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Frenet Frame With Respect to Conformable Derivative

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Abstract. Conformable fractional derivative is introduced by the authors Khalil at al in 2014. In this study, we investigate the frenet frame with respect to conformable fractional derivative. Curvature and torsion of a conformable curve are defined and the geometric interpretation of these two functions is studied. Also, fundamental theorem of curves is expressed for the conformable curves and an example of the curve corresponding to a fractional differential equation is given.

1. Introduction

The differential geometry of curves and surfaces has two aspects. One of them is called classical differential geometry and the other one is called global differential geometry.The classical geometry is the study of local properties of curves and surfaces. By local properties we mean those properties which depend only on the behavior of the curve or surface in the neighbourhood of a point. The methods which have shown themselves the adequate in the study of such properties are the methods of differential calculus.

Recently, in [9] the authors Khalil at al. introduced a new definition of the fractional derivative called conformable fractional derivative. Moreover, in that paper, the definition of fractional integral is also defined. This paper pioneered many new studies. In [11, 13], the authors investigate Lyapunov-type inequalities in the frame of conformable derivatives. In the papers [4, 12], conformable fractional operators and semigroup operators are studied. Further studies about conformable derivatives and its applications are found in [1, 3, 5–8]. In addition to this, conformable fractional derivative for multivariable functions is given in [1, 3, 8]. These papers give a chance to introduce the notion of conformable curve which is a generalized form of a curve.

In this study, the classical differential geometry of curves is investigated with respect to conformable fractional derivative and fractional integral. In this sense, conformable curve is defined and $\alpha-$Frenet formulas are given. If $\alpha = 1$, the theory of conformable curves coincides with the classical theory of curves. Especially, the fundamental theorem of the local theory of the conformable curves is given. And, for $\alpha-$differentiable functions $\kappa > 0$ and τ, a system of fractional order differential equations is formed, then a conformable curve is obtained by the solution of this system.

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2. Comformable Fractional Derivative

In this section, we give some basic definitions and properties of conformable fractional derivative introduced in [8–10].

Definition 2.1. *[9] Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. The conformable derivative of the function f of order α is defined by*

$$T_\alpha(f)(x) = \lim_{h\to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h} \tag{1}$$

for all $x > 0$, $\alpha \in (0,1)$.

Theorem 2.2. *[9] If a function $f : [0,\infty) \longrightarrow \mathbb{R}$ is $\alpha-$differentiable at $t_0 > 0$, $\alpha \in (0,1]$, then f is continuous at t_0.*

Theorem 2.3. *[9] Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*

(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.

(2) $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.

(3) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

(4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(5) $T_\alpha(\frac{f}{g}) = \dfrac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(6) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\dfrac{d}{dt}f(t)$.

Theorem 2.4. *[10] Assume $f, g : (0,\infty) \longrightarrow \mathbb{R}$ be two $\alpha-$differentiable functions where $\alpha \in (0,1]$. Then $g \circ f$ is $\alpha-$differentiable and for all t with $t \neq 0$ and $f(t) \neq 0$ we have*

$$T_\alpha(g \circ f)(t) = T_\alpha(g)(f(t))T_\alpha(f)(t)f(t)^{\alpha-1}. \tag{2}$$

Definition 2.5. *[8] Let f be a vector valued function with n real variables such that $f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$. Then we say that f is $\alpha-$differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$ where each $a_i > 0$, if there is a linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$\lim_{h\to 0} \frac{\|f(a_1 + h_1a_1^{1-\alpha}, ..., a_n + h_na_n^{1-\alpha}) - f(a_1, ..., a_n) - L(h)\|}{\|h\|} = 0 \tag{3}$$

where $h = (h_1, ..., h_n)$ and $\alpha \in (0,1]$. The linear transformation is denoted by $D^\alpha f(a)$ and called the conformable derivative of f of order α at a.

