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On Bazilevič Functions and Umezawa's Lemma

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Abstract. We consider some properties on $|z| = r < 1$ of analytic functions in the unit disk $|z| < 1$. Applying Umezawa's lemma, *On the theory of univalent functions*, Tohoku Math J. 7(1955) 212–228, we prove some sufficient conditions for functions to be in the class of Bazilevič functions and some related results.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}_p$ denote the class of all functions analytic in the unit disk $\mathbb{D}$ which have the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad z \in \mathbb{D}. \tag{1}$$

A function $f(z)$ meromorphic in a domain $D \subset \mathbb{C}$ is said to be p-valent in D if for each w the equation $f(z) = w$ has at most p roots in D, where roots are counted in accordance with their multiplicity, and there is some v such that the equation $f(z) = v$ has exactly p roots in D. In [6] S. Ozaki proved that if $f(z)$ of the form (1) is analytic in a convex domain $D \subset \mathbb{C}$ and for some real α we have

$$\Re\mathfrak{e}\{\exp(i\alpha) f^{(p)}(z)\} > 0 \quad z \in D,$$

then $f(z)$ is at most p-valent in D. Ozaki's condition is a generalization of the well known Noshiro-Warschawski univalence condition, [4], [12]. In recent paper [10] there are some other conditions for a function to be p-valent in D. Further, a function $f \in \mathcal{A}_p$, $p = 1, 2, 3, \ldots$, is said to be p-valently starlike, if

$$\Re\mathfrak{e}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

The class of all such functions is usually denoted by $\mathcal{S}^*_p$. For $p = 1$ we receive the well known class of normalized starlike univalent functions. Recall that $f(z)$ of the form (1) is called the p-valently Bazilevič function of type β if there exists a p-valently starlike function

$$g(z) = z^p + \sum_{n=p+1} b_n z^n, \quad z \in \mathbb{D}$$

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such that

$$\Re\mathfrak{e}\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\} > 0 \quad z \in \mathbb{D},$$

where $\beta > 0$. Let $\mathcal{P}$ denote the class of analytic functions $q(z)$ in $\mathbb{D}$ of the form

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n, \quad z \in \mathbb{D} \tag{2}$$

such that $\Re\mathfrak{e}\{q(z)\} > 0$, for $z \in \mathbb{D}$. Functions in $\mathcal{P}$ are sometimes called Carathéodory functions.

Lemma 1.1. *[7, Lemma 2] see also [11, pp.224-225] Let us denote by D_z a simply connected closed domain including $z = 0$ inside and by C_z the boundary of D_z. Let*

$$w = f(z) = z^p + \sum_{n=p+1} a_n z^n \tag{3}$$

be regular on D_z and $f(z)/z^p \neq 0$, $f'(z) \neq 0$ on D_z. If $f(z)$ is at least $(p+1)$-valent then C_z has at least one arc C_z' such that

$$\int_{C_z'} \frac{\partial}{\partial\theta}\left[\arg\{zf'(z)\}\right] \mathrm{d}\theta \leq -\pi. \tag{4}$$

and

$$\int_{C_z'} \frac{\partial}{\partial\theta} \arg\{f(z)\} \,\mathrm{d}\theta = 0, \quad z \in C_z'. \tag{5}$$

hold, and $f(z_1) = f(z_2)$, where $z_1 = re^{i\theta_1}$, $z_1 = re^{i\theta_2}$, $\theta_1 < \theta_2$ are the initial and the end point of C_z' respectively.

Lemma 1.2. *[11, p.224–225] Let $f(z)$ be analytic in a simply connected domain D where boundary Γ_z consists of a regular curve and $f'(z) \neq 0$ on Γ_z. Suppose that*

$$\int_{\Gamma_z} \frac{\partial}{\partial\theta}\left[\arg\{zf'(z)\}\right] \mathrm{d}\theta = \int_{\Gamma_z} \Re\mathfrak{e}\left(1 + \frac{zf''(z)}{f'(z)}\right) \mathrm{d}\theta = 2k\pi.$$

If we have for arbitrary $p-k+1$ arcs $C_1, C_2, \ldots, C_{p-k+1}$ on the boundary Γ_z of D which doesn't overlap one another

$$\int_{C_1+C_2+\ldots+C_{p-k+1}} \frac{\partial}{\partial\theta}\left[\arg\{zf'(z)\}\right] \mathrm{d}\theta > -\pi \tag{6}$$

or, if for arbitrary $p-k+1$ arcs $C_1, C_2, \ldots, C_{p-k+1}$ on the boundary Γ_z of D which doesn't overlap one another

$$\int_{C_1+C_2+\ldots+C_{p-k+1}} \frac{\partial}{\partial\theta}\left[\arg\{zf'(z)\}\right] \mathrm{d}\theta < (p+k+1)\pi, \tag{7}$$

then $f(z)$ is at most p-valent in D.

