobjective programming is, therefore, the search for a solution that best manages trade-offs criteria that conflict and that cannot be converted to a common measure. In recent years, multiobjective programming has grown remarkably in different directions in the settings of optimality conditions and duality theory. As a special case of a vector optimization problem, which appears repeatedly in the literature, is a nondifferentiable multiobjective programming problem containing a certain square root of a quadratic form in each component of the objective function. It has been enriched by the applications of various types of generalizations of convexity theory (see, for example, [4], [9], [18], [23], [25], [26], [28], [33], [34], [37], [38], [39], and others). In [16], Hanson and Mond introduced the so-called classes of (generalized) type I functions in nonlinear scalar optimization problems as a generalization of invexity introduced by Hanson [14]. The notion of type I functions has been generalized in several directions. Later, Kaul et al. [22] investigated Karush-Kuhn-Tucker type necessary and sufficient conditions and obtained duality results for differentiable multiobjective programming problems involving generalized type I functions. The class of higher-order type I functions for scalar optimization problems was introduced by Mishra and Rueda [26].

Mangasarian [24] introduced the concepts of second- and higher-order duality for nonlinear optimization problems. He has also indicated that the study of such dual problems is significant due to the computational advantage over the first-order duality as it provides tighter bounds for the value of the objective function when approximations are used. Motivated by the foregoing concepts introduced in [24], several researchers have worked in this field (see, for instance, [3], [4], [5], [6], [15], [21], [26], [31], [32], [35], [36], [41], [42], [43], and others). Mond [33] was the first who defined second order convexity and he used it to prove second-order duality results. Jeyakumar [21] discussed second-order Mangasarian type duality under ρ-convexity. In [42], Zhang and Mond proved some duality theorems for second-order duality in nonlinear programming under generalized second-order B-invexity. Various duality results for a mathematical programming problem has been established under higher-order invexity by Mond and Zang [35]. In [44], Zhang introduced higher-order (F, ρ)-convexity and he established higher-order duality results for multiobjective programming problems under introduced concept of generalized convexity. Mishra and Rueda [26] generalized the results of Zhang and Mond [42] and they proved various duality results between the considered nondifferentiable mathematical programming problem and its higher-order duals under the concept of higher-order type I functions. Later on, Yang et al. [41] discussed higher-order duality results under generalized convexity assumptions for multiobjective programming problems involving support functions. Mishra et al. [32] extended the class of generalized type I functions introduced by Aghezzaf and Hachimi [1] to the context of a higher-order case. Further, they formulated a number of higher-order duals to a nondifferentiable multiobjective programming problem and established higher-order duality results under the introduced higher-order generalized type I functions. In [40], Yang et al. established a converse duality theorem for higher-order Mond-Weir type multiobjective programming problems involving cones. Ahmad et al. [4] derived optimality conditions and Mond-Weir duality results for a nondifferentiable multiobjective programming problem containing a certain square root of a quadratic form in each component of the objective function in the presence of equality and inequality constraints. Recently, Jayswal et al. [20] have established weak, strong and strict converse duality theorems for higher-order Wolfe and Mond-Weir type multiobjective dual programs in order to relate efficient solutions of primal and dual problems under assumption that the involved functions are (generalized) higher-order (F, α, ρ, d)-V-type I functions.

In this paper, we consider a nonsmooth multiobjective programming problem and, following Mishra and Rueda [26] and Caristi et al. [8], we introduce the concept of higher-order (Φ, ρ)-type I objective and constraint functions for such a vector optimization problem. Further, we formulate two higher-order dual-order case problems for the considered nondifferentiable multiobjective programming problem, that are, higher-order dual problem in the sense of Mangasarian and higher-order mixed dual problem. Using the concept of higher-order (Φ, ρ)-type I objective and constraint functions, we prove weak, strong and strict converse duality theorems between the considered nonsmooth multiobjective programming problem and its higher-order dual problems formulated in the paper. Since the concept of (Φ, ρ)-type I objective and constraint functions generalizes a lot of other generalized type I notions previously defined in the literature, therefore, the results established in the paper are more general than those existing in the literature.

2. Nondifferentiable multiobjective programming and higher-order (Φ,ρ)-type I objective and constraint functions

The following convention for equalities and inequalities will be used in the paper.
For any $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, we define:
(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

We begin our considerations by introducing the definition of higher-order (Φ, ρ)-type I objective and constraint functions for a usual vector optimization problem, which we obtain if we set $B_i \equiv 0$ for all $i \in I$ in the considered nondifferentiable multiobjective programming problem (VP) (see Remark 1.1). Hence, we now consider a usual nonlinear vector optimization problem as follows

$$\begin{aligned} f(x) = \left(f_1(x), ..., f_q(x)\right) \to \min \\ \text{subject to} \quad g_j(x) \leqq 0, \ j = 1, ..., m, \qquad (\mathrm{VP}_0) \\ x \in X, \end{aligned}$$

where the functions f_i, $i \in I$, g_j, $j \in J$, and the set X are defined in the similar way as for the problem (VP). Throughout the paper, we shall write $g = (g_1, ..., g_m) : X \to R^m$.

Let $k = \left(k_1, ..., k_q\right) : X \times R^n \to R^q$ and $h = (h_1, ..., h_m) : X \times R^n \to R^m$ be differentiable functions, p any vector in R^n.

Definition 2.1. *If there exist $\rho = \left(\rho_{f_1}, ..., \rho_{f_q}, \rho_{g_1}, ..., \rho_{g_m}\right) \in R^{q+m}$ and a function $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in X$ and any $a \in R_+$, such that, the following inequalities*

$$f_i(x) - f_i(u) - k_i(u, p) + p^T \nabla_p k_i(u, p) \geqq \Phi\left(x, u, \left(\nabla_p k_i(u, p), \rho_{f_i}\right)\right), \ i \in I, \tag{1}$$

$$-g_j(u) - h_j(u, p) + p^T \nabla_p h_j(u, p) \geqq \Phi\left(x, u, \left(\nabla_p h_j(u, p), \rho_{g_j}\right)\right), \ j \in J \tag{2}$$

hold for all $x \in X$, then (f, g),is said to be higher-order (Φ, ρ)-type I objective and constraint functions at $u \in X$ on X.
If inequalities (1) and (2) are satisfied at each $u \in X$, then (f, g) is said to be higher-order (Φ, ρ)-type I objective and constraint functions on X.
If inequalities (1) are strict for all $x \in X$, $(x \neq u)$ and $i \in I$, then (f, g), is said to be higher-order strictly (Φ, ρ)-type I objective and constraint functions at $u \in X$ on X.

Now, we give an example of such a nondifferentiable vector optimization problem in which the involved functions are higher-order (Φ, ρ)-type I objective and constraint functions on the set of all feasible solutions.

Example 2.2. *Consider the following nonconvex vector optimization problem:*

$$\begin{aligned} f(x) = \left(x^3, x^2\right) \to \min \\ \text{subject to} \quad x^2 - 1 \leqq 0, \qquad (VP1_0) \\ -x - 1 \leqq 0, \\ x \in R. \end{aligned}$$

Note that the set of all feasible solutions in the considered nonconvex vector optimization problem (VP1$_0$) is $D = \left\{x \in R : x^2 - 1 \leqq 0 \wedge -x - 1 \leqq 0\right\} = [-1, 1]$ *and* $u = 0$ *is a feasible solution. It can be shown, by Definition 2.1, that* (f, g) *are higher-order* (Φ, ρ)*-type I objective and constraint functions at* $u = 0$ *on* D*. Indeed, let* $\Phi : D \times D \times R^2 \to R$ *be defined by* $\Phi(x, u, (\vartheta, \rho)) = \vartheta\left(u - x^2\right) + \rho$*,* $k_i : D \times R \to R$*,* $i = 1, 2$*, be defined by* $k_i(u, p) = \frac{p}{u^2+1} - 1$*,* $h_j : D \times R \to R$*,* $j = 1, 2$*, be defined by* $h_j(x, u) = \frac{p}{u^2+1} - 1$*,* $\rho_{f_1} = 1$*,* $\rho_{f_2} = 1$*,* $\rho_{g_1} = 2$*,* $\rho_{g_2} = 2$*. Then, by Definition 2.1, we have*

$$
\begin{aligned}
& f_1(x) - f_1(u) - k_1(u, p) + p^T \nabla_p k_1(u, p) - \Phi\left(x, u, \left(\nabla_p k_1(u, p), \rho_{f_1}\right)\right) \\
& = x^3 - u^3 - \left(\frac{p}{u^2 + 1} - 1\right) + p\left(\frac{1}{u^2 + 1}\right) - \Phi\left(x, u, \left(\frac{1}{u^2 + 1}, \rho_{f_1}\right)\right) \\
& = x^3 + x^2 \geqq 0 \ \ \forall x \in D
\end{aligned}
$$

$$
\begin{aligned}
& f_2(x) - f_2(u) - k_2(u, p) + p^T \nabla_p k_2(u, p) - \Phi\left(x, u, \left(\nabla_p k_2(u, p), \rho_{f_2}\right)\right) \\
& = x^2 - u^2 - \left(\frac{p}{u^2 + 1} - 1\right) + p\left(\frac{1}{u^2 + 1}\right) - \Phi\left(x, u, \left(\frac{1}{u^2 + 1}, \rho_{f_2}\right)\right) \\
& = 2x^2 \geqq 0 \ \ \forall x \in D g_1(x) - g_1(u) - h_1(u, p) + p^T \nabla_p h_1(u, p) - \Phi\left(x, u, \left(\nabla_p h_1(u, p), \rho_{g_1}\right)\right) \\
& = -\left(u^2 - 1\right) - \left(\frac{p}{u^2 + 1} - 1\right) + p\left(\frac{1}{u^2 + 1}\right) - \Phi\left(x, u, \left(\frac{1}{u^2 + 1}, \rho_{g_1}\right)\right) \\
& = x^2 \geqq 0 \ \ \forall x \in D
\end{aligned}
$$

$$
\begin{aligned}
& g_2(x) - g_2(u) - h_2(u, p) + p^T \nabla_p h_2(u, p) - \Phi\left(x, u, \left(\nabla_p h_2(u, p), \rho_{g_2}\right)\right) \\
& = -(-u - 1) - \left(\frac{p}{u^2 + 1} - 1\right) + p\left(\frac{1}{u^2 + 1}\right) - \Phi\left(x, u, \left(\frac{1}{u^2 + 1}, \rho_{g_2}\right)\right) \\
& = x^2 \geqq 0 \ \ \forall x \in D
\end{aligned}
$$

Hence, by Definition 2.1, it follows that (f, g) *are higher-order* (Φ, ρ)*-type I objective and constraint functions at* $u = 0$ *on* D*.*

Remark 2.3. *Note that the functions constituting the considered nonconvex vector optimization problem (VP1$_0$) are not higher-order type I at* $u = 0$ *on* D *with respect to any function* $\eta : D \times D \to R$ *in the sense of the definition given by Mishra and Rueda [26]. Further, it can be shown that all functions involved in the nonconvex vector optimization problem (VP1$_0$) considered in Example 2.2 are not* (Φ, ρ)*-invex at* $u = 0$ *on* D *(see [8]) with respect to* Φ *and* ρ *defined above. Also note that not all functions constituting the nonconvex vector optimization problem (VP1$_0$) considered in Example 2.2 are higher order* (Φ, ρ)*-invex at* $u = 0$ *on* D *with respect to the functions* k_i*,* $i = 1, 2$*, and* h_j*,* $j = 1, 2$ *(see [19]). Indeed, none of the constraint functions is higher order* (Φ, ρ)*-invex at* $u = 0$ *on* D *with respect to the functions* Φ*,* h_j *and scalars* ρ_{g_j}*,* $j = 1, 2$ *given above. As it follows even from this example, the class of higher-order* (Φ, ρ)*-type I objective and constraint functions is a larger class of functions than many classes of higher order generalized convex functions, previously defined in the literature. Hence, various higher order duality results established in the paper are applicable to a large class of nonconvex vector optimization problems in comparison to the similar results earlier established in the literature under other concepts of higher order generalized convexity.*

Lemma 2.4. *(Generalized Schwartz inequality): Let* B *be a positive semidefinite symmetric matrix of order* n*. Then, for all* $x, z \in R^n$*,*

$$x^T B z \leqq \left(x^T B x\right)^{1/2} \left(z^T B z\right)^{1/2}. \tag{3}$$

Note that the equality holds, if $Bx = \alpha B w$ *for some* $\alpha \in R^n$ *with* $\alpha \geqq 0$*. Moreover, if* $\left(z^T B z\right)^{1/2} \leqq 1$*, then*

$$x^T B z \leqq \left(x^T B x\right)^{1/2}. \tag{4}$$

3. Optimality

In this section, we give necessary optimality conditions for properly efficiency in the considered nondifferentiable multiobjective programming problem (VP) which are useful in proving higher-order dual results for (VP) in next sections.