Theorem 2.6. *[8] Let f be a vector valued function with n variables. If f is $\alpha-$differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, each $a_i > 0$, then there is a unique linear transformation $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$\lim_{h\to 0} \frac{\|f(a_1 + h_1a_1^{1-\alpha}, ..., a_n + h_na_n^{1-\alpha}) - f(a_1, ..., a_n) - L(h)\|}{\|h\|} = 0.$$

Theorem 2.7. *[8] If a vector valued function f with n variables is $\alpha-$differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, each $a_i > 0$, then f is continuous at $a \in \mathbb{R}^n$.*

Theorem 2.8. *[8] (Chain Rule) Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. If $f(x) = (f_1(x), ..., f_m(x))$ is $\alpha-$differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, each $a_i > 0$ such that $\alpha \in (0, 1]$ and $g(y) = (g_1(y), ..., g_p(y))$ is $\alpha-$differentiable at $f(a) \in \mathbb{R}^m$, all $f_i(a) > 0$ such that $\alpha \in (0, 1]$. Then the composition $g \circ f$ is $\alpha-$differentiable at a and*

$$D^\alpha(g \circ f)(a) = D^\alpha g(f(a)) \circ f(a)^{\alpha-1} \circ D^\alpha f(a) \tag{4}$$

where $f(a)^{\alpha-1}$ is the linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^m$ defined by $f(a)^{\alpha-1}(x_1, ..., x_m) = (x_1 f_1(a)^{\alpha-1}, ..., x_m f_m(a)^{\alpha-1})$.

Theorem 2.9. *[8] Let f be a vector valued function with n variables such that $f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$. Then f is $\alpha-$differentiable at $a = (a_1, ..., a_n) \in \mathbb{R}^n$, each $a_i > 0$ if and only if each f_i is, and*

$$D^\alpha f(a) = (D^\alpha f_1(a), ..., D^\alpha f_m(a)).$$

3. Conformable Curves

In this section, we are going to introduce conformable curves and their basic properties.

Definition 3.1. *The function $\gamma : (0, \infty) \to \mathbb{R}^3$ is called a conformable curve in $\mathbb{R}^3$ if γ is $\alpha-$differentiable.*

Notation: Along the work, for a conformable curve $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, we use the notation

$$D^\alpha \gamma(t) = T_\alpha \gamma(t) = (T_\alpha \gamma_1(t), T_\alpha \gamma_2(t), T_\alpha \gamma_3(t)).$$

Definition 3.2. *Let $\gamma : (0, \infty) \to \mathbb{R}^3$ be a conformable curve. Velocity vector of γ is determined by*

$$\frac{T_\alpha \gamma(t)}{t^{1-\alpha}}, \tag{5}$$

for all $t \in (0, \infty)$.

Definition 3.3. *Let $\gamma : (0, \infty) \to \mathbb{R}^3$ be a conformable curve. If the function $h : (0, \infty) \to (0, \infty)$ is $\alpha-$differentiable, the conformable curve $\beta = \gamma(h) : (0, \infty) \to \mathbb{R}^3$ is called the reparametrization of γ with h.*

Definition 3.4. *Let $\gamma : (0, \infty) \to \mathbb{R}^3$ be a conformable curve. Then the velocity function v of γ is defined by*

$$v(t) = \frac{||T_\alpha \gamma(t)||}{t^{1-\alpha}} \tag{6}$$

for all $t \in (0, \infty)$.

Definition 3.5. *Let $\gamma : (0, \infty) \to \mathbb{R}^3$ be a conformable curve. The arc length function s of γ is defined by*

$$s(t) = \mathbf{I}_\alpha^0 ||T_\alpha \gamma(t)|| \tag{7}$$

for all $t \in (0, \infty)$. If $v(t) = 1$ for all $t \in (0, \infty)$, it's said that γ has unit speed.