Here, $\arg \mathrm{d}f(z)$ means the argument of the tangent to the curve $f(re^{i\theta})$, $0 \leq \theta \leq 2\pi$ or $\arg\{izf'(z)\}$. Applying Umezawa's Lemma 1.2, we can have the following contraposition of it.

Theorem 1.3. *Let $f(z)$ be of the form* (1) *be analytic in $\mathbb{D}$ and $f'(z) \neq 0$ in $\mathbb{D}$ and let for arbitrary r, $0 < r < 1$, $f(z)$ satisfies*

$$\int_{|z|=r} \frac{\partial}{\partial \theta} [\arg\{zf'(z)\}] \, d\theta = 2p\pi.$$

Then, if $f(z)$ is at lest $(p+1)$-valent in $\mathbb{D}$, then there exists an arc Γ on the circle $|z| = r$, $0 < r < 1$, for which

$$\int_{\Gamma} \frac{\partial}{\partial \theta} [\arg\{zf'(z)\}] \, d\theta \leq -\pi \tag{8}$$

or

$$\int_{\Gamma} \frac{\partial}{\partial \theta} [\arg\{zf'(z)\}] \, d\theta \geq (2p+1)\pi. \tag{9}$$

2. Results and Discussion

Applying Theorem 1.3 gives the following theorem.

Theorem 2.1. *Let*

$$f(z) = z^p + \sum_{n=p+1} a_n z^n, \quad z \in \mathbb{D} \tag{10}$$

be analytic in $\mathbb{D}$. Assume that there exists a p-valently starlike function

$$g(z) = z^p + \sum_{n=p+1} b_n z^n, \quad z \in \mathbb{D} \tag{11}$$

such that

$$\Re\mathfrak{e} \left\{ \frac{zf'(z)}{f^{1-\beta}(z) g^{\beta}(z)} \right\} > 0 \quad z \in \mathbb{D}, \tag{12}$$

where $\beta > 0$. Then $f(z)$ is p-valent in $\mathbb{D}$.

Proof. From the hypothesis (12), we have

$$\begin{aligned} & \int_{|z|=r} \frac{\partial}{\partial \theta} \arg \left\{ \frac{zf'(z)}{f^{1-\beta}(z) g^{\beta}(z)} \right\} d\theta \qquad (13) \\ = & \int_{|z|=r} \left(\frac{\partial}{\partial \theta} \arg \{zf'(z)\} - \frac{\partial}{\partial \theta} \arg \left\{ f^{1-\beta}(z) \right\} d\theta - \frac{\partial}{\partial \theta} \arg \left\{ g^{\beta}(z) \right\} \right) \\ = & \int_{|z|=r} \left(\frac{\partial}{\partial \theta} \arg \{zf'(z)\} - (1-\beta) \frac{\partial}{\partial \theta} \arg \{f(z)\} - \beta \frac{\partial}{\partial \theta} \arg \{g(z)\} \right) d\theta \\ > & -\pi. \end{aligned}$$

It is trivial that $f(z)$ is at least p-valent in $\mathbb{D}$ because

$$f(z) = z^p + \sum_{n=p+1} a_n z^n$$

is at least p-valent in at the neighborhood of the origin. Then if $f(z)$ is not p-valent in $\mathbb{D}$ or $f(z)$ is at least $(p+1)$-valent in $\mathbb{D}$, then by Lemma 1.1, there exists an arc on the circle $|z| = r$, $0 < r < 1$, for which we have the following picture Fig. 1. which is a part of the image of $w = f(z)$, $|z| = r$.

