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Resistance Distance and Kirchhoff Index in Generalized R-Vertex and R-Edge Corona for Graphs

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Abstract. For a graph G, the graph $R(G)$ of a graph G is the graph obtained by adding a new vertex for each edge of G and joining each new vertex to both end vertices of the corresponding edge. Let $I(G)$ be the set of newly added vertices, *i.e* $I(G) = V(R(G)) \setminus V(G)$. The generalized R-vertex corona of G and H_i for $i = 1, 2, ..., n$, denoted by $R(G) \boxdot \wedge_{i=1}^{n} H_i$, is the graph obtained from $R(G)$ and H_i by joining the i-th vertex of $V(G)$ to every vertex in H_i. The generalized R-edge corona of G and H_i for $i = 1, 2, ..., m$, denoted by $R(G) \ominus \wedge_{i=1}^{m} H_i$, is the graph obtained from $R(G)$ and H_i by joining the i-th vertex of $I(G)$ to every vertex in H_i. In this paper, we derive closed-form formulas for resistance distance and Kirchhoff index of $R(G) \boxdot \wedge_{i=1}^{n} H_i$ and $R(G) \ominus \wedge_{i=1}^{m} H_i$ whenever G and H_i are arbitrary graphs. These results generalize the existing results.

1. Introduction

All graphs considered in this paper are simple and undirected. The resistance distance between vertices u and v of G was defined by Klein and Randić [9] to be the effective resistance between nodes u and v as computed with Ohm's law when all the edges of G are considered to be unit resistors. The Kirchhoff index $Kf(G)$ was defined in [9] as $Kf(G) = \sum_{u<v} r_{uv}(G)$, where $r_{uv}(G)$ denotes the resistance distance between u and v in G. These novel parameters are in fact intrinsic to the graph theory and has some nice properties and applications in chemistry. For the study of resistance distance and Kirchhoff index, one may be referred to the recent works ([2], [4], [5],), [7]-[25]) and the references therein.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let d_i be the degree of vertex i in G and $D_G = diag(d_1, d_2, \cdots d_{|V(G)|})$ the diagonal matrix with all vertex degrees of G as its diagonal entries. For a graph G, let A_G and B_G denote the adjacency matrix and vertex-edge incidence matrix of G, respectively. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of G, where D_G is the diagonal matrix of vertex degrees of G. We use $\mu_1(G) \geq u_2(G) \geq \cdots \geq \mu_n(G) = 0$ to denote the eigenvalues of L_G. For other undefined notations and terminology from graph theory, the readers may refer to [6] and the references therein.

In [14], Lu et.al generalized the corona operation and defined the generalized R-vertex corona. For a graph G, the graph $R(G)$ of a graph G is the graph obtained by adding a new vertex for each edge of G and

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joining each new vertex to both end vertices of the corresponding edge. Let $I(G)$ be the set of newly added vertices, *i.e* $I(G) = V(R(G)) \setminus V(G)$.

Definition 1.1([14]) The generalized R-vertex corona of G and H_i for $i = 1, 2, ..., n$, denoted by $R(G) \boxdot \wedge_{i=1}^{n} H_i$, is the graph obtained from $R(G)$ and H_i by joining the ith vertex of $V(G)$ to every vertex in H_i.

Definition 1.2 The generalized R-edge corona of G and H_i for $i = 1, 2, ..., m$, denoted by $R(G) \ominus \wedge_{i=1}^{m} H_i$, is the graph obtained from $R(G)$ and H_i by joining the ith vertex of $I(G)$ to every vertex in H_i.

Bu et al. investigated resistance distance in subdivision-vertex join and subdivision-edge join of graphs [2]. Liu et al. [12] gave the resistance distance and Kirchhoff index of R-vertex join and R-edge join of two graphs. In [11], the resistance distance of subdivision-vertex and subdivision-edge corona are obtained. Motivated by the results, in this paper we consider the generalization of the R-vertex corona and the R-edge corona to the case of $n(m)$ different graphs and we obtain the resistances distance and the Kirchhoff index in terms of the corresponding parameters of the factors. These results generalize the existing results in [13].

2. Preliminaries

The {1}-inverse of M is a matrix X such that $MXM = M$. If M is singular, then it has infinite {1}- inverse [1]. For a square matrix M, the group inverse of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is known that $M^{\#}$ exists if and only if $rank(M) = rank(M^2)$ ([1],[3]). If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric {1}- inverse of M. Actually, $M^{\#}$ is equal to the Moore-Penrose inverse of M since M is symmetric [3].

It is known that resistance distances in a connected graph G can be obtained from any {1}- inverse of G ([4]). We use $M^{(1)}$ to denote any {1}- inverse of a matrix M, and let $(M)_{uv}$ denote the (u, v)- entry of M.

Lemma 2.1 ([3]) Let G be a connected graph. Then

$$r_{uv}(G) = (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} = (L_G^{\#})_{uu} + (L_G^{\#})_{vv} - 2(L_G^{\#})_{uv}.$$

Let $\mathbf{1_n}$ denotes the column vector of dimension n with all the entries equal one. We will often use $\mathbf{1}$ to denote an all-ones column vector if the dimension can be read from the context.

Lemma 2.2 ([2]) For any graph G, we have $L_G^{\#}\mathbf{1} = 0$.

Lemma 2.3 ([24]) Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a nonsingular matrix. If A and D are nonsingular, then

$$\begin{aligned} M^{-1} &= \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \end{aligned}$$

where $S = D - CA^{-1}B$.

For a square matrix M, let $tr(M)$ denote the trace of M.

Lemma 2.4 ([15]) Let G be a connected graph on n vertices. Then

$$Kf(G) = ntr(L_G^{(1)}) - 1^T L_G^{(1)} 1 = ntr(L_G^{\#}).$$

Lemma 2.5([10]) Let G be a connected graph of order n with edge set E. Then

$$\sum_{u<v, uv \in E} r_{uv}(G) = n - 1.$$

For a vertex i of a graph G, let $T(i)$ denote the set of all neighbors of i in G.