Lemma 3.6. *Let the function $f : (0, \infty) \to \mathbb{R}$ be continuous and increasing (or decreasing) on $(0, \infty)$. If the function f is $\alpha-$differentiable for all $x \in (0, \infty)$ and $T_\alpha f(x) \neq 0$, then the inverse function $f^{-1} : (f(0^+), f(b^-)) \to \mathbb{R}$ (or $f^{-1} : (f(b^-), f(0^+)) \to \mathbb{R}$) is $\alpha-$differentiable at $y = f(x)$ where $b \in (0, \infty)$. Furthermore,*

$$(T_\alpha f^{-1})(y) = \frac{(xy)^{1-\alpha}}{T_\alpha f(x)}.$$

Proof. Since $x, x + hx^{1-\alpha} \in (0, \infty)$, $\alpha \in (0, 1)$, $y = f(x)$ and $f(x + hx^{1-\alpha}) = y + Hy^{1-\alpha}$, it's clear that $x = f^{-1}(y)$ and $x + hx^{1-\alpha} = f^{-1}(y + Hy^{1-\alpha})$. Since the function f is increasing (or decreasing),
$H = \dfrac{f(x + hx^{1-\alpha}) - f(x)}{y^{1-\alpha}} \neq 0$ for all $h \neq 0$. Hence, for $H \to 0$, $h \to 0$. Therefore, we have

$$\begin{aligned}\lim_{H\to 0} \frac{f^{-1}(y + Hy^{1-\alpha}) - f^{-1}(y)}{H} &= \lim_{h\to 0} \frac{(xy)^{1-\alpha}}{\dfrac{f(x + hx^{1-\alpha}) - f(x)}{h}} \\ &= \frac{(xy)^{1-\alpha}}{\lim_{h\to 0} \dfrac{f(x + hx^{1-\alpha}) - f(x)}{h}} = \frac{(xy)^{1-\alpha}}{T_\alpha f(x)}.\end{aligned}$$

□

Definition 3.7. *Let γ be a conformable curve. If $T_\alpha \gamma(t) \neq 0$ for all $t \in (0, \infty)$, γ is called a conformable regular curve.*

Theorem 3.8. *If γ is a conformable regular curve, there is a reparametrization β of γ such that β has unit speed.*

Proof. Let $a \in (0, \infty)$ be fixed. Consider the arc length function

$$s(t) = \mathbf{I}_\alpha^a \|T_\alpha \gamma(t)\|.$$

Since γ is conformable regular, $T_\alpha s(t) = \|T_\alpha \gamma(t)\| \neq 0$. By Lemma (3.6), $s(t)$ has an inverse function $t = t(s)$ and $t = t(s)$ is also differentiable. On the other hand, β is a reparametrization of γ such that $\beta(s) = \gamma(t(s))$. Let us show that β has unit speed: We know that $T_\alpha\beta(s) = (T_\alpha\gamma)(t(s))T_\alpha t(s)t(s)^{\alpha-1}$ by Theorem (2.8). Therefore,

$$\begin{aligned}\frac{\|T_\alpha\beta(s)\|}{s^{1-\alpha}} &= \frac{\|(T_\alpha\gamma)(t(s))\|T_\alpha t(s)t(s)^{\alpha-1}}{s^{1-\alpha}} \\ &= \frac{(T_\alpha s)(t(s))T_\alpha t(s)t(s)^{\alpha-1}}{s^{1-\alpha}} = \frac{T_\alpha(s(t(s)))}{s^{1-\alpha}} = \frac{T_\alpha s}{s^{1-\alpha}} = 1.\end{aligned}$$

This completes the proof. □

4. $\alpha-$Frenet Formulas

4.1. Conformable Curves With Unit Speed

Let β be a unit speed conformable curve, so $\dfrac{\|T_\alpha\beta(s)\|}{s^{1-\alpha}} = 1$ for each $s \in (0, \infty)$. Then $E_1(s) = \dfrac{T_\alpha\beta(s)}{s^{1-\alpha}}$ is called the unit tangent vector field on β. Since E_1 has constant length 1, we call $T_\alpha E_1$ the curvature vector field of β. $\alpha-$differentiation of $E_1 \cdot E_1 = 1$ gives $2T_\alpha E_1 \cdot E_1 = 0$, so $T_\alpha E_1$ is always orthogonal to E_1, that is, normal to β.