$$\begin{aligned}\Gamma &= \Big\{f(z) : f(re^{i\theta}), 0 \le \theta_1 \le \theta \le \theta_2,\ z_j = re^{i\theta_j},\ j = 1,2,\ f(z_1) = f(z_2)\\ &\quad \text{and}\ \ \frac{\partial}{\partial\theta}\arg\{zf'(z)\}\Big|_{z_2} = \frac{\partial}{\partial\theta}\arg\{zf'(z)\}\Big|_{z_1} - \pi\Big\}.\end{aligned}$$

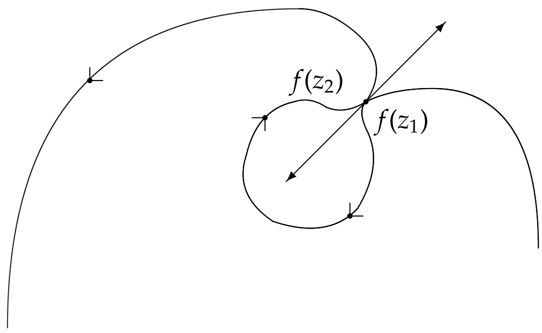

Fig.1. $w = f(z)$–plane

Then, we have

$$\int_\Gamma \frac{\partial}{\partial\theta}\arg\{zf'(z)\}\,\mathrm{d}\theta = -\pi. \tag{14}$$

From (13), we must have

$$\begin{aligned} &\int_\Gamma \frac{\partial}{\partial\theta}\arg\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\}\mathrm{d}\theta \qquad (15)\\ =\ &\int_\Gamma\left(\frac{\partial}{\partial\theta}\arg\{zf'(z)\} - (1-\beta)\frac{\partial}{\partial\theta}\arg\{f(z)\} - \beta\frac{\partial}{\partial\theta}\arg\{g(z)\}\right)\mathrm{d}\theta\\ =\ &\int_\Gamma\left(\frac{\partial}{\partial\theta}\arg\{zf'(z)\} - \beta\frac{\partial}{\partial\theta}\arg\{g(z)\}\right)\mathrm{d}\theta\\ >\ &-\pi\end{aligned}$$

because $f(z_1) = f(z_2)$. Therefore, we have

$$\int_\Gamma \frac{\partial}{\partial\theta}\arg\{zf'(z)\}\,\mathrm{d}\theta > \int_\Gamma \beta\frac{\partial}{\partial\theta}\arg\{g(z)\}\,\mathrm{d}\theta - \pi > -\pi \tag{16}$$

because $0 < \beta$ and $g(z)$ is p-valently starlike in $\mathbb{D}$. This contradicts (14) and it completes the proof of Theorem 2.1. □

Corollary 2.2. *If $f(z) \in \mathcal{A}_p$ and there exist $g(z) \in \mathcal{S}^*_p$, $q(z) \in \mathcal{P}$ and a positive integer $k \ge 2$ such that*

$$f^k(z) = kp\int_0^z \frac{g^k(t)q(t)}{t}\mathrm{d}t \quad z \in \mathbb{D}, \tag{17}$$

then $f(z)$ is p-valent in $\mathbb{D}$.

Proof. Equality (17) may be written in the form

$$zf^{k-1}(z)f'(z) = pg^k(z)q(z)$$

or

$$\frac{zf'(z)}{f^{1-k}(z)g^k(z)} = pq(z),$$

and $\Re\mathfrak{e}\{pq(z)\} > 0$ in $\mathbb{D}$. This gives (12) hence $f(z)$ is p-valent in $\mathbb{D}$.
□

For $k = 2$, Corollary 2.2 becomes the following corollary.

Corollary 2.3. *Let If $f(z) \in \mathcal{A}_p$ and there exist $g(z) \in \mathcal{S}_p^*$ and $q(z) \in \mathcal{P}$ such that*

$$zf(z)f'(z) = pg^2(z)q(z) \;\; z \in \mathbb{D},$$

Then $f(z)$ is p-valent in $\mathbb{D}$.