Lemma 2.6([2]) Let G be a connected graph. For any $i, j \in V(G)$,

$$r_{ij}(G) = d_i^{-1}(1 + \sum_{k\in T(i)} r_{kj}(G) - d_i^{-1} \sum_{k,l\in T(i)} r_{kl}(G)).$$

Lemma 2.7 ([12]) Let G be a graph of order n. For any $a, b > 0$ satisfying $b \neq a$, we have

$$(L_G + aI_n - \frac{a}{b} j_{n\times n})^{-1} = (L_G + aI_n)^{-1} + \frac{1}{a(b-n)} j_{n\times n},$$

where $j_{n\times n}$ denotes the $n \times n$ matrix with all entries equal to one.

Lemma 2.8 ([13]) Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be a symmetric block matrix. If D is nonsingular, then

$$X = \begin{pmatrix} H^{\#} & -H^{\#}BD^{-1} \\ -D^{-1}B^TH^{\#} & D^{-1} + D^{-1}B^TH^{\#}BD^{-1} \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of L, where $H = A - BD^{-1}B^T$.

Lemma 2.9 ([9]) Let k be a cut-vertex of a graph, and let i and j be vertices occurring in different components which arise upon deletion of k. Then $r_{ij} = r_{ik} + r_{kj}$.

3. The resistance distance and Kirchhoff index of $R(G) \boxdot \wedge_{i=1}^{n} H_i$

In this section, we focus on determing the resistance distance and Kirchhoff index of generalized R-vertex corona $R(G) \boxdot \wedge_{i=1}^{n} H_i$ whenever G and $H_i (i = 1, 2, ..., n)$ be an arbitrary graph.

Theorem 3.1 Let G be a connected graph with n vertices and m edges and let H_i be a graph with t_i vertices for $i = 1, 2, ..., n$. Then $R(G) \boxdot \wedge_{i=1}^{n} H_i$ have the resistance distance and Kirchhoff index as follows:

(i) For any $i, j \in V(G)$, we have

$$r_{ij}(R(G) \boxdot \wedge_{i=1}^{n} H_i) = \frac{2}{3}(L_G^{\#})_{ii} + \frac{2}{3}(L_G^{\#})_{jj} - \frac{4}{3}(L_G^{\#})_{ij} = \frac{2}{3} r_{ij}(G).$$

(ii) For any $i, j \in V(H_k)(k = 1, 2, ..., n)$, we have

$$r_{ij}(R(G) \boxdot \wedge_{i=1}^{n} H_i) = ((L_{H_k} + I_{t_k})^{-1})_{ii} + ((L_{H_k} + I_{t_k})^{-1})_{jj} - 2((L_{H_k} + I_{t_k})^{-1})_{ij}.$$

(iii) For any $i, j \in R(G)$, we have

$$r_{ij}(R(G) \boxdot \wedge_{i=1}^{n} H_i) = \frac{2}{3} r_{ij}(G).$$

(iv) For any $i \in V(G)$, $j \in V(H_k)(k = 1, 2, ..., n)$, we have

$$r_{ij}(R(G) \boxdot \wedge_{i=1}^{n} H_i) = r_{ik}(R(G)) + r_{kj}(F_k),$$

where $F_k = H_k \vee \{v\}$, i.e, F_k is the graph obtained by adding new edges from an isolated vetrtex v to every vertex of H_k.

(v) For any $i \in V(H_k)$, $j \in V(H_l)$, we have

$$r_{ij}(R(G) \boxdot \wedge_{i=1}^{n} H_i) = r_{kl}(R(G)) + r_{ik}(F_k) + r_{jl}(F_l),$$

where $F_k = H_k \vee \{v\}$, i.e, F_k is the graph obtained by adding new edges from an isolated vetrtex v to every vertex of H_k.

(vi) $Kf(R(G) \boxdot \wedge_{i=1}^{n} H_i)$

$$= (n+2m+\sum_{i=1}^{n} t_i)\Bigg(\frac{2}{3n}Kf(G)+\frac{m}{2}+\frac{1}{2}tr(D_G L_G^{\#})-\frac{n-1}{4}+\sum_{i=1}^{n}\sum_{j=1}^{t_i}\frac{1}{\mu_i(H_j)+1}$$
$$+2tr(Q^T L_G^{\#} Q)\Bigg)-\left(\frac{m}{2}+\frac{1}{4}\pi^T L_G^{\#}\pi+\pi^T L_G^{\#}\delta+\sum_{i=1}^{n} t_i+\delta^T L_G^{\#}\delta\right),$$

where Q equals (1), $\pi^T=(d_1,d_2,...,d_n)$, $\delta^T=(t_1,t_2,...,t_n)$.

Proof Let $R(G)$ and D_G be the incidence matrix and degree matrix of G. With a suitable labeling for vertices of $R(G) \boxdot \wedge_{i=1}^{n} H_i$, the Laplacian matrix of $R(G) \boxdot \wedge_{i=1}^{n} H_i$ can be written as follows:

$$L_{R(G)\boxdot\wedge_{i=1}^{n}H_i}=\begin{pmatrix} P+L_G & -R(G) & -Q \\ -R^T(G) & 2I_m & 0 \\ -Q^T & 0 & T \end{pmatrix},$$

where

$$P=\begin{pmatrix} d_1+t_1 & 0 & 0 & ... & 0 \\ 0 & d_2+t_2 & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & d_n+t_n \end{pmatrix},\quad Q=\begin{pmatrix} 1_{t_1}^T & 0 & 0 & ... & 0 \\ 0 & 1_{t_2}^T & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1_{t_n}^T \end{pmatrix}, \tag{1}$$

$$T=\begin{pmatrix} L_{H_1}+I_{t_1} & 0 & 0 & ... & 0 \\ 0 & L_{H_2}+I_{t_2} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & L_{H_n}+I_{t_n} \end{pmatrix}.$$

First we begin with the computation of $\{1\}$-inverse of $R(G) \boxdot \wedge_{i=1}^{n} H_i$.