The length of the curvature vector field $T_\alpha E_1$ gives a numerical measurement of the turning of β. The real valued function κ such that $\kappa(s) = \|T_\alpha E_1(s)\|$ for all $s \in (0, \infty)$ is called the curvature function of β. Let $\kappa > 0$. The unit vector field $E_2 = \dfrac{T_\alpha E_1}{\kappa}$ on β is called the principal normal vector field of β. The unit vector field $E_3 = E_1 \times E_2$ on β is called binormal vector field of β. Therefore the orthonormal system of the vector fields $\{E_1, E_2, E_3\}$ on β is called the $\alpha-$frenet frame.

Remark 4.1. *Now we express the terms $T_\alpha E_1, T_\alpha E_2, T_\alpha E_3$ in terms of E_1, E_2, E_3. Since $E_1 = T_\alpha\beta$, we have $T_\alpha E_1 = \kappa E_2$. Let us consider $T_\alpha E_3$. Since $E_3 \cdot E_3 = 1$, $T_\alpha E_3$ is orthogonal to E_3. On the other hand, since $E_3 \cdot E_1 = 0$, $T_\alpha E_3 \cdot E_1 + E_3 \cdot T_\alpha E_1 = 0$. Hence, we have $T_\alpha E_3 \cdot E_1 = -E_3 \cdot T_\alpha E_1 = -E_3 \cdot \kappa E_2 = 0$, that is, $T_\alpha E_3$ is orthogonal to E_1. Finally, since $T_\alpha E_3$ is orthogonal to both E_1 and E_3, $T_\alpha E_3$ is, at each point, a scalar multiple of E_2. Thus, $T_\alpha E_3 = -\tau E_2$, where the real valued function τ is called the torsion function of the conformable curve β.*

Theorem 4.2. *If β is a unit speed conformable curve with curvature $\kappa > 0$ and torsion τ then,*

$$\begin{array}{rlll} T_\alpha E_1 = & & \kappa E_2 & \\ T_\alpha E_2 = & -\kappa E_1 & & +\tau E_3. \\ T_\alpha E_3 = & & -\tau E_2 & \end{array}$$

Proof. By Remark (4.1) first and third formulas are essentially just the definitions of curvature and torsion. For second equation, let use orthonormal expansion to express $T_\alpha E_2$ in terms of E_1, E_2, E_3:

$$T_\alpha E_2 = (T_\alpha E_2 \cdot E_1)E_1 + (T_\alpha E_2 \cdot E_2)E_2 + (T_\alpha E_2 \cdot E_3)E_3.$$

$\alpha-$differentiating of $E_2 \cdot E_1 = 0$ gives $T_\alpha E_2 \cdot E_1 = -E_2 \cdot T_\alpha E_1 = -E_2 \cdot \kappa E_2 = -\kappa$. As usual, $T_\alpha E_2 \cdot E_2 = 0$, since E_2 is a unit vector field.
Finally, $T_\alpha E_2 \cdot E_3 = -E_2 \cdot T_\alpha E_3 = -E_2 \cdot -\tau E_2 = \tau$. □

Remark 4.3. *The equations obtained in Theorem 4.2 are the same as the equations in Theorem 3.2 in [2].*

4.2. Conformable Curves With Arbitrary Speed

Remark 4.4. *Let γ be a conformable regular curve with arbitrary speed and β be the unit speed reparametrization of γ. If s is an arc length function for γ as in Theorem (3.8), then $\gamma(t) = \beta(s(t))$ for all $t \in (0, \infty)$. If $\kappa^\beta > 0$, τ^β, $E_1^\beta, E_2^\beta, E_3^\beta$ are defined for β as in Subsection 4.1, we define for γ the curvature function as $\kappa = \kappa^\beta(s)$, torsion function as $\tau = \tau^\beta(s)$, unit tangent vector field as $E_1 = E_1^\beta(s)$, principal normal vector field as $E_2 = E_2^\beta(s)$ and binormal vector field as $E_3 = E_3^\beta(s)$.*