Theorem 2.4. *Let*

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

be analytic in $\mathbb{D}$. Assume that there exists a p-valently starlike function

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$$

such that

$$\frac{zf'(z)}{pf^{1-\beta}(z)g^{\beta}(z)} \prec \left(\frac{1+z}{1-z}\right)^2. \tag{18}$$

Then

$$\int_0^{2\pi} \left|\Re\mathfrak{e}\left\{\frac{zf'(z)}{pf^{1-\beta}(z)g^{\beta}(z)}\right\}\right| \mathrm{d}\theta \le 2\pi, \quad |z| < \sqrt{2} - 1.$$

Proof. If $\Phi(z) \prec \Phi_0(z)$, then [1]

$$\int_0^{2\pi} \left|\Re\mathfrak{e}\{\Phi(\rho e^{i\theta})\}\right| \mathrm{d}\theta \le \int_0^{2\pi} \left|\Re\mathfrak{e}\{\Phi_0(\rho e^{i\theta})\}\right| \mathrm{d}\theta \quad \text{for } 0 < \rho < 1. \tag{19}$$

From (18) and from (19), for all $z = \rho e^{i\theta}$, $\rho \in (0, 1)$, we have

$$\int_0^{2\pi} \left|\Re\mathfrak{e}\left\{\frac{zf'(z)}{pf^{1-\beta}(z)g^{\beta}(z)}\right\}\right| \mathrm{d}\theta \le \int_0^{2\pi} \left|\Re\mathfrak{e}\left\{\left(\frac{1+z}{1-z}\right)^2\right\}\right| \mathrm{d}\theta.$$

If $0 < r \le \sqrt{2} - 1$, then $(1 - r^2)^2 - 4r^2 \sin^2\theta \ge 0$ and we have

$$\begin{aligned}
\int_0^{2\pi} \left|\Re\mathfrak{e}\left(\frac{1+z}{1-z}\right)^2\right| \mathrm{d}\theta &= 2\int_0^{\pi} \left|\frac{(1-r^2)^2 - 4r^2\sin^2\theta}{(1+r^2-2r\cos\theta)^2}\right| \mathrm{d}\theta \\
&= 2\left[\frac{4r\sin\theta}{r^2 - 2r\cos\theta + 1} + \theta\right]_0^{\pi} \\
&= 2\pi.
\end{aligned}$$

where $z = re^{i\theta}$. □

Theorem 2.5. *Assume that $f(z) \in \mathcal{A}_p$, $g(z) \in \mathcal{A}_p$. If there are positive integer $m, n \in \{1, \dots, p\}$ such that*

$$\left|\arg\left\{\frac{zf^{(m)}(z)}{f^{(m-1)}(z)}\right\}\right| < \frac{\gamma\pi}{2}, \quad z \in \mathbb{D}, \tag{20}$$

for some $\gamma \in (0, 1)$,

$$\left|\arg\left\{\frac{zg^{(n)}(z)}{g^{(n-1)}(z)}\right\}\right| < \frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{21}$$

and

$$\left|\arg\left\{\frac{f^{(n)}(z)}{g^{(n)}(z)}\right\}\right| < \frac{(1-\gamma)\pi}{2\beta}, \quad z \in \mathbb{D}, \tag{22}$$

for some $\beta > 1 - \gamma$, then

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{23}$$

This means that $f(z)$ is a p-valently Bazilevič function of type β.

Proof. Let

$$q(z) = \left\{\frac{zf^{(m-1)}(z)}{(p-m+2)f^{(m-2)}(z)}\right\}, \quad q(0) = 1.$$

If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{q(z)\}| < \frac{\pi\gamma}{2} \tag{24}$$

for $|z| < |z_0|$ and

$$|\arg\{q(z_0)\}| = \frac{\pi\gamma}{2} \tag{25}$$

for some $\gamma \in (0, 1)$, then from [5], we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{2ik \arg\{q(z_0)\}}{\pi}, \tag{26}$$

for some $k \geq (a + a^{-1})/2 \geq 1$, where $\{q(z_0)\}^{1/\gamma} = \pm ia$, and $a > 0$. If we consider (25) for the case $\arg\{q(z_0)\} = \pi\gamma/2$, then from (26) we have

$$\begin{aligned}\left|\arg\left\{\frac{zf^{(m)}(z)}{f^{(m-1)}(z)}\right\}\right| &= \left|\arg\left\{(p-m+2)q(z_0) - 1 + \frac{z_0 q'(z_0)}{q(z_0)}\right\}\right| \\ &= \left|\arg\{(p-m+2)q(z_0) - 1 + ik\gamma\}\right| \\ &\geq \arg\{(p-m+2)q(z_0)\} = \pi\gamma/2.\end{aligned}$$