Definition 3.1. *A feasible point $\overline{x}$ is said to be a Pareto solution (an efficient solution) for (VP) if and only if there is no another $x \in D$ such that $f(x) \leq f(\overline{x})$.*

Afterwards, Geoffrion [11] modified the efficiency concept and defined the proper efficient solution in a multiobjective programming problem as follows:

Definition 3.2. *An efficient solution $\overline{x}$ for (VP) is said to be properly efficient if there exists a scalar $M > 0$, such that for each $i \in I$ and $x \in D$ satisfying $f_i(x) < f_i(\overline{x})$, we have $\frac{f_i(\overline{x})-f_i(x)}{f_k(x)-f_k(\overline{x})} \leqq M$ for at least one k satisfying $f_k(\overline{x}) < f_k(x)$.*

Remark 3.3. *Note that if $\overline{x}^T B_i \overline{x}$, $i \in I$, is not zero, then the corresponding function involved in the objective function of the considered nondifferentiable multiobjective programming problem (VP) is not differentiable. In order to derive necessary optimality conditions in such a case, for a feasible solution $\overline{x}$, we define the set $\Omega(\overline{x}) = \bigcup_{i=1}^{q} \Omega_i(\overline{x})$, where*

$$\Omega_i(\overline{x}) = \Big\{\omega \in R^n : \omega^T \nabla g_j(\overline{x}) \leqq 0,\ j \in J(\overline{x}) \text{ and } \omega^T \nabla f_i(\overline{x}) + \frac{\omega^T B_i \overline{x}}{\left(\overline{x}^T B_i \overline{x}\right)^{1/2}} < 0, \text{ if } \overline{x}^T B_i \overline{x} > 0,$$
$$\omega^T \nabla f_i(\overline{x}) + \left(\omega^T B_i \omega\right)^{1/2} < 0, \text{ if } \overline{x}^T B_i \overline{x} = 0\Big\},\ i \in I.$$

In [7], Bhatia and Jain established necessary optimality conditions for a feasible solution to be a properly efficient solution for the considered nondifferentiable multiobjective fractional programming problem in which the numerator of each component of the objective function contains a term involving square root of a certain positive semi-definite quadratic form. Necessary optimality conditions for the considered nondifferentiable multiobjective programming problem (VP) can be also found, for example, in [2], [4], [34].

Theorem 3.4. *(Necessary optimality conditions): Let $\overline{x}$ be a properly efficient solution in the considered nondifferentiable multiobjective programming problem (VP), the set $\Omega(\overline{x})$ be empty and a suitable constraint qualification be satisfied at $\overline{x}$. Then, there exist $\overline{\lambda} \in R^q$, $\overline{\xi} \in R^m$ and $\overline{w}_i \in R^n$, $i \in I$, such that*

$$\sum_{i=1}^{q} \overline{\lambda}_i \{\nabla f_i(\overline{x}) + B_i \overline{w}_i\} + \sum_{j=1}^{m} \overline{\xi}_j \nabla g_j(\overline{x}) = 0, \tag{5}$$

$$\overline{\xi}_j g_j(\overline{x}) = 0,\ j \in J, \tag{6}$$

$$\overline{\lambda}_i > 0,\ i = 1, ..., q,\ \sum_{i=1}^{q} \overline{\lambda}_i = 1,\ \overline{\xi}_j \geqq 0,\ j \in J, \tag{7}$$

$$\left(\overline{x}^T B_i \overline{x}\right)^{1/2} = \overline{x}^T B_i \overline{w}_i,\ i \in I, \tag{8}$$

$$\overline{w}_i^T B_i \overline{w}_i \leqq 1,\ i \in I. \tag{9}$$

4. Mangasarian duality

In this section, a higher-order dual problem in the sense of Mangasarian is formulated for the considered nondifferentiable multiobjective programming problem (VP) and several duality theorems are established under assumption that the functions constituting the problem (VP) are (generalized) higher-order (Φ, ρ)-type I objective and constraint functions.

Consider the following dual problem (MVD) in the sense of Mangasarian related to problem (VP):

$$\text{Maximize } \Big(f_1(u) + [u+p]^T B_1 w_1 + k_1(u,p) + \xi^T g(u) + \xi^T h(u,p), ..., f_q(u) + [u+p]^T B_q w_q + k_q(u,p) + \xi^T g(u) + \xi^T h(u,p)\Big) \quad (MVD)$$

$$\text{s.t. } \sum_{i=1}^{q} \lambda_i \left(\nabla_p k_i(u,p) + B_i w_i\right) + \sum_{j=1}^{m} \xi_j \nabla_p h_j(u,p) = 0, \tag{10}$$

$$w_i^T B w_i \leqq 1, \, i \in I, \tag{11}$$

$$u \in X, p \in R^n, w_i \in R^n, \lambda_i > 0, i = 1, ..., q, \sum_{i=1}^{q} \lambda_i = 1, \xi_j \geqq 0, j \in J. \tag{12}$$

Let

$$\Omega_{MVD} = \left\{\left(u, \lambda, \xi, \left(w_1, ..., w_q\right), p\right) \in X \times R_+^q \times R_+^m \times (\underbrace{R^n \times ... \times R^n}_{q \text{ times}}) \times R^n : \text{verifying the constraints of (MVD)}\right\}$$

be the set of all feasible solutions in problem (MVD). Further, $U = \left\{u \in X : \left(u, \lambda, \xi, \left(w_1, ..., w_q\right), p\right) \in \Omega_{MVD}\right\}$.

Theorem 4.1. *(Weak duality). Let x and (u, λ, ξ, w, p) be any feasible solutions for the vector optimization problems (VP) and (MVD), respectively. Further, assume that $\left(f_i(\cdot) + (\cdot)^T B_i w_i, g(\cdot)\right)$, $i \in I$, is higher-order (Φ, ρ)-type I objective and constraint functions at u on $D \cup U$. If $\sum_{i=1}^{q} \lambda_i \rho_{f_i} + \sum_{j}^{m} \xi_j \rho_{g_j} \geqq 0$, then the following cannot hold:*

$$f_i(x) + \left(x^T B_i x\right)^{1/2} \leqq f_i(u) + [u+p]^T B_i w_i + k_i(u,p) + \xi^T g(u) + \xi^T h(u,p), \; i \in I, \tag{13}$$

and

$$f_i(x) + \left(x^T B_i x\right)^{1/2} < f_i(u) + [u+p]^T B_i w_i + k_i(u,p) + \xi^T g(u) + \xi^T h(u,p) \;\; \textit{for at least one } i \in I. \tag{14}$$

Proof. Let x and (u, λ, ξ, w, p) be feasible solutions for the vector optimization problems (VP) and (MVD), respectively. We proceed by contradiction. Suppose, contrary to the result, that the inequalities (13) and (14) hold. Then, by the generalized Schwartz inequality (see Lemma 2.4), (13) and (14) give, respectively,

$$f_i(x) + x^T B_i w_i \leqq f_i(u) + [u+p]^T B_i w_i + k_i(u,p) + \xi^T g(u) + \xi^T h(u,p), i \in I \tag{15}$$

and

$$f_i(x) + x^T B_i w_i < f_i(u) + [u+p]^T B_i w_i + k_i(u,p) + \xi^T g(u) + \xi^T h(u,p) \;\; \text{for at least one } i \in I. \tag{16}$$

Since $(u, \lambda, \xi, w, p) \in \Omega_{MVD}$, we have that $\lambda_i > 0$, $i = 1, ..., q$, $\sum_{i=1}^{q} \lambda_i = 1$. Thus, (15) and (16) yield

$$\lambda_i \left[f_i(x) + x^T B_i w_i\right] \leqq \lambda_i \left(f_i(u) + [u+p]^T B_i w_i + k_i(u,p)\right) + \lambda_i \left[\xi^T g(u) + \xi^T h(u,p)\right], \;\; i \in I \tag{17}$$

and

$$\lambda_i \left[f_i(x) + x^T B_i w_i\right] < \lambda_i \left(f_i(u) + [u+p]^T B_i w_i + k_i(u,p)\right) + \lambda_i \left[\xi^T g(u) + \xi^T h(u,p)\right] \text{ for at least one } i \in I. \tag{18}$$

Adding both sides of (17) and (18) and using $\sum_{i=1}^{q}\lambda_i = 1$, we get

$$\sum_{i=1}^{q}\lambda_i\left[f_i(x)+x^TB_iw_i\right] < \sum_{i=1}^{q}\lambda_i\left(f_i(u)+[u+p]^TB_iw_i+k_i(u,p)\right)+\xi^Tg(u)+\xi^Th(u,p). \tag{19}$$

By assumption, $\left(f_i(\cdot)+(\cdot)^TB_iw_i, g(\cdot)\right)$, $i\in I$, is higher-order (Φ,ρ)-type I objective and constraint functions at u on $D\cup U$. Then, by Definition 2.1, the following inequalities

$$f_i(z)+z^TB_iw_i-f_i(u)-u^TB_iw_i-k_i(u,p)+p^T\nabla_pk_i(u,p)\geqq\Phi\left(z,u,\left(\nabla_pk_i(u,p)+B_iw_i,\rho_{f_i}\right)\right),\ i\in I, \tag{20}$$

$$-g_j(u)-h_j(u,p)+p^T\nabla_ph_j(u,p)\geqq\Phi\left(z,u,\left(\nabla_ph_j(u,p),\rho_{g_j}\right)\right),\ j\in J \tag{21}$$

hold for all $z\in D\cup U$. Therefore, they are also satisfied for $z=x\in D$. Hence, multiplying each inequality (20) by $\lambda_i>0$, $i\in I$, and each inequality (21) by $\xi_j\geqq 0$, $j\in J$, we get

$$\lambda_i\left[f_i(x)+x^TB_iw_i-f_i(u)-u^TB_iw_i-k_i(u,p)+p^T\nabla_pk_i(u,p)\right]\geqq\lambda_i\Phi\left(x,u,\left(\nabla_pk_i(u,p)+B_iw_i,\rho_{f_i}\right)\right),\ i\in I, \tag{22}$$

$$\xi_j\left[-g_j(u)-h_j(u,p)+p^T\nabla_ph_j(u,p)\right]\geqq\xi_j\Phi\left(x,u,\left(\nabla_ph_j(u,p),\rho_{g_j}\right)\right),\ j\in J. \tag{23}$$