Lemma 4.5. *If γ is a conformable regular curve in $\mathbb{R}^3$ with $\kappa > 0$ then*

$$\begin{array}{rlll} T_\alpha E_1 = & & \kappa v \lambda^{1-\alpha} E_2 & \\ T_\alpha E_2 = & -\kappa v \lambda^{1-\alpha} E_1 & & +\tau v \lambda^{1-\alpha} E_3 \\ T_\alpha E_3 = & & -\tau v \lambda^{1-\alpha} E_2 & \end{array}$$

where $\lambda = \dfrac{t}{s(t)}$.

Proof. Let β be a unit speed reparametrization of γ, then by Remark 4.4 $E_1 = E_1^\beta(s)$, where s is an arc length function for γ. The $\alpha-$chain rule as applied to differentiation of vector fields gives

$$\begin{aligned} T_\alpha E_1(t) = T_\alpha(E_1^\beta(s(t))) &= (T_\alpha E_1^\beta)(s(t)) T_\alpha s(t) s(t)^{\alpha-1} \\ &= \kappa^\beta(s(t)) E_2^\beta(s(t)) v t^{1-\alpha} s(t)^{\alpha-1} \\ &= \kappa(t) E_2(t) v \frac{t^{1-\alpha}}{s(t)^{1-\alpha}} = \kappa v \lambda^{1-\alpha} E_2 \end{aligned}$$

where $\lambda = \dfrac{t}{s(t)}$.
The formulas for $T_\alpha E_2$ and $T_\alpha E_3$ are derived in the same way. □

Theorem 4.6. *Let γ be a conformable curve in $\mathbb{R}^3$. Then*

$$\begin{aligned}
E_1 &= \frac{T_\alpha\gamma}{\|T_\alpha\gamma\|}\\
E_3 &= \frac{T_\alpha\gamma \times T_\alpha^2\gamma}{\|T_\alpha\gamma \times T_\alpha^2\gamma\|}\\
E_2 &= E_3 \times E_1\\
\kappa &= (\frac{t}{\lambda})^{1-\alpha}\frac{\|T_\alpha\gamma \times T_\alpha^2\gamma\|}{\|T_\alpha\gamma\|^3}\\
\tau &= (\frac{t}{\lambda})^{1-\alpha}\frac{(T_\alpha\gamma \times T_\alpha^2\gamma)\cdot T_\alpha^3\gamma}{\|T_\alpha\gamma \times T_\alpha^2\gamma\|^2}.
\end{aligned}$$

Proof. Since $T_\alpha\gamma = vt^{1-\alpha}E_1$, we have $E_1 = \dfrac{T_\alpha\gamma}{vt^{1-\alpha}}$. We know $v = \dfrac{\|T_\alpha\gamma\|}{t^{1-\alpha}}$, then $E_1 = \dfrac{T_\alpha\gamma}{\|T_\alpha\gamma\|}$.
$T_\alpha^2\gamma(t) = T_\alpha(vt^{1-\alpha}E_1) = (t^{1-\alpha}T_\alpha v + (1-\alpha)t^{1-2\alpha}v)E_1 + \kappa v^2(t\lambda)^{1-\alpha}E_2$.
So we have

$$\begin{aligned}
T_\alpha\gamma \times T_\alpha^2\gamma &= vt^{1-\alpha}E_1 \times [(t^{1-\alpha}T_\alpha v + (1-\alpha)t^{1-2\alpha}v)E_1 + \kappa v^2(t\lambda)^{1-\alpha}E_2]\\
&= vt^{1-\alpha}(t^{1-\alpha}T_\alpha v + (1-\alpha)t^{1-2\alpha}v)E_1 \times E_1 + \kappa v^2(t\lambda)^{1-\alpha}E_1 \times E_2\\
&= \kappa v^3(\lambda t^2)^{1-\alpha}E_3.
\end{aligned}$$