By Lemma 2.8, we have

$$\begin{aligned} H &= L_G+P-\begin{pmatrix} -R(G) & -Q \end{pmatrix}\begin{pmatrix} \frac{1}{2}I_m & 0 \\ 0 & T^{-1} \end{pmatrix}\begin{pmatrix} -R^T(G) \\ -Q^T \end{pmatrix} \\ &= L_G+P-\begin{pmatrix} -\frac{1}{2}R(G) & -QT^{-1} \end{pmatrix}\begin{pmatrix} -R^T(G) \\ -Q^T \end{pmatrix} \\ &= L_G+D_G+\begin{pmatrix} t_1 & 0 & 0 & ... & 0 \\ 0 & t_2 & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & t_n \end{pmatrix}-\frac{1}{2}(D_G+A_G)-\begin{pmatrix} t_1 & 0 & 0 & ... & 0 \\ 0 & t_2 & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & t_n \end{pmatrix} \\ &= \frac{3}{2}L_G, \end{aligned}$$

so $H^{\#}=\frac{2}{3}L_G^{\#}$.

According to Lemma 2.8, we calculate $-H^{\#}BD^{-1}$ and $-D^{-1}B^TH^{\#}$.

$$\begin{aligned} -H^{\#}BD^{-1} &= -\tfrac{2}{3}L_G^{\#}\begin{pmatrix} -R(G) & -Q \end{pmatrix}\begin{pmatrix} \frac{1}{2}I_m & 0 \\ 0 & T^{-1} \end{pmatrix} \\ &= -\tfrac{2}{3}L_G^{\#}\begin{pmatrix} -\frac{1}{2}R(G) & -QT^{-1} \end{pmatrix}=\begin{pmatrix} \frac{1}{3}L_G^{\#}R(G) & \frac{2}{3}L_G^{\#}Q \end{pmatrix} \end{aligned}$$

and

$$-D^{-1}B^TH^{\#}=-(H^{\#}BD^{-1})^T=\begin{pmatrix} \frac{1}{3}R^T(G)L_G^{\#} \\ \frac{2}{3}Q^TL_G^{\#} \end{pmatrix}.$$

We are ready to compute the $D^{-1}B^TH^{\#}BD^{-1}$.

$$\begin{aligned} D^{-1}B^TH^{\#}BD^{-1} &= \begin{pmatrix} \frac{1}{2}I_m & 0 \\ 0 & T^{-1} \end{pmatrix}\begin{pmatrix} -R^T(G) \\ -Q^T \end{pmatrix}L_G^{\#}\begin{pmatrix} -R(G) & -Q \end{pmatrix}\begin{pmatrix} \frac{1}{2}I_m & 0 \\ 0 & T^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}R^T(G)L_G^{\#}R(G) & \frac{1}{2}R^T(G)L_G^{\#}Q \\ \frac{1}{2}Q^TL_G^{\#}R(G) & Q^TL_G^{\#}Q \end{pmatrix}. \end{aligned}$$

Based on Lemma 2.8, the following matrix

$$N = \begin{pmatrix} \frac{2}{3}L_G^{\#} & \frac{1}{3}L_G^{\#}R(G) & \frac{2}{3}L_G^{\#}Q \\ \frac{1}{3}R^T(G)L_G^{\#} & \frac{1}{2}I_m + \frac{1}{4}R^T(G)L_G^{\#}R(G) & \frac{1}{2}R^T(G)L_G^{\#}Q \\ \frac{2}{3}Q^TL_G^{\#} & \frac{1}{2}Q^TL_G^{\#}R(G) & T^{-1} + Q^TL_G^{\#}Q \end{pmatrix} \tag{2}$$

is a symmetric {1}- inverse of $L_{R(G)\boxdot\wedge_{i=1}^n H_i}$.

For any $i, j \in V(G)$, by Lemma 2.1 and the Equation (2), we have

$$r_{ij}(R(G)\boxdot\wedge_{i=1}^n H_i) = \frac{2}{3}(L_G^{\#})_{ii} + \frac{2}{3}(L_G^{\#})_{jj} - \frac{4}{3}(L_G^{\#})_{ij} = \frac{2}{3}r_{ij}(G),$$

as stated in (i).

For any $i, j \in V(H_k)(k = 1, 2, ..., n)$, by Lemma 2.1 and the Equation (2), we have

$$r_{ij}(R(G)\boxdot\wedge_{i=1}^n H_i) = ((L_{H_k} + I_{t_k})^{-1})_{ii} + ((L_{H_k} + I_{t_k})^{-1})_{jj} - 2((L_{H_k} + I_{t_k})^{-1})_{ij},$$

as stated in (ii).

From the left side of above equation, we can obviously have

$$r_{ij}(F_k) = ((L_{H_k} + I_{t_l})^{-1})_{ii} + ((L_{H_k} + I_{t_l})^{-1})_{jj} - 2((L_{H_k} + I_{t_l})^{-1})_{ij},$$

where $F_k = H_k \vee \{v\}$, i.e, F_k is the graph obtained by adding new edges from an isolated vetrtex v to every vertex of H_k.

For any $i, j \in R(G)$, by Lemma 2.1 and the Equation (2), we have

$$r_{ij}(R(G)\boxdot\wedge_{i=1}^n H_i) = r_{ij}(R(G)).$$

By Lemma 3.1 in [7], $r_{ij}(R(G)) = \frac{2}{3}r_{ij}(G)$, so $r_{ij}(R(G)\boxdot\wedge_{i=1}^n H_i) = \frac{2}{3}r_{ij}(G)$, as stated in (iii).