Adding both sides of (22) and (23), we get

$$\sum_{i=1}^{q}\lambda_i\left[f_i(x)+x^TB_iw_i-f_i(u)-u^TB_iw_i-k_i(u,p)+p^T\nabla_pk_i(u,p)\right]\geqq\sum_{i=1}^{q}\lambda_i\Phi\left(x,u,\left(\nabla_pk_i(u,p)+B_iw_i,\rho_{f_i}\right)\right), \tag{24}$$

$$\sum_{j=1}^{m}\xi_j\left[-g_j(u)-h_j(u,p)+p^T\nabla_ph_j(u,p)\right]\geqq\sum_{j=1}^{m}\xi_j\Phi\left(x,u,\left(\nabla_ph_j(u,p),\rho_{g_j}\right)\right). \tag{25}$$

Let us introduce the following notations

$$\widetilde{\lambda}_i=\frac{\lambda_i}{\sum_{i=1}^{q}\lambda_i+\sum_{j=1}^{m}\xi_j},\ i\in I,\ \widetilde{\xi}_j=\frac{\xi_j}{\sum_{i=1}^{q}\lambda_i+\sum_{j=1}^{m}\xi_j},\ j\in J. \tag{26}$$

Note that $\widetilde{\lambda}_i\in(0,1]$, $i\in I$, $\widetilde{\xi}_j\in[0,1]$, $j\in J$, and

$$\sum_{i=1}^{q}\widetilde{\lambda}_i+\sum_{j=1}^{m}\widetilde{\xi}_j=1. \tag{27}$$

Taking into account (26) in (24) and (25), and then adding both sides the resulting inequalities, we obtain

$$\sum_{i=1}^{q}\widetilde{\lambda}_i\left[f_i(x)+x^TB_iw_i-f_i(u)-u^TB_iw_i-k_i(u,p)+p^T\nabla_pk_i(u,p)\right]+\sum_{j=1}^{m}\widetilde{\xi}_j\left[-g_j(u)-h_j(u,p)+p^T\nabla_ph_j(u,p)\right]$$
$$\geqq\sum_{i=1}^{q}\widetilde{\lambda}_i\Phi\left(x,u,\left(\nabla_pk_i(u,p)+B_iw_i,\rho_{f_i}\right)\right)+\sum_{j=1}^{m}\widetilde{\xi}_j\Phi\left(x,u,\left(\nabla_ph_j(u,p),\rho_{g_j}\right)\right). \tag{28}$$

By Definition 2.1, it follows that $\Phi(x,u,\cdot)$ is convex on R^{n+1}. Since $\widetilde{\lambda}_i>0$, $i\in I$, $\widetilde{\xi}_j\in[0,1]$, $j\in J$, by (27) and the definition of a convex function, we have

$$\sum_{i=1}^{q}\widetilde{\lambda}_i\left[f_i(x)+x^TB_iw_i-\left[f_i(u)+u^TB_iw_i\right]-k_i(u,p)+p^T\nabla_pk_i(u,p)\right]+\sum_{j=1}^{m}\widetilde{\xi}_j\left[-g_j(u)-h_j(u,p)+p^T\nabla_ph_j(u,p)\right]$$
$$\geqq\Phi\left(x,u,\left(\sum_{i=1}^{q}\widetilde{\lambda}_i\left(\nabla_pk_i(u,p)+B_iw_i,\rho_{f_i}\right)+\sum_{j=1}^{m}\widetilde{\xi}_j\left(\nabla_ph_j(u,p),\rho_{g_j}\right)\right)\right). \tag{29}$$

Using (26) in (29), by the constraint (10), we get

$$\frac{1}{\sum_{i=1}^{q}\lambda_i+\sum_{j=1}^{m}\xi_j}\sum_{i=1}^{q}\lambda_i\left[f_i(x)+x^TB_iw_i-f_i(u)-u^TB_iw_i-k_i(u,p)+p^T\nabla_pk_i(u,p)\right]+$$

$$\frac{1}{\sum_{i=1}^{q}\lambda_i+\sum_{j=1}^{m}\xi_j}\sum_{j=1}^{m}\xi_j\left(-g_j(u)-h_j(u,p)+p^T\nabla_ph_j(u,p)\right)\geqq\Phi\left(x,u,\frac{1}{\sum_{i=1}^{q}\lambda_i+\sum_{j=1}^{m}\xi_j}\left(0\,,\ \sum_{i=1}^{q}\lambda_i\rho_{f_i}+\sum_{j=1}^{m}\xi_j\rho_{g_j}\right)\right). \tag{30}$$

By Definition 2.1, it follows that $\Phi(x,u,(0,a))\geqq 0$ for every $a\in R_+$. By assumption, $\sum_{i=1}^{q}\lambda_i\rho_{f_i}+\sum_{j=1}^{m}\xi_j\rho_{g_j}\geqq 0$. Thus, the following inequality

$$\Phi\left(x,u,\frac{1}{\sum_{i=1}^{q}\lambda_i+\sum_{j=1}^{m}\xi_j}\left(0\,,\ \sum_{i=1}^{q}\lambda_i\rho_{f_i}+\sum_{j=1}^{m}\xi_j\rho_{g_j}\right)\right)\geqq 0 \tag{31}$$

holds. Combining (30) and (31), we obtain that

$$\sum_{i=1}^{q}\lambda_i\left(f_i(x)+x^TB_iw_i\right)\geqq\sum_{i=1}^{q}\lambda_i\left(f_i(u)+u^TB_iw_i+k_i(u,p)-\ p^T\nabla_pk_i(u,p)\right)+\sum_{j=1}^{m}\xi_j\left(g_j(u)+h_j(u,p)-p^T\nabla_ph_j(u,p)\right).$$

Thus,

$$\sum_{i=1}^{q}\lambda_i\left(f_i(x)+x^TB_iw_i\right)\geqq\sum_{i=1}^{q}\lambda_i\left(f_i(u)+[u+p]^TB_iw_i+k_i(u,p)\right)+$$
$$\sum_{j=1}^{m}\xi_j\left(g_j(u)+h_j(u,p)\right)-p^T\sum_{i=1}^{q}\left[\nabla_pk_i(u,p)+B_iw_i+\sum_{j=1}^{m}\xi_j\nabla_ph_j(u,p)\right].$$

By the constraint (10), it follows that the following inequality

$$\sum_{i=1}^{q}\lambda_i\left(f_i(x)+x^TB_iw_i\right)\geqq\sum_{i=1}^{q}\lambda_i\left(f_i(u)+[u+p]^TB_iw_i+k_i(u,p)\right)+\sum_{j=1}^{m}\xi_j\left(g_j(u)+h_j(u,p)\right).$$

holds, contradicting (19). This completes the proof of the theorem. □

Theorem 4.2. *(Strong duality). Let $\overline{x}\in D$ be a properly efficient solution of the considered nondifferentiable multiobjective programming problem (VP) such that the set $\Omega(\overline{x})$ is empty and $\nabla g_j(\overline{x})$, $j\in J(\overline{x})$, be linearly independent. Further, assume that*

$$\begin{cases} k_i(\overline{x},0)=0\ \textit{for all}\ i\in I;\ \ \nabla_pk(\overline{x},0)=\nabla f(\overline{x}), \\ h_j(\overline{x},0)=0\ \textit{for all}\ j\in J;\ \ \nabla_ph(\overline{x},0)=\nabla g(\overline{x}). \end{cases} \tag{32}$$

Then there exist $\overline{\lambda}\in R^q$, $\overline{\xi}\in R^m$, $\overline{w}_i\in R^n$, $i\in I$, such that $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is feasible for (MVD) and the corresponding objective values of (VP) and (MVD) are equal. Further, if weak duality (Theorem 4.1) holds, then $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is a properly efficient solution of a maximum type in (MVD).

Proof. By assumption, $\overline{x}\in D$ is a properly efficient solution of the considered nondifferentiable multiobjective programming problem (VP), the set $\Omega(\overline{x})$ is empty and the Linear Independence Constraint Qualification is satisfied at $\overline{x}$. Then, by Theorem 3.4 and (32), it follows that $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is feasible in (MVD). Also the corresponding objective values of (VP) and (MVD) are equal as it follows by (8) and (32).

In order to prove that $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is a properly efficient of a maximum type for (MVD), first, we show that $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is efficient of a maximum type for (MVD). We proceed by contradiction. Suppose, contrary to the result, that it is not efficient of (MVD). Then, by Definition 3.1, there exists $\left(\widetilde{u},\widetilde{\lambda},\widetilde{\xi},\left(\widetilde{w}_1,...,\widetilde{w}_q\right),\widetilde{p}\right)\in\Omega_{MVD}$ such that

$$f_i(\widetilde{u})+[\widetilde{u}+\widetilde{p}]^TB_i\widetilde{w}_i+k_i(\widetilde{u},\widetilde{p})+\widetilde{\xi}^Tg(\widetilde{u})+\widetilde{\xi}^Th(\widetilde{u},\widetilde{p})\geqq f_i(\overline{x})+\overline{x}^TB_i\overline{w}_i+k_i(\overline{x},0)+\overline{\xi}^Tg(\overline{x})+\overline{\xi}^Th(\overline{x},0)\,,\ i\in I, \tag{33}$$

$$f_{i^*}(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_{i^*}\widetilde{w}_{i^*} + k_{i^*}(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}) > f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*}\overline{w}_{i^*} + k_{i^*}(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0)$$
$$\text{for at least one } i^* \in I. \tag{34}$$

Then, (32), (33) and (34) yield, respectively,

$$f_i(\overline{x}) + \overline{x}^T B_i\overline{w}_i + \overline{\xi}^T g(\overline{x}) \leqq f_i(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_i\widetilde{w}_i + k_i(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}),\ i \in I, \tag{35}$$

$$f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*}\overline{w}_{i^*} + \overline{\xi}^T g(\overline{x}) < f_{i^*}(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_{i^*}\widetilde{w}_{i^*} + k_{i^*}(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}) \text{ for at least one } i^* \in I. \tag{36}$$

Hence, by (6) and (8), it follows that the following inequalities

$$f_i(\overline{x}) + \left(\overline{x}^T B_i\overline{x}\right)^{1/2} \leqq f_i(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_i\widetilde{w}_i + k_i(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}),\ i \in I, \tag{37}$$

$$f_{i^*}(\overline{x}) + \left(\overline{x}^T B_{i^*}\overline{x}\right)^{1/2} < f_{i^*}(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_{i^*}\widetilde{w}_{i^*} + k_{i^*}(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}) \text{ for at least one } i^* \in I \tag{38}$$

hold, contradicting weak duality (Theorem 4.1). This means that $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is efficient of a maximum type for (MVD).