If we take norms, we get $\|T_\alpha\gamma \times T_\alpha^2\gamma\| = \kappa v^3(\lambda t^2)^{1-\alpha}$.
Then,

$$\frac{T_\alpha\gamma \times T_\alpha^2\gamma}{\|T_\alpha\gamma \times T_\alpha^2\gamma\|}.$$

On the other hand,

$$\kappa = \frac{\|T_\alpha\gamma \times T_\alpha^2\gamma\|}{v^3(\lambda t^2)^{1-\alpha}} = \frac{\|T_\alpha\gamma \times T_\alpha^2\gamma\|}{(vt^{1-\alpha})^3(\frac{\lambda}{t})^{1-\alpha}} = (\frac{t}{\lambda})^{1-\alpha}\frac{\|T_\alpha\gamma \times T_\alpha^2\gamma\|}{\|T_\alpha\gamma\|^3}.$$

Now we need only to find E_3 component of $T_\alpha^3\gamma$. Then,

$$T_\alpha^3\gamma(t) = \kappa v^3(\lambda^2 t)^{1-\alpha}\tau E_3 + ...$$

Consequently,

$$(T_\alpha\gamma \times T_\alpha^2\gamma)\cdot T_\alpha^3\gamma = \kappa^2 v^6(\lambda t)^{3(1-\alpha)}\tau.$$

Thus,

$$\begin{aligned}
\tau = \frac{(T_\alpha\gamma \times T_\alpha^2\gamma)\cdot T_\alpha^3\gamma}{\kappa^2 v^6(\lambda t)^{3(1-\alpha)}} &= \frac{(T_\alpha\gamma \times T_\alpha^2\gamma)\cdot T_\alpha^3\gamma}{(\kappa v^3(\lambda t^2)^{1-\alpha})^2}(\frac{\lambda}{t})^{1-\alpha}\\
&= (\frac{t}{\lambda})^{1-\alpha}\frac{(T_\alpha\gamma \times T_\alpha^2\gamma)\cdot T_\alpha^3\gamma}{\|T_\alpha\gamma \times T_\alpha^2\gamma\|^2}.
\end{aligned}$$

□

Remark 4.7. *Although the equations in the case of unit speed conformable curves are structurally identical with usual Frenet equations , arbitrary speed curves are different equations from usual case.*

Example 4.8. *We compute the $\alpha-$Frenet apparatus of the conformable curve*

$$\gamma(t) = \Big(3\cos(\frac{t^\alpha}{\alpha}), 3\sin(\frac{t^\alpha}{\alpha}), 4\frac{t^\alpha}{\alpha}\Big).$$

The $\alpha-$derivatives are

$$\begin{aligned} T_\alpha\gamma(t) =& \Big(-3\sin(\frac{t^\alpha}{\alpha}), 3\cos(\frac{t^\alpha}{\alpha}), 4\Big)\\ T_\alpha^2\gamma(t) =& \Big(-3\cos(\frac{t^\alpha}{\alpha}), -3\sin(\frac{t^\alpha}{\alpha}), 0\Big)\\ T_\alpha^3\gamma(t) =& \Big(3\sin(\frac{t^\alpha}{\alpha}), -3\cos(\frac{t^\alpha}{\alpha}), 0\Big). \end{aligned}$$

Then, $\|T_\alpha\gamma(t)\| = 5$. Applying the definition of cross product yields

$$T_\alpha\gamma(t) \times T_\alpha^2\gamma(t) = \Big(12\sin(\frac{t^\alpha}{\alpha}), -12\cos(\frac{t^\alpha}{\alpha}), 9\Big).$$

Hence, $\|T_\alpha\gamma(t) \times T_\alpha^2\gamma(t)\| = 15$. The expression above for $T_\alpha\gamma(t) \times T_\alpha^2\gamma(t)$ and $T_\alpha^3\gamma(t)$ yield

$$(T_\alpha\gamma(t) \times T_\alpha^2\gamma(t)) \cdot T_\alpha^3\gamma(t) = 6.$$