For any $i \in V(G)$, $j \in V(H_k)(k = 1, 2, ..., n)$, since i and j belong to different components, then by Lemma 2.9, we have

$$r_{ij}(R(G)\boxdot\wedge_{i=1}^n H_i) = r_{ik}(R(G)) + r_{kj}(F_k),$$

as stated in (iv).

For any $i \in V(H_k)$, $j \in V(H_l)$, by Lemma 2.9, we have

$$r_{ij}(R(G)\boxdot\wedge_{i=1}^n H_i) = r_{kl}(R(G)) + r_{ik}(F_k) + r_{jl}(F_l),$$

as stated in (v).

By Lemma 2.4, we have

$$\begin{aligned} Kf(R(G)\boxdot\wedge_{i=1}^{nH_i}) &= (n + m + \sum_{i=1}^n t_i)tr(N) - \mathbf{1}^{\mathbf{T}}N\mathbf{1}^{\mathbf{T}} \\ &= (n + m + \sum_{i=1}^n t_i)\left(\frac{2}{3}tr(L_G^{\#}) + tr\left(\frac{1}{2}I_m + \frac{1}{4}R^T(G)L_G^{\#}R(G)\right) + \right. \\ &\quad \left. + tr(T^{-1} + Q^TL_G^{\#}Q)\right) - \mathbf{1}^{\mathbf{T}}N\mathbf{1}^{\mathbf{T}} \\ &= (n + m + \sum_{i=1}^n t_i)\left(\frac{2}{3n}Kf(G) + \frac{m}{2} + \frac{1}{4}\sum_{i<j,i,j\in E(G)}[(L_G^{\#})_{ii} + (L_G^{\#})_{jj}\right. \\ &\quad \left. + 2(L_G^{\#})_{ij}] + tr\left(T^{-1} + Q^TL_G^{\#}Q\right)\right) - \mathbf{1}^{\mathbf{T}}N\mathbf{1}^{\mathbf{T}}. \end{aligned}$$

By Lemma 2.4, we get

$$\begin{aligned} Kf(R(G)\boxdot \wedge_{i=1}^{n}H_i) &= (n+m+\sum_{i=1}^{n}t_i)\Bigg(\frac{2}{3n}Kf(G)+\frac{m}{2}+\frac{1}{4}\sum_{i<j,i,j\in E(G)}[2(L_G^{\#})_{ii}+2(L_G^{\#})_{jj} \\ &\quad -r_{ij}(G)]+tr\left(T^{-1}+Q^TL_G^{\#}Q\right)\Bigg)-\mathbf{1^T}N\mathbf{1^T} \\ &= (n+m+\sum_{i=1}^{n}t_i)\Big(\frac{2}{3n}Kf(G)+\frac{m}{2}+\frac{1}{2}tr(D_GL_G^{\#})-\frac{n-1}{4} \\ &\quad +tr\left(T^{-1}+Q^TL_G^{\#}Q\right)\Big)-\mathbf{1^T}N\mathbf{1^T}. \end{aligned}$$

Note that the eigenvalues of $(L(H_i)+I_{t_i})$ $(i=1,2,...,n)$ are $\mu_1(H_i)+1,\mu_2(H_i)+1,...,\mu_{t_i}(H_i)+1$. Then

$$tr(T^{-1})=\sum_{i=1}^{n}\sum_{j=1}^{t_i}\frac{1}{\mu_i(H_j)+1}. \tag{3}$$

By Lemma 2.2, $L_G^{\#}\mathbf{1}=0$ and $(\mathbf{1^T}\left(R^T(G)L_G^{\#}Q\right)\mathbf{1})^T=\mathbf{1^T}\left(Q^TL_G^{\#}R(G)\right)\mathbf{1}$, then

$$\begin{aligned} \mathbf{1^T}N\mathbf{1} &= \frac{m}{2}+\frac{1}{4}\mathbf{1^T}\left(R^T(G)L_G^{\#}R(G)\right)\mathbf{1}+\mathbf{1^T}\left(R^T(G)L_G^{\#}Q\right)\mathbf{1} \\ &\quad +\mathbf{1^T}T^{-1}\mathbf{1}+\mathbf{1^T}\left(Q^TL_G^{\#}Q\right)\mathbf{1}. \end{aligned}$$

Note that $R(G)\mathbf{1}=\pi$, where $\pi^T=(d_1,d_2,...,d_n)$, then $\mathbf{1^T}\left(R^T(G)L_G^{\#}R(G)\right)\mathbf{1}=\pi^TL_G^{\#}\pi$, so

$$\mathbf{1^T}N\mathbf{1} = \frac{m}{2}+\frac{1}{4}\pi^TL_G^{\#}\pi+\pi^TL_G^{\#}Q\mathbf{1}+\mathbf{1^T}T^{-1}\mathbf{1}+\mathbf{1^T}\left(Q^TL_G^{\#}Q\right)\mathbf{1}. \tag{4}$$