This contradicts (20), so supposition (25) is false and (24) holds true in whole unit disc $\mathbb{D}$. The same argumentation shows that (24) holds true if we consider (25) for the case $\arg\{q(z_0)\} = -\pi\gamma/2$. Applying this method again and again we obtain that (20) implies the same inequality for all smaller numbers than m namely

$$\left|\arg\left\{\frac{zf^{(m)}(z)}{f^{(m-1)}(z)}\right\}\right| < \frac{\gamma\pi}{2} \quad \Rightarrow \quad \forall k \in \{1, \dots, m\} : \quad \left|\arg\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\}\right| < \frac{\gamma\pi}{2}.$$

Also, in the same way, from (21) we have

$$\left|\arg\left\{\frac{zg^{(n)}(z)}{g^{(n-1)}(z)}\right\}\right| < \frac{\pi}{2} \quad \Rightarrow \quad \forall k \in \{1,\dots,n\}: \quad \left|\arg\left\{\frac{zg^{(k)}(z)}{g^{(k-1)}(z)}\right\}\right| < \frac{\pi}{2}.$$

Furthermore, it is known, [3, p.200], that if q is convex univalent in $\mathbb{D}$ and $F(z), G(z)$ are analytic in $\mathbb{D}$, $G(0) = F(0)$ and

$$\mathfrak{Re}\left\{\frac{zG'(z)}{G(z)}\right\} > 0, \quad (z \in \mathbb{D}),$$

then we have

$$\frac{F'(z)}{G'(z)} \prec q(z) \quad \Rightarrow \quad \frac{F(z)}{G(z)} \prec q(z), \quad (z \in \mathbb{D}). \tag{27}$$

If we put

$$F(z) = f^{(n-1)}(z), \quad G(z) = g^{(n-1)}(z), \quad q(z) = \left\{\frac{1+z}{1-z}\right\}^{\alpha}, \quad \alpha = \frac{(1-\gamma)\pi}{2\beta} \tag{28}$$

then by (25) and (27), we have

$$\left|\arg\left\{\frac{f^{(n)}(z)}{g^{(n)}(z)}\right\}\right| < \frac{(1-\gamma)\pi}{2\beta} \quad \Rightarrow \quad \left|\arg\left\{\frac{f^{(n-1)}(z)}{g^{n-1}(z)}\right\}\right| < \frac{(1-\gamma)\pi}{2\beta}, \quad (z \in \mathbb{D}). \tag{29}$$

Applying this method again and again we obtain that

$$\left|\arg\left\{\frac{f^{(n)}(z)}{g^{(n)}(z)}\right\}\right| < \frac{(1-\gamma)\pi}{2\beta} \quad \Rightarrow \forall k \in \{0,\dots,n\}: \quad \left|\arg\left\{\frac{f^{(k-1)}(z)}{g^{k-1}(z)}\right\}\right| < \frac{(1-\gamma)\pi}{2\beta}, \quad (z \in \mathbb{D}).$$

Note that (27) is an improvement of the earlier Pommerenke's result [8, Lemma 1, p.180]: If $f(z)$ is analytic and $g(z)$ is convex in $\mathbb{D}$, then

$$\left|\arg\frac{f'(z)}{g'(z)}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \quad \Rightarrow \quad \left|\arg\frac{f(z)}{g(z)}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}, \tag{30}$$

where $0 < \alpha \le 1$.

From the above considerations, we can see that inequality (20) holds true for $m = 1$, inequality (21) holds true for $n = 1$ and inequality (22) holds true for $n = 0$. Therefore, we have

$$\begin{aligned}
\left|\arg\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\}\right| &= \left|\arg\left\{\frac{zf'(z)}{f(z)}\left[\frac{f(z)}{g(z)}\right]^{\beta}\right\}\right| \\
&= \left|\arg\left\{\frac{zf'(z)}{f(z)}\right\} + \beta\arg\left\{\frac{f(z)}{g(z)}\right\}\right| \\
&\le \left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| + \beta\left|\arg\left\{\frac{f(z)}{g(z)}\right\}\right| \\
&\le \frac{\gamma\pi}{2} + \beta\frac{(1-\gamma)\pi}{2\beta} \\
&= \frac{\pi}{2}
\end{aligned}$$

This is (23). □

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