By Theorem 4.6,

$$\begin{aligned} E_1(t) =& \Big(-\frac{3}{5}\sin(\frac{t^\alpha}{\alpha}), \frac{3}{5}\cos(\frac{t^\alpha}{\alpha}), \frac{4}{5}\Big)\\ E_2(t) =& \Big(-\cos(\frac{t^\alpha}{\alpha}), -\sin(\frac{t^\alpha}{\alpha}), 0\Big)\\ E_3(t) =& \Big(\frac{12}{15}\sin(\frac{t^\alpha}{\alpha}), -\frac{12}{15}\cos(\frac{t^\alpha}{\alpha}), \frac{9}{15}\Big)\\ \kappa(t) =& \frac{3}{25}(\frac{t}{\lambda})^{1-\alpha}\\ \tau(t) =& \frac{4}{25}(\frac{t}{\lambda})^{1-\alpha}. \end{aligned}$$

5. Fundamental Theorem of the Local Theory of Conformable Curves

Theorem 5.1. *Given $\alpha-$differentiable functions $\kappa(s) > 0$ and $\tau(s)$, $s \in (0,\infty)$, there exists a conformable regular parametrized curve $\gamma : (0,\infty) \to \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature and $\tau(s)$ is the torsion of γ.*

Proof. The starting point is to observe that $\alpha-$Frenet equations

$$\begin{array}{rlll} T_\alpha E_1 = & & \kappa E_2 & \\ T_\alpha E_2 = & -\kappa E_1 & & +\tau E_3 \\ T_\alpha E_3 = & & -\tau E_2 & \end{array} \tag{8}$$

may be considered as a system of fractional differential equations in $(0,\infty) \times \mathbb{R}^9$,

$$\left.\begin{array}{rl} T_\alpha\rho_1(s) = & f_1(s, \rho_1, ..., \rho_9) \\ \vdots & \\ T_\alpha\rho_9(s) = & f_9(s, \rho_1, ..., \rho_9) \end{array}\right., \tag{9}$$

where $(\rho_1, \rho_2, \rho_3) = E_1$, $(\rho_4, \rho_5, \rho_6) = E_2$, $(\rho_7, \rho_8, \rho_9) = E_3$, and f_i, $i = 1, ..., 9$ are linear functions (with coefficients that depend on s) of the coordinates ρ_i.

Given initial conditions $s_0 \in (0,\infty)$, $\rho_1(s_0),...,\rho_9(s_0)$, there exists an open interval $\mathbf{I} \subset (0,\infty)$ containing s_0 and a unique $\alpha-$differentiable mapping $\gamma : \mathbf{I} \to \mathbb{R}^9$, with $\gamma(s_0) = (\rho_1(s_0),...,\rho_9(s_0))$ and $T_\alpha\gamma(s) = (f_1,...,f_9)$, where each f_i, $i = 1,...,9$ is calculated in $(s,\gamma(s)) \in \mathbf{I}\times\mathbb{R}^9$. Furthermore, if the system is linear, $\mathbf{I} = (0,\infty)$. It follows that given an orthonormal, positively oriented trihedron $\{E_1(s_0), E_2(s_0), E_3(s_0)\}$ in $\mathbb{R}^3$ and a value $s_0 \in (0,\infty)$, there exists a family of trihedrons $\{E_1(s), E_2(s), E_3(s)\}$, $s \in (0,\infty)$.
We shall first show that the family $\{E_1(s), E_2(s), E_3(s)\}$ thus obtained remains orthonormal for every $s \in (0,\infty)$. By using the system (8) to express the $\alpha-$derivatives relative to s of the six quantities

$$E_1\cdot E_2,\quad E_1\cdot E_3,\quad E_2\cdot E_3,\quad E_1\cdot E_1,\quad E_2\cdot E_2,\quad E_3\cdot E_3$$

as functions of these same quantities, we obtain that the system of $\alpha-$differential equations