Let $R_i=L(H_i)+I_{t_i}(i=1,2,...,n)$, then

$$\mathbf{1^T}T^{-1}\mathbf{1^T} = \begin{pmatrix}\mathbf{1_{t_1}^T} & \mathbf{1_{t_2}^T} & \cdots & \mathbf{1_{t_n}^T}\end{pmatrix}\begin{pmatrix}R_1^{-1} & 0 & 0 & ... & 0\\ 0 & R_2^{-1} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & R_n^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{1_{t_1}}\\ \mathbf{1_{t_2}}\\ \cdots\\ \mathbf{1_{t_n}}\end{pmatrix}$$

$$= \sum_{i=1}^{n}\mathbf{1_{t_i}^T}(L(H_i)+I_{t_i})^{-1}\mathbf{1_{t_i}}=\sum_{i=1}^{n}t_i, \tag{5}$$

and

$$\mathbf{1^T}Q^T = \begin{pmatrix}\mathbf{1_{t_1}^T} & \mathbf{1_{t_2}^T} & \cdots & \mathbf{1_{t_n}^T}\end{pmatrix}\begin{pmatrix}\mathbf{1_{t_1}} & 0 & 0 & ... & 0\\ 0 & \mathbf{1_{t_2}} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & \mathbf{1_{t_n}}\end{pmatrix}$$

$$= (t_1,t_2,...,t_n)=\delta^T. \tag{6}$$

Plugging (3), (4), (5) and (6) into $Kf(R(G)\boxdot\wedge_{i=1}^{n}H_i)$, we obtain the required result in (*vi*).

4. The resistance distance and Kirchhoff index of $R(G) \ominus \wedge_{i=1}^{m} H_i$

In this section, we focus on determing the resistance distance and Kirchhoff index of generalized R-edge corona $R(G) \ominus \wedge_{i=1}^{m} H_i$ whenever G and $H_i (i = 1, 2, ..., n)$ be an arbitrary graph.

Theorem 4.1 Let G be a connected graph with n vertices and m edges, Let H_i be a graph with t_i vertices for $i = 1, 2, ..., m$. Then $R(G) \ominus \wedge_{i=1}^{m} H_i$ have the resistance distance and Kirchhoff index as follows:

(i) For any $i, j \in V(G)$, we have

$$r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) = \frac{2}{3}(L_G^{\#})_{ii} + \frac{2}{3}(L_G^{\#})_{jj} - \frac{4}{3}(L_G^{\#})_{ij} = \frac{2}{3} r_{ij}(G).$$

(ii) For any $i, j \in V(H_k)(k = 1, 2, ..., m)$, we have

$$\begin{aligned} r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) &= (L_{H_k} + I_{t_k} - \frac{1}{2 + t_k} j_{t_k})_{ii}^{-1} + (L_{H_k} + I_{t_k} - \frac{1}{2 + t_k} j_{t_k})_{jj}^{-1} \\ &\quad -2(L_{H_1} + I_{t_k} - \frac{1}{2 + t_k} j_{t_k})_{ij}^{-1}. \end{aligned}$$

(iii) For any $i, j \in R(G)$, we have

$$r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) = \frac{2}{3} r_{ij}(G).$$

(iv) For any $i \in V(G)$, $j \in V(H_k)(k = 1, 2, ..., n)$, we have

$$r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) = r_{ik}(R(G)) + r_{kj}(F_k),$$

where $F_k = H_k \vee \{v\}$, i.e, F_k is the graph obtained by adding new edges from an isolated vetrtex v to every vertex of H_k.

(v) For any $i \in V(H_k)$, $j \in V(H_l)$, we have

$$r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) = r_{kl}(R(G)) + r_{ik}(F_k) + r_{jl}(F_l),$$

where $F_k = H_k \vee \{v\}$, i.e, F_k is the graph obtained by adding new edges from an isolated vetrtex v to every vertex of H_k.

(vi) $Kf(R(G) \ominus \wedge_{i=1}^{m} H_i)$

$$\begin{aligned} &= (n + 2m + \sum_{i=1}^{n} t_i)\Bigg(\frac{2}{3n} Kf(G) + \frac{m}{2} + \frac{1}{3} tr(D_G L_G^{\#}) - \frac{n-1}{2} + \sum_{i=1}^{n} \sum_{j=1}^{t_i} \frac{1}{\mu_i(H_j) + 1} \\ &\quad + \frac{2}{3} tr(F^T R^T(G) L_G^{\#} R(G) F)\Bigg) - \Bigg(\frac{m}{2} + \frac{1}{6} \pi^T L_G^{\#} \pi + \sum_{i=1}^{m} t_i + \frac{2}{3} \pi^T(G) L_G^{\#} R(G) \delta + \pi^T L_G^{\#} \delta \\ &\quad + \frac{1}{2} \sum_{i=1}^{m} t_i(2 + t_i) + \frac{2}{3} \delta^T R^T(G) L_G^{\#} R(G) \delta\Bigg), \end{aligned}$$

where F equals (7), $\pi^T = (d_1, d_2, ..., d_n)$, $\delta^T = (t_1, 0, ..., 0, t_2, 0, ..., 0..., t_m)$.

Proof Let $R(G)$ and D_G be the incidence matrix and degree matrix of G. With a suitable labeling for vertices of $R(G) \ominus \wedge_{i=1}^{m} H_i$, the Laplacian matrix of $R(G) \ominus \wedge_{i=1}^{m} H_i$ can be written as follows:

$$L_{R(G) \ominus \wedge_{i=1}^{m} H_i} = \begin{pmatrix} L_G + D_G & -R(G) & 0 \\ -R^T(G) & P & -M \\ 0 & -M^T & Q \end{pmatrix},$$

where

$$P=\begin{pmatrix} 2+t_1 & 0 & 0 & ... & 0 \\ 0 & 2+t_2 & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 2+t_m \end{pmatrix}_{m\times m}, \quad M=\begin{pmatrix} 1_{t_1}^T & 0 & 0 & ... & 0 \\ 0 & 1_{t_2}^T & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1_{t_m}^T \end{pmatrix}_{m\times(t_1+t_2+\cdots+t_n)},$$

$$Q=\begin{pmatrix} L_{H_1}+I_{t_1} & 0 & 0 & ... & 0 \\ 0 & L_{H_2}+I_{t_2} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & L_{H_n}+I_{t_n} \end{pmatrix}.$$

Let $A=L_G+D_G$, $B=\begin{pmatrix} -R(G) & 0 \end{pmatrix}$, $B^T=\begin{pmatrix} -R^T(G) \\ 0 \end{pmatrix}$ and $D=\begin{pmatrix} P & -M \\ -M^T & Q \end{pmatrix}$.