Now, we shall prove that $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is a properly efficient solution of a maximum type in (MVD) by the method of contradiction. Suppose that $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is not so. Then, by Definition 3.2, it follows that there exist $\left(\widetilde{u},\widetilde{\lambda},\widetilde{\xi},\left(\widetilde{w}_1,...,\widetilde{w}_q\right),\widetilde{p}\right) \in \Omega_{MVD}$ and $i^* \in I$ satisfying

$$f_{i^*}(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_{i^*}\widetilde{w}_{i^*} + k_{i^*}(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}) > f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*}\overline{w}_{i^*} + k_{i^*}(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0) \tag{39}$$

such that the inequality

$$f_{i^*}(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_{i^*}\widetilde{w}_{i^*} + k_{i^*}(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}) - \left(f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*}\overline{w}_{i^*} + k_{i^*}(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0)\right) >$$
$$M\left(f_t(\overline{x}) + \overline{x}^T B_t\overline{w}_t + k_t(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0) - \left(f_t(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_t\widetilde{w}_t + k_t(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p})\right)\right). \tag{40}$$

holds for each scalar $M > 0$ and all $t \in I$ satisfying

$$f_t(\overline{x}) + \overline{x}^T B_t\overline{w}_t + k_t(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0) > f_t(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_t\widetilde{w}_t + k_t(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}). \tag{41}$$

We divide the index set I and denote by I_1 the set of indexes of objective functions satisfying the inequality (41). By I_2 we denote the set of indexes of objective functions defining by $I_2 = I\setminus(I_1 \cup i^*)$. Let $M > \frac{\lambda_{i^*}}{\lambda_t}|I_1|$, where $|I_1|$ denotes the number of elements in the set I_1. Hence, by (40) and (41), it follows that

$$\lambda_{i^*}\left(f_{i^*}(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_{i^*}\widetilde{w}_{i^*} + k_{i^*}(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p}) - \left(f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*}\overline{w}_{i^*} + k_{i^*}(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0)\right)\right) >$$
$$\sum_{t\in I_1}\lambda_t\left(f_t(\overline{x}) + \overline{x}^T B_t\overline{w}_t + k_t(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0) - \left(f_t(\widetilde{u}) + [\widetilde{u}+\widetilde{p}]^T B_t\widetilde{w}_t + k_t(\widetilde{u},\widetilde{p}) + \widetilde{\xi}^T g(\widetilde{u}) + \widetilde{\xi}^T h(\widetilde{u},\widetilde{p})\right)\right). \tag{42}$$

Using the definition of the set I_2 together with (42), we obtain

$$\sum_{i=1}^{q}\lambda_i\left(f_i(\overline{x}) + \overline{x}^T B_i\overline{w}_i + k_i(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0)\right) = \lambda_{i^*}\left(f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*}\overline{w}_{i^*} + k_{i^*}(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0)\right) +$$
$$\sum_{i\in I_1}\lambda_i\left(f_i(\overline{x}) + \overline{x}^T B_i\overline{w}_i + k_i(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0)\right) + \sum_{i\in I_2}\lambda_i\left(f_i(\overline{x}) + \overline{x}^T B_i\overline{w}_i + k_i(\overline{x},0) + \overline{\xi}^T g(\overline{x}) + \overline{\xi}^T h(\overline{x},0)\right) <$$

$$\lambda_{i^*}\left(f_{i^*}(\widetilde{u})+[\widetilde{u}+\widetilde{p}]^T B_{i^*}\widetilde{w}_{i^*}+k_{i^*}(\widetilde{u},\widetilde{p})+\widetilde{\xi}^T g(\widetilde{u})+\widetilde{\xi}^T h(\widetilde{u},\widetilde{p})\right)+$$

$$\sum_{t\in I_1}\lambda_t\left(f_t(\widetilde{u})+[\widetilde{u}+\widetilde{p}]^T B_t\widetilde{w}_t+k_t(\widetilde{u},\widetilde{p})+\widetilde{\xi}^T g(\widetilde{u})+\widetilde{\xi}^T h(\widetilde{u},\widetilde{p})\right)+$$

$$\sum_{t\in I_2}\lambda_t\left(f_t(\widetilde{u})+[\widetilde{u}+\widetilde{p}]^T B_t\widetilde{w}_t+k_t(\widetilde{u},\widetilde{p})+\widetilde{\xi}^T g(\widetilde{u})+\widetilde{\xi}^T h(\widetilde{u},\widetilde{p})\right)=$$

$$\sum_{i=1}^{q}\lambda_i\left(f_i(\widetilde{u})+[\widetilde{u}+\widetilde{p}]^T B_i\widetilde{w}_i+k_i(\widetilde{u},\widetilde{p})+\widetilde{\xi}^T g(\widetilde{u})+\widetilde{\xi}^T h(\widetilde{u},\widetilde{p})\right).$$

By (6), (8) and (32), it follows that the following inequality

$$\sum_{i=1}^{q}\lambda_i\left(f_i(\overline{x})+\left(\overline{x}^T B_i\overline{x}\right)^{1/2}\right)<\sum_{i=1}^{q}\lambda_i\left(f_i(\widetilde{u})+[\widetilde{u}+\widetilde{p}]^T B_i\widetilde{w}_i+k_i(\widetilde{u},\widetilde{p})+\widetilde{\xi}^T g(\widetilde{u})+\widetilde{\xi}^T h(\widetilde{u},\widetilde{p})\right)$$

holds, contradicting weak duality (Theorem 4.1). This means that $\left(\overline{x},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}=0\right)$ is a properly efficient solution in (MVD) and completes the proof of theorem. □

Theorem 4.3. *(Restricted converse duality). Let $\overline{x}$ and $\left(\overline{u},\overline{\lambda},\overline{\xi},\left(\overline{w}_1,...,\overline{w}_q\right),\overline{p}\right)$ be feasible solutions in the vector optimization problems (VP) and (MVD), respectively, such that*

$$f_i(\overline{x})+\overline{x}^T B_i\overline{w}_i \leqq f_i(\overline{u})+[\overline{u}+\overline{p}]^T B_i\overline{w}_i+k_i(\overline{u},\overline{p})+\overline{\xi}^T g(\overline{u})+\overline{\xi}^T h(\overline{u},\overline{p}),\ i\in I. \tag{43}$$

Further, assume that $\left(f_i(\cdot)+(\cdot)^T B_i\overline{w}_i,\ i\in I,\ g_j(\cdot),\ j\in J\right)$, is higher-order strictly (Φ,ρ)-type I objective and constraint functions at $\overline{u}$ on $D\cup U$. If $\sum_{i=1}^{q}\overline{\lambda}_i\rho_{f_i}+\sum_{j}^{m}\overline{\xi}_j\rho_{g_j}\geqq 0$, then $\overline{x}=\overline{u}$.

Proof. Since $\overline{\lambda}_i>0$, $i\in I$, $\sum_{i=1}^{q}\overline{\lambda}_i=1$, (43) gives

$$\textstyle\sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{x})+\overline{x}^T B_i\overline{w}_i\right)\leqq\sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{u})+[\overline{u}+\overline{p}]^T B_i\overline{w}_i+k_i(\overline{u},\overline{p})\right)+\overline{\xi}^T g(\overline{u})+\overline{\xi}^T h(\overline{u},\overline{p}). \tag{44}$$

We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}\neq\overline{u}$. By assumption, $\left(f_i(\cdot)+(\cdot)^T B_i\overline{w}_i,g(\cdot)\right)$, $i\in I$, is higher-order (Φ,ρ)-type I objective and constraint functions at $\overline{u}$ on $D\cup U$. Since $\overline{\lambda}_i>0$, $i\in I$, and $\overline{\xi}_j\geqq 0$, $j\in J$, by Definition 2.1, we have

$$\overline{\lambda}_i\left(f_i(\overline{x})+\overline{x}^T B_i\overline{w}-f_i(\overline{u})-\overline{u}^T B_i\overline{w}_i-k_i(\overline{u},\overline{p})+\overline{p}^T\nabla_p k_i(\overline{u},\overline{p})\right)>\overline{\lambda}_i\Phi\left(\overline{x},\overline{u},\left(\nabla_p k_i(\overline{u},\overline{p})+B_i\overline{w}_i,\rho_{f_i}\right)\right),\ i\in I, \tag{45}$$

$$\overline{\xi}_j\left[-g_j(\overline{u})-h_j(\overline{u},\overline{p})+p^T\nabla_p h_j(\overline{u},\overline{p})\right]\geqq\overline{\xi}_j\Phi\left(\overline{x},\overline{u},\left(\nabla_p h_j(\overline{u},\overline{p}),\rho_{g_j}\right)\right),\ j\in J. \tag{46}$$

Adding both sides of (45) and (46), we get

$$\textstyle\sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{x})+\overline{x}^T B_i\overline{w}_i-f_i(\overline{u})-\overline{u}^T B_i\overline{w}_i-k_i(\overline{u},\overline{p})+\overline{p}^T\nabla_p k_i(\overline{u},\overline{p})\right)>\sum_{i=1}^{q}\overline{\lambda}_i\Phi\left(\overline{x},\overline{u},\left(\nabla_p k_i(\overline{u},\overline{p})+B_i\overline{w}_i,\rho_{f_i}\right)\right), \tag{47}$$

$$\textstyle\sum_{j=1}^{m}\overline{\xi}_j\left[-g_j(\overline{u})-h_j(\overline{u},\overline{p})+\overline{p}^T\nabla_p h_j(\overline{u},\overline{p})\right]\geqq\sum_{j=1}^{m}\overline{\xi}_j\Phi\left(\overline{x},\overline{u},\left(\nabla_p h_j(\overline{u},\overline{p}),\rho_{g_j}\right)\right). \tag{48}$$

Let us introduce the following notations

$$\lambda_i^*=\frac{\overline{\lambda}_i}{\sum_{i=1}^{q}\overline{\lambda}_i+\sum_{j=1}^{m}\overline{\xi}_j},\ i\in I,\ \xi_j^*=\frac{\overline{\xi}_j}{\sum_{i=1}^{q}\overline{\lambda}_i+\sum_{j=1}^{m}\overline{\xi}_j},\ j\in J. \tag{49}$$

Note that $\lambda_i^* \in (0,1]$, $i \in I$, $\xi_j^* \in [0,1]$, $j \in J$, and

$$\sum_{i=1}^{q} \lambda_i^* + \sum_{j=1}^{m} \xi_j^* = 1. \tag{50}$$

Using (47) and (48) together with (49), we get, respectively,

$$\sum_{i=1}^{q} \lambda_i^* \left[f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i - \left(f_i(\overline{u}) + \overline{u}^T B_i \overline{w}_i \right) - k_i(\overline{u},\overline{p}) + \overline{p}^T \nabla_p k_i(\overline{u},\overline{p}) \right] > \sum_{i=1}^{q} \lambda_i^* \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p k_i(\overline{u},\overline{p}) + B_i \overline{w}_i, \rho_{f_i} \right) \right), \tag{51}$$

$$\sum_{j=1}^{m} \xi_j^* \left[-g_j(\overline{u}) - h_j(\overline{u},\overline{p}) + \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \right] \geqq \sum_{j=1}^{m} \xi_j^* \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p h_j(\overline{u},\overline{p}), \rho_{g_j} \right) \right). \tag{52}$$

Adding both sides of (51) and (52), respectively, and then adding both sides of the obtained inequalities, we get

$$\begin{gathered} \sum_{i=1}^{q} \lambda_i^* \left[f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i - \left(f_i(\overline{u}) + \overline{u}^T B_i \overline{w}_i \right) - k_i(\overline{u},\overline{p}) + \overline{p}^T \nabla_p k_i(\overline{u},\overline{p}) \right] + \\ \sum_{j=1}^{m} \xi_j^* \left[-g_j(\overline{u}) - h_j(\overline{u},\overline{p}) + \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \right] > \\ \sum_{i=1}^{q} \lambda_i^* \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p k_i(\overline{u},\overline{p}) + B_i \overline{w}_i, \rho_{f_i} \right) \right) + \sum_{j=1}^{m} \xi_j^* \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p h_j(\overline{u},\overline{p}), \rho_{g_j} \right) \right). \end{gathered} \tag{53}$$

By Definition 2.1, it follows that $\Phi(\overline{x}, \overline{u}, \cdot)$ is convex on R^{n+1}. Since $\lambda_i^* \in (0,1]$, $i \in I$, $\xi_j^* \in [0,1]$, $j \in J$, and (50) is satisfied, by the definition of a convex function, we have

$$\begin{gathered} \sum_{i=1}^{q} \lambda_i^* \left[f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i - \left(f_i(\overline{u}) + \overline{u}^T B_i \overline{w}_i \right) - k_i(\overline{u},\overline{p}) + \overline{p}^T \nabla_p k_i(\overline{u},\overline{p}) \right] + \\ \sum_{j=1}^{m} \xi_j^* \left[-g_j(\overline{u}) - h_j(\overline{u},\overline{p}) + \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \right] > \\ \Phi\left(\overline{x}, \overline{u}, \left(\sum_{i=1}^{q} \lambda_i^* \left(\nabla_p k_i(\overline{u},\overline{p}) + B_i \overline{w}_i, \rho_{f_i} \right) + \sum_{j=1}^{m} \xi_j^* \left(\nabla_p h_j(\overline{u},\overline{p}), \rho_{g_j} \right) \right) \right). \end{gathered} \tag{54}$$