First, we will compute D^{-1}. By Lemma 2.3, we have

$$S=\begin{pmatrix} L_{H_1}+I_{t_1} & 0 & 0 & ... & 0 \\ 0 & L_{H_2}+I_{t_2} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & L_{H_m}+I_{t_m} \end{pmatrix}-\begin{pmatrix} 1_{t_1} & 0 & 0 & ... & 0 \\ 0 & 1_{t_2} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1_{t_m} \end{pmatrix}$$
$$\begin{pmatrix} 2+t_1 & 0 & 0 & ... & 0 \\ 0 & 2+t_2 & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 2+t_n \end{pmatrix}^{-1}\begin{pmatrix} 1_{t_1}^T & 0 & 0 & ... & 0 \\ 0 & 1_{t_2}^T & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1_{t_m}^T \end{pmatrix}.$$

$$S^{-1}=\begin{pmatrix} (L_{H_1}+I_{t_1}-\frac{1}{2+t_1}j_{t_1})^{-1} & 0 & 0 & ... & 0 \\ 0 & (L_{H_2}+I_{t_2}-\frac{1}{2+t_2}j_{t_2})^{-1} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & (L_{H_m}+I_{t_m}-\frac{1}{2+t_m}j_{t_m})^{-1} \end{pmatrix}.$$

According to Lemma 2.3, we have

$$P-MQ^{-1}M^T=\begin{pmatrix} 2+t_1 & 0 & 0 & ... & 0 \\ 0 & 2+t_2 & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 2+t_m \end{pmatrix}-\begin{pmatrix} 1_{t_1}^T & 0 & 0 & ... & 0 \\ 0 & 1_{t_2}^T & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1_{t_m}^T \end{pmatrix}$$
$$\begin{pmatrix} (L_{H_1}+I_{t_1})^{-1} & 0 & 0 & ... & 0 \\ 0 & (L_{H_2}+I_{t_2})^{-1} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & (L_{H_m}+I_{t_m})^{-1} \end{pmatrix}\begin{pmatrix} 1_{t_1} & 0 & 0 & ... & 0 \\ 0 & 1_{t_2} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1_{t_m} \end{pmatrix}$$
$$=2I_m,$$

so $(P-MQ^{-1}M^T)^{-1}=\frac{1}{2}I_m$.

By Lemma 2.3, we have

$$-P^{-1}MS^{-1}=-\begin{pmatrix} \frac{1}{2+t_1} & 0 & 0 & ... & 0 \\ 0 & \frac{1}{2+t_2} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & \frac{1}{2+t_m} \end{pmatrix}\begin{pmatrix} 1_{t_1}^T & 0 & 0 & ... & 0 \\ 0 & 1_{t_2}^T & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1_{t_m}^T \end{pmatrix}$$
$$\begin{pmatrix} (L_{H_1}+I_{t_1}-\frac{1}{2+t_1}J_{t_1})^{-1} & 0 & 0 & ... & 0 \\ 0 & (L_{H_2}+I_{t_2}-\frac{1}{2+t_2}J_{t_2})^{-1} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & (L_{H_m}+I_{t_m}-\frac{1}{2+t_m}J_{t_m})^{-1} \end{pmatrix}$$

$$= -\begin{pmatrix} \frac{1}{2}1_{t_1}^T & 0 & 0 & ... & 0 \\ 0 & \frac{1}{2}1_{t_2}^T & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & \frac{1}{2}1_{t_m}^T \end{pmatrix} = F, \tag{7}$$

Similarly, $-S^{-1}M^TP^{-1} = N^T$, so $D^{-1} = \begin{pmatrix} \frac{1}{2}I_m & F \\ F^T & S^{-1} \end{pmatrix}$.

Next we begin with the computation of {1}-inverse of $L_{R(G)\ominus\wedge_{i=1}^m H_i}$.

By Lemma 2.8, we have

$$\begin{aligned} H &= L_G + D_G - \begin{pmatrix} R(G) & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}I_m & F \\ F^T & T^{-1} \end{pmatrix}\begin{pmatrix} R^T(G) \\ 0 \end{pmatrix} \\ &= L_G + D_G - \begin{pmatrix} \frac{1}{2}R(G) & R(G)F \end{pmatrix}\begin{pmatrix} R^T(G) \\ 0 \end{pmatrix} \\ &= L_G + D_G - \tfrac{1}{2}(D_G + A_G) \\ &= \tfrac{3}{2}L_G, \end{aligned}$$

so $H^\# = \frac{2}{3}L_G^\#$.

According to Lemma 2.8, we calculate $-H^\#BD^{-1}$ and $-D^{-1}B^TH^\#$.

$$\begin{aligned} -H^\#BD^{-1} &= -\tfrac{2}{3}L_G^\#\begin{pmatrix} -R(G) & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}I_m & F \\ F^T & S^{-1} \end{pmatrix} \\ &= -\tfrac{2}{3}L_G^\#\begin{pmatrix} -\frac{1}{2}R(G) & -R(G)F \end{pmatrix} = \begin{pmatrix} \frac{1}{3}L_G^\#R(G) & \frac{2}{3}L_G^\#R(G)F \end{pmatrix} \end{aligned}$$

and

$$-D^{-1}B^TH^\# = -(H^\#BD^{-1})^T = \begin{pmatrix} \frac{1}{3}R^T(G)L_G^\# \\ \frac{2}{3}F^TR^T(G)L_G^\# \end{pmatrix}.$$

We are ready to compute the $D^{-1}B^TH^\#BD^{-1}$.

$$\begin{aligned} D^{-1}B^TH^\#BD^{-1} &= \tfrac{2}{3}\begin{pmatrix} \frac{1}{2}I_m & F \\ F^T & S^{-1} \end{pmatrix}\begin{pmatrix} -R^T(G) \\ 0 \end{pmatrix}L_G^\#\begin{pmatrix} -R(G) & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}I_m & F \\ F^T & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{6}R^T(G)L_G^\#R(G) & \frac{1}{3}R^T(G)L_G^\#R(G)F \\ \frac{1}{3}F^TR^TL_G^\#R(G) & \frac{2}{3}F^TR^TL_G^\#R(G)F \end{pmatrix}. \end{aligned}$$

Based on Lemma 2.3 and 2.8, the following matrix

$$N = \begin{pmatrix} \frac{2}{3}L_G^\# & \frac{1}{3}L_G^\#R(G) & \frac{2}{3}L_G^\#R(G)F \\ \frac{1}{3}R^T(G)L_G^\# & \frac{1}{2}I_m + \frac{1}{6}R^T(G)L_G^\#R(G) & F + \frac{1}{3}R^T(G)L_G^\#R(G)F \\ \frac{2}{3}F^TR^T(G)L_G^\# & F^T + \frac{1}{3}F^TR^T(G)L_G^\#R(G) & S^{-1} + \frac{2}{3}F^TR^T(G)L_G^\#R(G)F \end{pmatrix} \tag{8}$$

is a symmetric {1}- inverse of $L_{R(G)\ominus\wedge_{i=1}^m H_i}$.

For any $i, j \in V(G)$, by Lemma 2.1 and the Equation (8), we have

$$r_{ij}(R(G)\ominus\wedge_{i=1}^m H_i) = \frac{2}{3}(L_G^\#)_{ii} + \frac{2}{3}(L_G^\#)_{jj} - \frac{4}{3}(L_G^\#)_{ij} = \frac{2}{3}r_{ij}(G),$$

as stated in (i).

For any $i, j \in V(H_k)(k = 1, 2, ..., m)$, by Lemma 2.1 and the Equation (8), we have

$$\begin{aligned} r_{ij}(R(G)\ominus\wedge_{i=1}^m H_i) &= (L_{H_k} + I_{t_k} - \frac{1}{2+t_k}j_{t_k})_{ii}^{-1} + (L_{H_k} + I_{t_k} - \frac{1}{2+t_k}j_{t_k})_{jj}^{-1} \\ &\quad -2(L_{H_1} + I_{t_k} - \frac{1}{2+t_k}j_{t_k})_{ij}^{-1}, \end{aligned}$$

as stated in (ii).

From the left side of above equation, we can obviously have

$$r_{ij}(F_k) = ((L_{H_k} + I_{t_l})^{-1})_{ii} + ((L_{H_k} + I_{t_l})^{-1})_{jj} - 2((L_{H_k} + I_{t_l})^{-1})_{ij},$$

where $F_k = H_k \vee \{v\}$, i.e, F_k is the graph obtained by adding new edges from an isolated vetrtex v to every vertex of H_k.

For any $i, j \in R(G)$, by Lemma 2.1 and the Equation (8), we have

$$r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) = r_{ij}(R(G)).$$

By Lemma 3.1 in [7], $r_{ij}(R(G)) = \frac{2}{3} r_{ij}(G)$, so $r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) = \frac{2}{3} r_{ij}(G)$, as stated in (iii).

For any $i \in V(G)$, $j \in V(H_k)(k = 1, 2, ..., m)$, since i and j belong to different components, then by Lemma 2.9, we have

$$r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) = r_{ik}(R(G)) + r_{kj}(F_k),$$

as stated in (iv).

For any $i \in V(H_k)$, $j \in V(H_l)$, by Lemma 2.9, we have

$$r_{ij}(R(G) \ominus \wedge_{i=1}^{m} H_i) = r_{kl}(R(G)) + r_{ik}(F_k) + r_{jl}(F_l),$$

as stated in (v).

By Lemma 2.4, we have

$$\begin{aligned}
Kf(R(G) \ominus \wedge_{i=1}^{m} H_i) &= (n + m + \sum_{i=1}^{m} t_i) tr(N) - \mathbf{1}^{\mathbf{T}} N \mathbf{1} \\
&= (n + m + \sum_{i=1}^{m} t_i) \left(\frac{2}{3} tr(L_G^{\#}) + tr\left(\frac{1}{2} I_m + \frac{1}{6} R^T(G) L_G^{\#} R(G) \right) + \right. \\
&\quad \left. + tr(S^{-1} + \frac{2}{3} F^T R^T L_G^{\#} R(G) F) \right) - \mathbf{1}^{\mathbf{T}} N \mathbf{1} \\
&= (n + m + \sum_{i=1}^{m} t_i) \left(\frac{2}{3n} Kf(G) + \frac{m}{2} + \frac{1}{6} \sum_{i<j, i,j \in E(G)} [(L_G^{\#})_{ii} + (L_G^{\#})_{jj} \right. \\
&\quad \left. + 2(L_G^{\#})_{ij}] + tr\left(S^{-1} + \frac{2}{3} F^T R^T(G) L_G^{\#} R(G) F \right) \right) - \mathbf{1}^{\mathbf{T}} N \mathbf{1}.
\end{aligned}$$