Using (54) together with (49), we get

$$\begin{gathered} \frac{1}{\sum_{i=1}^{q} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\xi}_j} \sum_{i=1}^{q} \overline{\lambda}_i \left[f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i - \left(f_i(\overline{u}) + \overline{u}^T B_i \overline{w}_i \right) - k_i(\overline{u},\overline{p}) + \overline{p}^T \nabla_p k_i(\overline{u},\overline{p}) \right] + \\ \frac{1}{\sum_{i=1}^{q} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\xi}_j} \sum_{j=1}^{m} \overline{\xi}_j \left[-g_j(\overline{u}) - h_j(\overline{u},\overline{p}) + \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \right] > \\ \Phi\left(\overline{x}, \overline{u}, \frac{1}{\sum_{i=1}^{q} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\xi}_j} \left(\sum_{i=1}^{q} \overline{\lambda}_i \left[\nabla_p k_i(\overline{u},\overline{p}) + B_i \overline{w}_i \right] + \sum_{j=1}^{m} \overline{\xi}_j \nabla_p h_j(\overline{u},\overline{p}) \ , \ \sum_{i=1}^{q} \overline{\lambda}_i \rho_{f_i} + \sum_{j=1}^{m} \overline{\xi}_j \rho_{g_j} \right) \right). \end{gathered} \tag{55}$$

By the constraint (10), the inequality (55) gives

$$\begin{gathered} \frac{1}{\sum_{i=1}^{q} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\xi}_j} \sum_{i=1}^{q} \overline{\lambda}_i \left[f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i - \left(f_i(\overline{u}) + \overline{u}^T B_i \overline{w}_i \right) - k_i(\overline{u},\overline{p}) + \overline{p}^T \nabla_p k_i(\overline{u},\overline{p}) \right] + \\ \frac{1}{\sum_{i=1}^{q} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\xi}_j} \sum_{j=1}^{m} \overline{\xi}_j \left(-g_j(\overline{u}) - h_j(\overline{u},\overline{p}) + \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \right) > \\ \Phi\left(\overline{x}, \overline{u}, \frac{1}{\sum_{i=1}^{q} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\xi}_j} \left(0 \ , \ \sum_{i=1}^{q} \overline{\lambda}_i \rho_{f_i} + \sum_{j=1}^{m} \overline{\xi}_j \rho_{g_j} \right) \right). \end{gathered} \tag{56}$$

By Definition 2.1, it follows that $\Phi(\overline{x}, \overline{u}, (0, a)) \geqq 0$ for every $a \in R_+$. By assumption, $\sum_{i=1}^{q} \overline{\lambda}_i \rho_{f_i} + \sum_{j=1}^{m} \xi_j \rho_{g_j} \geqq 0$. Thus, the following inequality

$$\Phi\left(\overline{x}, \overline{u}, \frac{1}{\sum_{i=1}^{q} \overline{\lambda}_i + \sum_{j=1}^{m} \overline{\xi}_j} \left(0 \ , \ \sum_{i=1}^{q} \overline{\lambda}_i \rho_{f_i} + \sum_{j=1}^{m} \overline{\xi}_j \rho_{g_j} \right) \right) \geqq 0 \tag{57}$$

holds. Hence, (56) and (57) yield

$$\begin{aligned}\sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{x})+\overline{x}^TB_i\overline{w}_i\right) \quad > \quad & \sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{u})+\overline{u}^TB_i\overline{w}_i+k_i\left(\overline{u},\overline{p}\right)-\overline{p}^T\nabla_pk_i\left(\overline{u},\overline{p}\right)\right)\\ &+\sum_{j=1}^{m}\overline{\xi}_j\left(g_j(\overline{u})+h_j\left(\overline{u},\overline{p}\right)-\overline{p}^T\nabla_ph_j\left(\overline{u},\overline{p}\right)\right).\end{aligned}$$

Thus,

$$\sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{x})+\overline{x}^TB_i\overline{w}_i\right)>\sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{u})+\left[\overline{u}+\overline{p}\right]^TB_i\overline{w}_i+k_i\left(\overline{u},\overline{p}\right)\right)+$$
$$\sum_{j=1}^{m}\overline{\xi}_j\left(g_j(\overline{u})+h_j\left(\overline{u},\overline{p}\right)\right)-\overline{p}^T\sum_{i=1}^{q}\overline{\lambda}_i\left[\nabla_pk_i\left(\overline{u},\overline{p}\right)+B_i\overline{w}_i+\sum_{j=1}^{m}\overline{\xi}_j\nabla_ph_j\left(\overline{u},\overline{p}\right)\right].$$

By the constraint (10), it follows that the following inequality

$$\sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{x})+\overline{x}^TB_i\overline{w}_i\right)>\sum_{i=1}^{q}\overline{\lambda}_i\left(f_i(\overline{u})+\left[\overline{u}+\overline{p}\right]^TB_i\overline{w}_i+k_i\left(\overline{u},\overline{p}\right)\right)+\sum_{j=1}^{m}\overline{\xi}_j\left(g_j(\overline{u})+h_j\left(\overline{u},\overline{p}\right)\right)$$

holds, contradicting (44). This completes the proof of the theorem. □

5. Mixed duality

In this section, a higher-order mixed dual problem is formulated for the considered nondifferentiable multiobjective programming problem (VP) and several mixed duality theorems are established under assumption that the functions constituting the problem (VP) are (generalized) higher-order (Φ,ρ)-type I objective and constraint functions.

Consider the following higher-order mixed dual problem (MXVD) related to problem (VP):

$$\text{Maximize}\left(f_1(u)+u^TB_1w_1+k_1(u,p)-p^T\nabla_pk_1(u,p)+\sum_{j\in J_0}\left[\xi_jg_j(u)+\xi_jh_j(u,p)-p^T\nabla_p\xi_jh_j(u,p)\right],...,\right.$$

$$\left.f_q(u)+u^TB_qw_q+k_q(u,p)-p^T\nabla_pk_q(u,p)+\sum_{j\in J_0}\left[\xi_jg_j(u)+\xi_jh_j(u,p)-p^T\nabla_p\xi_jh_j(u,p)\right]\right) \qquad \text{(MXVD)}$$

$$\text{s.t.} \sum_{i=1}^{q}\lambda_i\left(\nabla_pk_i(u,p)+B_iw_i\right)+\sum_{j=1}^{m}\xi_j\nabla_ph_j(u,p)=0, \tag{58}$$

$$\sum_{j\in J_\beta}\left(\xi_jg_j(u)+\xi_jh_j(u,p)-p^T\nabla_p\xi_jh_j(u,p)\right)\geqq 0,\ \ \beta=1,...,s, \tag{59}$$

$$w_i^TBw_i\leq 1,\ i\in I, \tag{60}$$

$$u\in X,\ p\in R^n,\ w_i\in R^n,\ \lambda_i>0,\ i=1,...,q,\ \sum_{i=1}^{q}\lambda_i=1,\ \xi_j\geqq 0,\ j\in J, \tag{61}$$

where $J_\beta\subseteq J$, $\beta=0,1,...,s$ with $J_\beta\cap J_\gamma=\varnothing$, $\beta\neq\gamma$ and $\bigcup_{\beta=0}^{s}J_\beta=J$.

Let

$$\Omega_{MXVD}=\left\{\left(u,\lambda,\xi,\left(w_1,...,w_q\right),p\right)\in X\times R_+^q\times R_+^m\times(\underbrace{R^n\times...\times R^n}_{q\text{ times}})\times R^n:\text{verifying the constraints of (MXVD)}\right\}$$

be the set of all feasible solutions in problem (MXVD). Further, $U_{MXVD}=\left\{u\in X:\left(u,\lambda,\xi,\left(w_1,...,w_q\right),p\right)\in\Omega_{MXVD}\right\}$.

Theorem 5.1. *(Weak duality). Let x and $\left(u, \lambda, \xi, \left(w_1, ..., w_q\right), p\right)$ be any feasible solutions for the problems (VP) and (MXVD), respectively. Further, assume that $\left(\left(f_i(\cdot) + (\cdot)^T B_i w_i + \sum_{j\in J_0} \xi_j g_j(u),\ i \in I\right),\ \left(g_j(u),\ j \in J_\beta, \beta = 1, ..., s\right)\right)$ is higher-order (Φ, ρ)-type I objective and constraint functions at u on $D \cup U_{MXVD}$. If $\sum_{i=1}^{q} \lambda_i \rho_{f_i} + \sum_{j\in J_\beta} \xi_j \rho_{g_j} \geqq 0$, then the following cannot hold*

$$f_i(x) + \left(x^T B_i x\right)^{1/2} \leqq f_i(u) + u^T B_i w_i + k_i(u,p) - p^T \nabla_p k_i(u,p) + \sum_{j\in J_0} \left[\xi_j g_j(u) + \xi_j h_j(u,p) - p^T \nabla_p \xi_j h_j(u,p)\right],\ i \in I, \tag{62}$$

and

$$f_i(x) + \left(x^T B_i x\right)^{1/2} < f_i(u) + u^T B_i w_i + k_i(u,p) - p^T \nabla_p k_i(u,p) + \sum_{j\in J_0} \left[\xi_j g_j(u) + \xi_j h_j(u,p) - p^T \nabla_p \xi_j h_j(u,p)\right]$$
$$\text{for at least one } \in I. \tag{63}$$

Proof. Let x and $\left(u, \lambda, \xi, \left(w_1, ..., w_q\right), p\right)$ be feasible solutions for the vector optimization problems (VP) and (MXVD), respectively. We proceed by contradiction. Suppose, contrary to the result, that the inequalities (62) and (63) are satisfied. Then, by the generalized Schwartz inequality (see Lemma 2.4), (62) and (63) yield, respectively,

$$f_i(x) + x^T B_i w_i \leqq f_i(u) + u^T B_i w_i + k_i(u,p) - p^T \nabla_p k_i(u,p) + \sum_{j\in J_0} \left[\xi_j g_j(u) + \xi_j h_j(u,p) - p^T \nabla_p \xi_j h_j(u,p)\right],\ i \in I, \tag{64}$$

and

$$f_i(x) + x^T B_i w_i < f_i(u) + u^T B_i w_i + k_i(u,p) - p^T \nabla_p k_i(u,p) + \sum_{j\in J_0} \left[\xi_j g_j(u) + \xi_j h_j(u,p) - p^T \nabla_p \xi_j h_j(u,p)\right]$$
$$\text{for at least one } i \in I. \tag{65}$$

Multiplying each inequality (64) by $\lambda_i > 0$, $i = 1, ..., q$, and then using $\sum_{i=1}^{q} \lambda_i = 1$, we get

$$\begin{aligned} \sum_{i=1}^{q} \lambda_i \left(f_i(x) + x^T B_i w_i\right) < \sum_{i=1}^{q} \lambda_i \left(f_i(u) + u^T B_i w_i + k_i(u,p) - p^T \nabla_p k_i(u,p)\right) + \\ \sum_{j\in J_0} \left[\xi_j g_j(u) + \xi_j h_j(u,p) - p^T \nabla_p \xi_j h_j(u,p)\right]. \end{aligned} \tag{66}$$

By assumption, $\left(\left(f_i(\cdot) + (\cdot)^T B_i w_i + \sum_{j\in J_0} \xi_j g_j(\cdot),\ i \in I\right),\ \left(g_j(\cdot),\ j \in J_\beta, \beta = 1, ..., s\right)\right)$ are higher-order (Φ, ρ)-type I objective and constraint functions at u on $D \cup U_{MXVD}$. Then, by Definition 2.1, the following inequalities

$$\begin{aligned} f_i(x) + x^T B_i w_i + \sum_{j\in J_0} \xi_j g_j(x) - f_i(u) - u^T B_i w_i - \sum_{j\in J_0} \xi_j g_j(u) - k_i(u,p) + p^T \nabla_p k_i(u,p) \geqq \\ \Phi\left(x, u, \left(\nabla_p k_i(u,p) + B_i w_i + \sum_{j\in J_0} \xi_j \nabla_p g_j(u,p), \rho_{f_i}\right)\right),\ i \in I, \end{aligned} \tag{67}$$

$$-g_j(u) - h_j(u,p) + p^T \nabla_p h_j(u,p) \geqq \Phi\left(z, u, \left(\nabla_p h_j(u,p), \rho_{g_j}\right)\right),\ j \in J_\beta,\ \beta = 1, ..., s \tag{68}$$

hold. Hence, multiplying each inequality (67) by $\lambda_i > 0$, $i \in I$, and each inequality (68) by $\xi_j \geqq 0$, $j \in J$, and then using $\sum_{i=1}^{q} \lambda_i = 1$, we get

$$\begin{aligned} \sum_{i=1}^{q} \lambda_i \left(f_i(x) + x^T B_i w_i - f_i(u) - u^T B_i w_i - k_i(u,p) + p^T \nabla_p k_i(u,p)\right) \sum_{j\in J_0} \xi_j g_j(x) - \sum_{j\in J_0} \xi_j g_j(u) \geqq \\ \sum_{i=1}^{q} \lambda_i \Phi\left(x, u, \left(\nabla_p k_i(u,p) + B_i w_i + \sum_{j\in J_0} \xi_j \nabla_p g_j(u,p), \rho_{f_i}\right)\right), \end{aligned} \tag{69}$$

$$\sum_{j\in J_\beta} \xi_j \left(-g_j(u) - h_j(u,p) + p^T \nabla_p h_j(u,p)\right) \geqq \sum_{j\in J_\beta} \xi_j \Phi\left(x, u, \left(\nabla_p h_j(u,p), \rho_{g_j}\right)\right). \tag{70}$$

Combining (59) and (70), we obtain

$$\sum_{j\in J_\beta} \xi_j \Phi\left(x, u, \left(\nabla_p h_j(u,p), \rho_{g_j}\right)\right) \leqq 0,\ \beta = 1, ..., s. \tag{71}$$

Hence, (69) and (71) yield

$$\begin{gathered}\sum_{i=1}^{q} \lambda_i \left(f_i(x) + x^T B_i w_i - f_i(u) - u^T B_i w_i - k_i(u,p) + p^T \nabla_p k_i(u,p)\right) + \\ \sum_{j\in J_0} \xi_j g_j(x) - \sum_{j\in J_0} \xi_j g_j(u) - \sum_{j\in J_\beta} \xi_j \left(-g_j(u) - h_j(u,p) + p^T \nabla_p h_j(u,p)\right) \geqq \\ \sum_{i=1}^{q} \lambda_i \Phi\left(x,u,\left(\nabla_p k_i(u,p) + B_i w_i + \sum_{j\in J_0} \xi_j \nabla_p g_j(u,p), \rho_{f_i}\right)\right) + \sum_{j\in J_\beta} \xi_j \Phi\left(x,u,\left(\nabla_p h_j(u,p), \rho_{g_j}\right)\right).\end{gathered} \tag{72}$$

Let us introduce the following notations

$$\widetilde{\lambda}_i = \frac{\lambda_i}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j},\ i \in I,\ \widetilde{\xi}_j = \frac{\xi_j}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j},\ j \in J_\beta,\ \beta = 1, ..., s. \tag{73}$$

Note that $\widetilde{\lambda}_i \in (0,1]$, $i \in I$, $\widetilde{\xi}_j \in [0,1]$, $j \in J$, and

$$\sum_{i=1}^{q} \widetilde{\lambda}_i + \sum_{j\in J_\beta} \widetilde{\xi}_j = 1. \tag{74}$$

Using (73) in (72), we get

$$\begin{gathered}\sum_{i=1}^{q} \widetilde{\lambda}_i \left(f_i(x) + x^T B_i w_i - f_i(u) - u^T B_i w_i - k_i(u,p) + p^T \nabla_p k_i(u,p)\right) + \\ \frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left[\sum_{j\in J_0} \xi_j g_j(x) - \sum_{j\in J_0} \xi_j g_j(u) - \sum_{j\in J_0} \xi_j h_j(u,p) + p^T \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p)\right] \geqq \\ \sum_{i=1}^{q} \widetilde{\lambda}_i \Phi\left(x,u,\left(\nabla_p k_i(u,p) + B_i w + \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p), \rho_{f_i}\right)\right) + \sum_{j\in J_\beta} \widetilde{\xi}_j \Phi\left(x,u,\left(\nabla_p h_j(u,p), \rho_{g_j}\right)\right).\end{gathered} \tag{75}$$

By Definition 2.1, it follows that $\Phi(x,u,\cdot)$ is convex on R^{n+1}. Since $\widetilde{\lambda}_i \in (0,1]$, $i \in I$, $\widetilde{\xi}_j \in [0,1]$, $j \in J_\beta$, $\beta = 1, ..., s$, and (74) is satisfied, by the definition of a convex function, we have

$$\begin{gathered}\sum_{i=1}^{q} \widetilde{\lambda}_i \Phi\left(x,u,\left(\nabla_p k_i(u,p) + B_i w_i + \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p), \rho_{f_i}\right)\right) + \sum_{j\in J_\beta} \widetilde{\xi}_j \Phi\left(x,u,\left(\nabla_p h_j(u,p), \rho_{g_j}\right)\right) \geqq \\ \Phi\left(x,u,\left(\sum_{i=1}^{q} \widetilde{\lambda}_i \left[\nabla_p k_i(u,p) + B_i w_i + \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p), \rho_{f_i}\right] + \sum_{j\in J_\beta} \widetilde{\xi}_j \left(\nabla_p h_j(u,p), \rho_{g_j}\right)\right)\right).\end{gathered} \tag{76}$$

Combining (75) and (76), we obtain

$$\begin{gathered}\sum_{i=1}^{q} \widetilde{\lambda}_i \left(f_i(x) + x^T B_i w_i - f_i(u) - u^T B_i w_i - k_i(u,p) + p^T \nabla_p k_i(u,p)\right) + \\ \frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left[\sum_{j\in J_0} \xi_j g_j(x) - \sum_{j\in J_0} \xi_j g_j(u) - \sum_{j\in J_0} \xi_j h_j(u,p) + p^T \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p)\right] \geqq \\ \Phi\left(x,u,\left(\sum_{i=1}^{q} \widetilde{\lambda}_i \left[\nabla_p k_i(u,p) + B_i w_i + \frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p), \rho_{f_i}\right] + \sum_{j\in J_\beta} \widetilde{\xi}_j \left(\nabla_p h_j(u,p), \rho_{g_j}\right)\right)\right)\end{gathered} \tag{77}$$

Using (73) in (77), we obtain

$$\begin{gathered}\frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left\{\sum_{i=1}^{q} \lambda_i \left[f_i(x) + x^T B_i w_i - f_i(u) - u^T B_i w_i - k_i(u,p) + p^T \nabla_p k_i(u,p)\right] + \right. \\ \left. \sum_{j\in J_0} \xi_j g_j(x) - \sum_{j\in J_0} \xi_j g_j(u) - \sum_{j\in J_0} \xi_j h_j(u,p) + p^T \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p)\right\} \geqq \\ \Phi\left(x,u,\frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left(\sum_{i=1}^{q} \lambda_i \left[\nabla_p k_i(u,p) + B_i w_i + \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p)\right] + \sum_{j\in J_\beta} \xi_j \nabla_p h_j(u,p), \sum_{i=1}^{q} \lambda_i \rho_{f_i} + \sum_{j\in J_\beta} \xi_j \rho_{g_j}\right)\right).\end{gathered}$$

Since $\sum_{i=1}^{q} \lambda_i = 1$ and $\bigcup_{\beta=0}^{s} J_\beta = J$, the above inequality yields

$$\frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left\{ \sum_{i=1}^{q} \lambda_i \left[f_i(x) + x^T B_i w_i - f_i(u) - u^T B_i w_i - k_i(u,p) + p^T \nabla_p k_i(u,p) \right] + \sum_{j\in J_0} \xi_j g_j(x) - \sum_{j\in J_0} \xi_j g_j(u) - \sum_{j\in J_0} \xi_j h_j(u,p) + p^T \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p) \right\} \geqq \Phi\left(x, u, \frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left(\sum_{i=1}^{q} \lambda_i \left[\nabla_p k_i(u,p) + B_i w_i \right] + \sum_{j=1}^{m} \xi_j \nabla_p h_j(u,p) \ , \ \sum_{i=1}^{q} \lambda_i \rho_{f_i} + \sum_{j\in J_\beta} \xi_j \rho_{g_j} \right) \right). \tag{78}$$

By the constraint (58), the inequality (78) implies

$$\frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left\{ \sum_{i=1}^{q} \lambda_i \left[f_i(x) + x^T B_i w_i - f_i(u) - u^T B_i w_i - k_i(u,p) + p^T \nabla_p k_i(u,p) \right] + \sum_{j\in J_0} \xi_j g_j(x) - \sum_{j\in J_0} \xi_j g_j(u) - \sum_{j\in J_0} \xi_j h_j(u,p) + p^T \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p) \right\} \geqq \Phi\left(x, u, \frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left(0, \sum_{i=1}^{q} \lambda_i \rho_{f_i} + \sum_{j\in J_\beta} \xi_j \rho_{g_j} \right) \right). \tag{79}$$

By Definition 2.1, it follows that $\Phi(x, u, (0, a)) \geqq 0$ for every $a \in R_+$. By assumption, $\sum_{i=1}^{q} \lambda_i \rho_{f_i} + \sum_{j\in J_\beta} \xi_j \rho_{g_j} \geqq 0$. Thus, the following inequality

$$\Phi\left(x, u, \frac{1}{\sum_{i=1}^{q} \lambda_i + \sum_{j\in J_\beta} \xi_j} \left(0 \ , \ \sum_{i=1}^{q} \lambda_i \rho_{f_i} + \sum_{j\in J_\beta} \xi_j \rho_{g_j} \right) \right) \geqq 0 \tag{80}$$

holds. Combining (79) and (80), we get

$$\sum_{i=1}^{q} \lambda_i \left(f_i(x) + x^T B_i w_i - f_i(u) - u^T B_i w_i - k_i(u,p) + p^T \nabla_p k_i(u,p) \right) + \sum_{j\in J_0} \xi_j g_j(x) - \sum_{j\in J_0} \xi_j g_j(u) - \sum_{j\in J_0} \xi_j h_j(u,p) + p^T \sum_{j\in J_0} \xi_j \nabla_p h_j(u,p) \geqq 0.$$

Thus, by $x \in D$ and $\xi_j \geqq 0$, $j \in J$, it follows that the following inequality

$$\sum_{i=1}^{q} \lambda_i \left(f_i(x) + x^T B_i w_i \right) \geqq \sum_{i=1}^{q} \lambda_i \left(f_i(u) + u^T B_i w_i + k_i(u,p) - p^T \nabla_p k_i(u,p) \right) + \sum_{j\in J_0} \left(\xi_j g_j(u) + \xi_j h_j(u,p) - p^T \nabla_p \xi_j h_j(u,p) \right).$$

holds, contradicting (66). This completes the proof of the theorem. □

Theorem 5.2. *(Strong duality). Let $\overline{x} \in D$ be a properly efficient solution in the considered nondifferentiable multiobjective programming problem (VP) such that the set $\Omega(\overline{x})$ is empty and let $\nabla g_j(\overline{x})$, $j \in J(\overline{x})$, be linearly independent. Further, assume that*

$$\begin{cases} k_i(\overline{x}, 0) = 0 \text{ for all } i \in I; \ \ \nabla_p k(\overline{x}, 0) = \nabla f(\overline{x}), \\ h_j(\overline{x}, 0) = 0 \text{ for all } j \in J; \ \ \nabla_p h(\overline{x}, 0) = \nabla g(\overline{x}). \end{cases} \tag{81}$$

Then there exist $\overline{\lambda} \in R^q$, $\overline{\xi} \in R^m$, $\overline{w}_i \in R^n$, $i \in I$, such that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p} = 0\right)$ is feasible for (MXVD) and the corresponding objective values of (VP) and (MXVD) are equal. Further, if weak duality (Theorem 5.1) holds, then $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p} = 0\right)$ is a properly efficient solution of a maximum type for the higher-order mixed dual problem (MXVD).