By Lemma 2.5, we get

$$\begin{aligned}
Kf(R(G) \ominus \wedge_{i=1}^{m} H_i) &= (n + m + \sum_{i=1}^{m} t_i) \left(\frac{2}{3n} Kf(G) + \frac{m}{2} + \frac{1}{6} \sum_{i<j, i,j \in E(G)} [2(L_G^{\#})_{ii} + 2(L_G^{\#})_{jj} \right. \\
&\quad \left. - r_{ij}(G)] + tr\left(S^{-1} + \frac{2}{3} F^T R(G)^T L_G^{\#} R(G) F \right) \right) - \mathbf{1}^{\mathbf{T}} N \mathbf{1} \\
&= (n + m + \sum_{i=1}^{m} t_i) \left(\frac{2}{3n} Kf(G) + \frac{m}{2} + \frac{1}{3} tr(D_G L_G^{\#}) - \frac{n-1}{6} \right. \\
&\quad \left. + tr\left(S^{-1} + \frac{2}{3} F^T R(G)^T L_G^{\#} R(G) F \right) \right) - \mathbf{1}^{\mathbf{T}} N \mathbf{1}
\end{aligned}$$

Note that the eigenvalues of $(L_{H_i} + I_{t_i} - \frac{1}{2+t_i} j_{t_i})$ $(i = 1, 2, ..., m)$ are $\mu_1(H_i) + 1, \mu_2(H_i) + 1, ..., \mu_{t_i}(H_i) + 1$. Then

$$tr(S^{-1}) = \sum_{i=1}^{m} \sum_{j=1}^{t_i} \frac{1}{\mu_i(H_j) + 1}. \tag{9}$$

By Lemma 2.2, $L_G^{\#}\mathbf{1}=0$ and $(\mathbf{1}^{\mathbf{T}}\left(R^T(G)L_G^{\#}Q\right)\mathbf{1})^T=\mathbf{1}^{\mathbf{T}}\left(Q^TL_G^{\#}R(G)\right)\mathbf{1}$, then

$$\begin{aligned}\mathbf{1}^{\mathbf{T}}N\mathbf{1} &= \frac{m}{2}+\frac{1}{6}\mathbf{1}^{\mathbf{T}}\left(R^T(G)L_G^{\#}R(G)\right)\mathbf{1}+\mathbf{1}^{\mathbf{T}}F\mathbf{1}+\mathbf{1}^{\mathbf{T}}F^T\mathbf{1}\\ &\quad+\frac{2}{3}\mathbf{1}^{\mathbf{T}}R^T(G)L_G^{\#}R(G)F\mathbf{1}+\mathbf{1}^{\mathbf{T}}S^{-1}\mathbf{1}+\frac{2}{3}\mathbf{1}^{\mathbf{T}}\left(F^TR^T(G)L_G^{\#}R(G)F\right)\mathbf{1}.\end{aligned}$$

Note that $R(G)\mathbf{1}=\pi$, where $\pi^T=(d_1,d_2,...,d_n)$, then $\mathbf{1}^{\mathbf{T}}\left(R^T(G)L_G^{\#}R(G)\right)\mathbf{1}=\pi^TL_G^{\#}\pi$, so

$$\mathbf{1}^{\mathbf{T}}N\mathbf{1} = \frac{m}{2}+\frac{1}{6}\pi^TL_G^{\#}\pi+\pi^TL_G^{\#}Q(G)\mathbf{1}+\mathbf{1}^{\mathbf{T}}T^{-1}\mathbf{1}+\mathbf{1}^{\mathbf{T}}\left(Q^TL_G^{\#}Q\right)\mathbf{1}. \tag{10}$$

Let $R_i=L_{H_i}+I_{t_i}-\frac{1}{2+t_i}j_{t_i}(i=1,2,...,m)$. Then

$$\mathbf{1}^{\mathbf{T}}S^{-1}\mathbf{1}^{\mathbf{T}} = \begin{pmatrix}\mathbf{1}_{\mathbf{t_1}}^{\mathbf{T}} & \mathbf{1}_{\mathbf{t_2}}^{\mathbf{T}} & \cdots & \mathbf{1}_{\mathbf{t_m}}^{\mathbf{T}}\end{pmatrix}\begin{pmatrix}R_1^{-1} & 0 & 0 & ... & 0\\ 0 & R_2^{-1} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & R_m^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{1}_{\mathbf{t_1}}\\ \mathbf{1}_{\mathbf{t_2}}\\ \cdots\\ \mathbf{1}_{\mathbf{t_m}}\end{pmatrix}$$

$$= \sum_{i=1}^{m}\mathbf{1}_{\mathbf{t_i}}^{\mathbf{T}}(L_{H_i}+I_{t_i}-\frac{1}{2+t_i}j_{t_i})^{-1}\mathbf{1}_{\mathbf{t_i}}=\frac{1}{2}\sum_{i=1}^{m}t_i(2+t_i), \tag{11}$$

and

$$\mathbf{1}^{\mathbf{T}}F^T = \frac{1}{2}\begin{pmatrix}\mathbf{1}_{\mathbf{t_1}}^{\mathbf{T}} & \mathbf{1}_{\mathbf{t_2}}^{\mathbf{T}} & \cdots & \mathbf{1}_{\mathbf{t_m}}^{\mathbf{T}}\end{pmatrix}\begin{pmatrix}\mathbf{1}_{\mathbf{t_1}} & 0 & 0 & ... & 0\\ 0 & \mathbf{1}_{\mathbf{t_2}} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & \mathbf{1}_{\mathbf{t_m}}\end{pmatrix}$$

$$= \frac{1}{2}(t_1,0,...,0,t_2,0,...,0...,t_m)=\frac{1}{2}\delta^T. \tag{12}$$

Plugging (9), (10), (11) and (12) into $Kf(R(G)\ominus\wedge_{i=1}^{m}H_i)$, we obtain the required result in (*vi*).

5. Conclusion

In this paper, using the Laplacian generalized inverse approach, we obtained the resistance distance and Kirchhoff indices of $R(G)\boxdot\wedge_{i=1}^{n}H_i$ and $R(G)\ominus\wedge_{i=1}^{m}H_i$ whenever G and H_i are arbitrary graph. These results generalize the existing results in [13].

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