Proof. Since $\overline{x} \in D$ is assumed to be a properly efficient solution in the considered nondifferentiable multiobjective programming problem (VP) such that the set $\Omega(\overline{x})$ is empty and, moreover, the Linear Independence Constraint Qualification is satisfied at $\overline{x}$, by Theorem 3.4 and the assumption (81), it follows that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p} = 0\right)$ is feasible in (MXVD). Therefore, it is not difficult to see that the corresponding objective values of (VP) and (MXVD) are equal as it follows by (8) and (81).

In order to prove that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p} = 0\right)$ is a properly efficient solution in (MXVD), first, we show that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p} = 0\right)$ is an efficient solution of a maximum type in (MXVD). We proceed by contradiction. Suppose, contrary to the result, that it is not efficient in (MXVD). Then, by Definition 3.1, there exists $\left(\widetilde{u}, \widetilde{\lambda}, \widetilde{\xi}, \left(\widetilde{w}_1, ..., \widetilde{w}_q\right), \widetilde{p}\right) \in \Omega_{MXVD}$ such that

$$\begin{gathered} f_i(\widetilde{u}) + \widetilde{u}^T B_i \widetilde{w}_i + k_i(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_i(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \geqq \\ f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i + k_i(\overline{x}, 0) + \textstyle\sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right], \ i \in I. \end{gathered} \tag{82}$$

$$\begin{gathered} f_i(\widetilde{u}) + \widetilde{u}^T B_i \widetilde{w}_i + k_i(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_i(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] > \\ f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i + k_i(\overline{x}, 0) + \textstyle\sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] \text{ for at least one } i \in I. \end{gathered} \tag{83}$$

Hence, (6) and (81) yield, respectively,

$$f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i \leqq f_i(\widetilde{u}) + \widetilde{u}^T B_i \widetilde{w}_i + k_i(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_i(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right], \ i \in I,$$

$$f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i < f_i(\widetilde{u}) + \widetilde{u}^T B_i \widetilde{w}_i + k_i(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_i(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \text{ for at least one } i \in I.$$

Using (8) and (60), we get, respectively, that the following inequalities

$$f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x}\right)^{1/2} \leqq f_i(\widetilde{u}) + \widetilde{u}^T B_i \widetilde{w}_i + k_i(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_i(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right], \ i \in I,$$

$$f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x}\right)^{1/2} < f_i(\widetilde{u}) + \widetilde{u}^T B_i \widetilde{w}_i + k_i(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_i(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \text{ for at least one } i \in I$$

hold, contradicting weak duality (Theorem 5.1). Hence, $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p} = 0\right)$ is efficient of a maximum type in (MXVD).

We now prove that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p} = 0\right)$ is a properly efficient solution in (MXVD). We proceed by contradiction. Suppose, contrary to the result, that it is not properly efficient in (MXVD). Then, by Definition 3.2, it follows that, for each scalar $M > 0$, there exist $\left(\widetilde{u}, \widetilde{\lambda}, \widetilde{\xi}, \left(\widetilde{w}_1, ..., \widetilde{w}_q\right), \widetilde{p}\right) \in \Omega_{MXVD}$ and $i^* \in I$ satisfying

$$\begin{gathered} f_{i^*}(\widetilde{u}) + \widetilde{u}^T B_{i^*} \widetilde{w}_{i^*} + k_{i^*}(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_{i^*}(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] > \\ f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*} \overline{w}_{i^*} + k_{i^*}(\overline{x}, 0) + \textstyle\sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] \end{gathered} \tag{84}$$

such that, the inequality

$$\begin{gathered} f_{i^*}(\widetilde{u}) + \widetilde{u}^T B_{i^*} \widetilde{w}_{i^*} + k_{i^*}(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_{i^*}(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] - \\ \left(f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*} \overline{w}_{i^*} + k_{i^*}(\overline{x}, 0) + \textstyle\sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] \right) > M \left(f_t(\overline{x}) + \overline{x}^T B_t \overline{w}_t + k_t(\overline{x}, 0) + \textstyle\sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] - \right. \\ \left. \left(f_t(\widetilde{u}) + \widetilde{u}^T B_t \widetilde{w}_t + k_t(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_t(\widetilde{u}, \widetilde{p}) + \textstyle\sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \right) \right) \end{gathered} \tag{85}$$

holds for all $t \in I$ satisfying

$$f_t(\overline{x}) + \overline{x}^T B_t \overline{w}_t + k_t(\overline{x}, 0) + \sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] > \\ f_t(\widetilde{u}) + \widetilde{u}^T B_t \widetilde{w}_t + k_t(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_t(\widetilde{u}, \widetilde{p}) + \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right]. \tag{86}$$

Hence, by (6), (8) and (81), it follows that the inequality

$$f_{i^*}(\widetilde{u}) + \widetilde{u}^T B_{i^*} \widetilde{w}_{i^*} + k_{i^*}(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_{i^*}(\widetilde{u}, \widetilde{p}) + \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] - f_{i^*}(\overline{x}) - \overline{x}^T B_{i^*} \overline{w}_{i^*} > \\ M\Big(f_t(\overline{x}) + \overline{x}^T B_t \overline{w}_t - f_t(\widetilde{u}) - \widetilde{u}^T B_t \widetilde{w}_t - k_t(\widetilde{u}, \widetilde{p}) + \widetilde{p}^T \nabla_p k_t(\widetilde{u}, \widetilde{p}) - \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \Big) \tag{87}$$

holds for all $t \in I$ satisfying

$$f_t(\overline{x}) + \left(\overline{x}^T B_t \overline{x} \right)^{1/2} > f_t(\widetilde{u}) + \widetilde{u}^T B_t \widetilde{w}_t + k_t(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_t(\widetilde{u}, \widetilde{p}) + \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right]. \tag{88}$$

We divide the index set I and denote by I_1 the set of indexes of objective functions satisfying the inequality (88). By I_2 we denote the set of indexes of objective functions defining by $I_2 = I \backslash (I_1 \cup i^*)$. Let $M > \frac{\lambda_{i^*}}{\lambda_t} |I_1|$, where $|I_1|$ denotes the number of elements in the set I_1. Hence, by (87) and (88), it follows that

$$\lambda_{i^*} \Big(f_{i^*}(\widetilde{u}) + \widetilde{u}^T B_{i^*} \widetilde{w}_{i^*} + k_{i^*}(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_{i^*}(\widetilde{u}, \widetilde{p}) + \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] - f_{i^*}(\overline{x}) - \overline{x}^T B_{i^*} \overline{w}_{i^*} \Big) > \\ \sum_{t \in I_1} \lambda_t \Big(f_t(\overline{x}) + \overline{x}^T B_t \overline{w}_t - f_t(\widetilde{u}) - \widetilde{u}^T B_t \widetilde{w}_t - k_t(\widetilde{u}, \widetilde{p}) + \widetilde{p}^T \nabla_p k_t(\widetilde{u}, \widetilde{p}) - \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \Big). \tag{89}$$

Using the definition of the set I_2 together with (89), we get

$$\sum_{i=1}^{q} \lambda_i \left(f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i + k_i(\overline{x}, 0) + \sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] \right) = \lambda_{i^*} \left(f_{i^*}(\overline{x}) + \overline{x}^T B_{i^*} \overline{w}_{i^*} + k_{i^*}(\overline{x}, 0) + \sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] \right) +$$

$$\sum_{i \in I_1} \lambda_i \left(f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i + k_i(\overline{x}, 0) + \sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] \right) + \sum_{i \in I_2} \lambda_i \left(f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i + k_i(\overline{x}, 0) + \sum_{j \in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x}, 0) \right] \right) <$$

$$\lambda_{i^*} \left(f_{i^*}(\widetilde{u}) + \widetilde{u}^T B_{i^*} \widetilde{w}_{i^*} + k_{i^*}(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_{i^*}(\widetilde{u}, \widetilde{p}) + \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \right) +$$

$$\sum_{t \in I_1} \lambda_t \left(f_t(\widetilde{u}) + \widetilde{u}^T B_t \widetilde{w}_t + k_t(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_t(\widetilde{u}, \widetilde{p}) + \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \right) +$$

$$\sum_{t \in I_2} \lambda_t \left(f_t(\widetilde{u}) + \widetilde{u}^T B_t \widetilde{w}_t + k_t(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_t(\widetilde{u}, \widetilde{p}) + \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \right) =$$

$$\sum_{i=1}^{q} \lambda_i \left(f_i(\widetilde{u}) + \widetilde{u}^T B_i \widetilde{w}_i + k_i(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p k_i(\widetilde{u}, \widetilde{p}) + \sum_{j \in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u}, \widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u}, \widetilde{p}) \right] \right).$$

By (6), (8), (81), it follows that the following inequality

$$\sum_{i=1}^{q} \lambda_i \left(f_i(\overline{x}) + \left(\overline{x}^T B_i \overline{x} \right)^{1/2} \right) < \sum_{i=1}^{q} \lambda_i \left(f_i(\widetilde{u}) + \widetilde{u}^T B_i \widetilde{w}_i + k_i(\widetilde{u},\widetilde{p}) - \widetilde{p}^T \nabla_p k_i(\widetilde{u},\widetilde{p}) + \sum_{j\in J_0} \widetilde{\xi}_j \left[g_j(\widetilde{u}) + h_j(\widetilde{u},\widetilde{p}) - \widetilde{p}^T \nabla_p h_j(\widetilde{u},\widetilde{p}) \right] \right)$$

holds, contradicting weak duality (Theorem 5.1). This means that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p} = 0\right)$ is a properly efficient solution in (MXVD) and completes the proof of this theorem. □

A restricted version of the converse duality for (VP) and (MXVD) is the following:

Theorem 5.3. *(Restricted converse duality). Let $\overline{x}$ and $\left(\overline{u}, \overline{\lambda}, \overline{\xi}, \left(\overline{w}_1, ..., \overline{w}_q\right), \overline{p}\right)$ be feasible solutions in problems (VP) and (MXVD), respectively, such that*

$$\begin{aligned} f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i + \textstyle\sum_{j\in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x},\overline{p}) - \overline{p}^T \nabla_p h_j(\overline{x},\overline{p}) \right] \leqq f_i(\overline{u}) + \overline{u}^T B_i \overline{w}_i + \\ k_i(\overline{u},\overline{p}) - \overline{p}^T \nabla_p k_i(\overline{u},\overline{p}) + \textstyle\sum_{j\in J_0} \overline{\xi}_j \left[g_j(\overline{u}) + h_j(\overline{u},\overline{p}) - \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \right], \ i \in I. \end{aligned} \tag{90}$$

Further, assume that $\left(\left(f_i(\cdot) + (\cdot)^T B_i \overline{w}_i + \sum_{j\in J_0} \overline{\xi}_j g_j(\cdot),\ i \in I\right),\ \left(g_j(\cdot),\ j \in J_\beta,\ \beta = 1, ..., s\right)\right)$, is higher-order strictly (Φ, ρ)-type I objective and constraint functions at $\overline{u}$ on $D \cup U_{MXVD}$. If $\sum_{i=1}^{q} \overline{\lambda}_i \rho_{f_i} + \sum_{j\in J_\beta} \overline{\xi}_j \rho_{g_j}$, then $\overline{x} = \overline{u}$.

Proof. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \neq \overline{u}$. By assumption, $\left(\left(f_i(\cdot) + (\cdot)^T B_i \overline{w}_i + \sum_{j\in J_0} \overline{\xi}_j g_j(\cdot),\ i \in I\right),\ \left(g_j(\cdot),\ j \in J_\beta,\ \beta = 1, ..., s\right)\right)$ is higher-order strictly (Φ, ρ)-type I objective and constraint functions at $\overline{u}$ on $D \cup U$. By Definition 2.1, we have

$$\begin{aligned} f_i(\overline{x}) + \overline{x}^T B_i \overline{w}_i + \textstyle\sum_{j\in J_0} \overline{\xi}_j \left[g_j(\overline{x}) + h_j(\overline{x},\overline{p}) - \overline{p}^T \nabla_p h_j(\overline{x},\overline{p}) \right] - \Big(f_i(\overline{u}) + \overline{u}^T B_i \overline{w}_i + \\ k_i(\overline{u},\overline{p}) - \overline{p}^T \nabla_p k_i(\overline{u},\overline{p}) + \textstyle\sum_{j\in J_0} \overline{\xi}_j \left[g_j(\overline{u}) + h_j(\overline{u},\overline{p}) - \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \right] \Big) > \\ \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p k_i(\overline{u},\overline{p}) + B_i \overline{w}_i + \textstyle\sum_{j\in J_0} \overline{\xi}_j \nabla_p h_j(\overline{u},\overline{p}), \rho_{f_i}\right)\right), \ i \in I, \end{aligned} \tag{91}$$

$$-g_j(\overline{u}) - h_j(\overline{u},\overline{p}) + \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \geqq \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p h_j(\overline{u},\overline{p}), \rho_{g_j}\right)\right), \ j \in J_\beta. \tag{92}$$

Combining (90) and (91), we get

$$\Phi\left(\overline{x}, \overline{u}, \left(\nabla_p k_i(\overline{u},\overline{p}) + B_i \overline{w}_i + \sum_{j\in J_0} \overline{\xi}_j \nabla_p h_j(\overline{u},\overline{p}), \rho_{f_i}\right)\right) < 0, \ i \in I, \tag{93}$$

Since $\overline{\lambda}_i > 0$, $i \in I$, and $\overline{\xi}_j \geqq 0$, $j \in J$, (93) and (92) yield, respectively,

$$\sum_{i=1}^{q} \overline{\lambda}_i \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p k_i(\overline{u},\overline{p}) + B_i \overline{w}_i + \sum_{j\in J_0} \overline{\xi}_j \nabla_p h_j(\overline{u},\overline{p}), \rho_{f_i}\right)\right) < 0, \tag{94}$$

$$\sum_{j\in J_\beta} \overline{\xi}_j \left(-g_j(\overline{u}) - h_j(\overline{u},\overline{p}) + \overline{p}^T \nabla_p h_j(\overline{u},\overline{p}) \right) \geqq \sum_{j\in J_\beta} \overline{\xi}_j \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p h_j(\overline{u},\overline{p}), \rho_{g_j}\right)\right). \tag{95}$$

Hence, using the constraint (59) together with (95), we get

$$\sum_{j\in J_\beta} \overline{\xi}_j \Phi\left(\overline{x}, \overline{u}, \left(\nabla_p h_j(\overline{u},\overline{p}), \rho_{g_j}\right)\right) \leqq 0. \tag{96}$$

Let us introduce the following notations

$$\lambda_i^* = \frac{\overline{\lambda}_i}{\sum_{i=1}^{q}\overline{\lambda}_i + \sum_{j\in J_\beta}\overline{\xi}_j},\ i \in I,\ \xi_j^* = \frac{\overline{\xi}_j}{\sum_{i=1}^{q}\overline{\lambda}_i + \sum_{j\in J_\beta}\overline{\xi}_j},\ j \in J_\beta,\ \beta = 1,...,s. \tag{97}$$

Note that $\lambda_i^* \in (0,1]$, $i \in I$, $\xi_j^* \in [0,1]$, $j \in J$, and

$$\sum_{i=1}^{q}\lambda_i^* + \sum_{j\in J_\beta}\xi_j^* = 1. \tag{98}$$

Taking into account (97) in (94) and (96), we get, respectively,

$$\sum_{i=1}^{q}\lambda_i^*\Phi\left(\overline{x},\overline{u},\left(\nabla_p k_i(\overline{u},\overline{p}) + B_i\overline{w}_i + \sum_{j\in J_0}\overline{\xi}_j\nabla_p h_j(\overline{u},\overline{p}),\rho_{f_i}\right)\right) < 0, \tag{99}$$

$$\sum_{j\in J_\beta}\xi_j^*\Phi\left(\overline{x},\overline{u},\left(\nabla_p h_j(\overline{u},\overline{p}),\rho_{g_j}\right)\right) \leqq 0. \tag{100}$$

Adding both sides of (99) and (100), we obtain

$$\sum_{i=1}^{q}\lambda_i^*\Phi\left(\overline{x},\overline{u},\left(\nabla_p k_i(\overline{u},\overline{p}) + B_i\overline{w}_i + \sum_{j\in J_0}\overline{\xi}_j\nabla_p h_j(\overline{u},\overline{p}),\rho_{f_i}\right)\right) + \sum_{j\in J_\beta}\xi_j^*\Phi\left(\overline{x},\overline{u},\left(\nabla_p h_j(\overline{u},\overline{p}),\rho_{g_j}\right)\right) < 0. \tag{101}$$

By Definition 2.1, it follows that $\Phi(\overline{x},\overline{u},\cdot)$ is convex on R^{n+1}. Since $\lambda_i^* \in (0,1]$, $i \in I$, $\xi_j^* \in [0,1]$, $j \in J$, and (98) is satisfied, by the definition of a convex function, we have

$$\begin{gathered}\sum_{i=1}^{q}\lambda_i^*\Phi\left(\overline{x},\overline{u},\left(\nabla_p k_i(\overline{u},\overline{p}) + B_i\overline{w}_i + \sum_{j\in J_0}\overline{\xi}_j\nabla_p h_j(\overline{u},\overline{p}),\rho_{f_i}\right)\right) + \sum_{j\in J_\beta}\xi_j^*\Phi\left(\overline{x},\overline{u},\left(\nabla_p h_j(\overline{u},\overline{p}),\rho_{g_j}\right)\right) \geqq \\ \Phi\left(\overline{x},\overline{u},\left(\sum_{i=1}^{q}\lambda_i^*\left(\nabla_p k_i(\overline{u},\overline{p}) + B_i\overline{w}_i + \sum_{j\in J_0}\overline{\xi}_j\nabla_p h_j(\overline{u},\overline{p}),\rho_{f_i}\right)\right) + \sum_{j\in J_\beta}\xi_j^*\left(\nabla_p h_j(\overline{u},\overline{p}),\rho_{g_j}\right)\right).\end{gathered} \tag{102}$$

Combining (101) and (102) and using (97), we get

$$\begin{gathered}\Phi\Bigg(\overline{x},\overline{u},\frac{1}{\sum_{i=1}^{q}\overline{\lambda}_i+\sum_{j\in J_\beta}\overline{\xi}_j}\Big(\sum_{i=1}^{q}\overline{\lambda}_i\left[\nabla_p k_i(\overline{u},\overline{p}) + B_i\overline{w}_i + \sum_{j\in J_0}\overline{\xi}_j\nabla_p h_j(\overline{u},\overline{p})\right] \\ + \sum_{j\in J_\beta}\overline{\xi}_j\nabla_p h_j(\overline{u},\overline{p})\ ,\ \sum_{i=1}^{q}\overline{\lambda}_i\rho_{f_i} + \sum_{j\in J_\beta}\overline{\xi}_j\rho_{g_j}\Big) < 0.\end{gathered} \tag{103}$$

Since $\sum_{i=1}^{q}\overline{\lambda}_i = 1$ and $\bigcup_{\beta=0}^{s}J_\beta = J$, (103) yields

$$\Phi\Bigg(\overline{x},\overline{u},\frac{1}{\sum_{i=1}^{q}\overline{\lambda}_i+\sum_{j\in J_\beta}\overline{\xi}_j}\Big(\sum_{i=1}^{q}\overline{\lambda}_i\left[\nabla_p k_i(\overline{u},\overline{p}) + B_i\overline{w}_i\right] + \sum_{j=1}^{m}\overline{\xi}_j\nabla_p h_j(\overline{u},\overline{p})\ ,\ \sum_{i=1}^{q}\overline{\lambda}_i\rho_{f_i} + \sum_{j\in J_\beta}\overline{\xi}_j\rho_{g_j}\Big) < 0. \tag{104}$$

Hence, the constraint (58) implies

$$\Phi\left(\overline{x},\overline{u},\frac{1}{\sum_{i=1}^{q}\overline{\lambda}_i + \sum_{j\in J_\beta}\overline{\xi}_j}\left(0\ ,\ \sum_{i=1}^{q}\overline{\lambda}_i\rho_{f_i} + \sum_{j\in J_\beta}\overline{\xi}_j\rho_{g_j}\right)\right) < 0. \tag{105}$$

By Definition 2.1, it follows that $\Phi(\overline{x},\overline{u},(0,a)) \geqq 0$ for every $a \in R_+$. By assumption, $\sum_{i=1}^{q}\overline{\lambda}_i\rho_{f_i} + \sum_{j\in J_\beta}\xi_j\rho_{g_j} \geqq 0$. Thus, the following inequality

$$\Phi\left(\overline{x},\overline{u},\frac{1}{\sum_{i=1}^{q}\overline{\lambda}_i + \sum_{j\in J_\beta}\overline{\xi}_j}\left(0\ ,\ \sum_{i=1}^{q}\overline{\lambda}_i\rho_{f_i} + \sum_{j\in J_\beta}\overline{\xi}_j\rho_{g_j}\right)\right) \geqq 0$$

holds, contradicting (105). This completes the proof of the theorem. □

6. Conclusion

In the paper, a new concept of type I functions has been defined in the case of a nondifferentiable multiobjective programming problem. The class of so-called higher-order (Φ,ρ)-type I objective and constraint functions is a generalization and extension of many concepts of higher-order generalized convexity, including the class of higher-order type I functions introduced by Mishra and Rueda [26] and the class of (Φ,ρ)-invex functions introduced by Caristi et al. [8]. For the considered nondifferentiable multiobjective programming problem, its higher-order Mangasarian dual problem and its higher-order mixed dual problem have been defined and weak, strong and strict converse duality theorems have been established under assumptions that the involved functions are higher-order (strictly) (Φ, ρ)-type I objective and constraint functions. Since the concept of (Φ, ρ)-type I objective and constraint functions unify many other concepts of type I objective and constraint functions previously defined in the literature, therefore, the higher-order duality results established in the paper extend adequate results already existing in optimization theory.

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