$$\begin{aligned}
&T_\alpha(E_1(s)\cdot E_2(s)) = \kappa(s)E_2(s)\cdot E_2(s) - \kappa(s)E_1(s)\cdot E_1(s) + \tau(s)E_1(s)\cdot E_3(s),\\
&T_\alpha(E_1(s)\cdot E_3(s)) = \kappa(s)E_2(s)\cdot E_3(s) - \tau(s)E_1(s)\cdot E_2(s),\\
&T_\alpha(E_2(s)\cdot E_3(s)) = -\kappa(s)E_1(s)\cdot E_3(s) + \tau(s)E_3(s)\cdot E_3(s) - \tau(s)E_2(s)\cdot E_2(s),\\
&T_\alpha(E_1(s)\cdot E_1(s)) = 2\kappa(s)E_1(s)\cdot E_2(s),\\
&T_\alpha(E_2(s)\cdot E_2(s)) = -2\kappa(s)E_1(s)\cdot E_3(s) + 2\tau(s)E_3(s)\cdot E_3(s),\\
&T_\alpha(E_3(s)\cdot E_3(s)) = 2\tau(s)E_3(s)\cdot E_2(s).
\end{aligned}$$

It is easily checked that

$$E_1\cdot E_2 = 0,\quad E_1\cdot E_3 = 0,\quad E_2\cdot E_3 = 0,\quad E_1\cdot E_1 = 1,\quad E_2\cdot E_2 = 1,\quad E_3\cdot E_3 = 1$$

is a solution of above system with initial conditions $0,0,0,1,1,1$. By uniqueness, the family $\{E_1(s), E_2(s), E_3(s)\}$ is orthonormal for every $s \in (0,\infty)$, as we claimed.
From the family $\{E_1(s), E_2(s), E_3(s)\}$ it is possible to obtain a conformable curve by setting

$$\gamma(s) = \mathbf{I}_\alpha^{s_0} E_1(s).$$

□

Example 5.2. *Let $\alpha-$differentiable functions $\kappa(s) = 3$ and $\tau(s) = 4$. Then the associated fractional differential equations are*

$$\begin{array}{lccc}
T_\alpha E_1 = & & 3E_2 & \\
T_\alpha E_2 = & -3E_1 & & +4E_3. \\
T_\alpha E_3 = & & -4E_2 &
\end{array}$$

Let $T_\alpha E_1 = (e_{11}^{(\alpha)}, e_{12}^{(\alpha)}, e_{13}^{(\alpha)})$, $T_\alpha E_2 = (e_{21}^{(\alpha)}, e_{22}^{(\alpha)}, e_{23}^{(\alpha)})$ and $T_\alpha E_3 = (e_{31}^{(\alpha)}, e_{32}^{(\alpha)}, e_{33}^{(\alpha)})$ where $e_{ij}^{(\alpha)}$ is the $\alpha-$differentials of the coordinate functions of the $\alpha-$Frenet apparatus.
For simplicity, we choose $\alpha = 1/2$. Hence,

$$\begin{aligned}
e_{11}^{(1/2)}(s) &= 3e_{21}(s)\\
e_{12}^{(1/2)}(s) &= 3e_{22}(s)\\
e_{13}^{(1/2)}(s) &= 3e_{23}(s)\\
e_{21}^{(1/2)}(s) &= -3e_{11}(s) + 4e_{31}(s)\\
e_{22}^{(1/2)}(s) &= -3e_{12}(s) + 4e_{32}(s)\\
e_{23}^{(1/2)}(s) &= -3e_{13}(s) + 4e_{33}(s)\\
e_{31}^{(1/2)}(s) &= -4e_{21}(s)\\
e_{32}^{(1/2)}(s) &= -4e_{22}(s)\\
e_{33}^{(1/2)}(s) &= -4e_{23}(s),
\end{aligned} \tag{10}$$

and given initial conditions

$$\begin{aligned} E_1(\frac{\pi^2}{400}) =& (1,0,0) \\ E_2(\frac{\pi^2}{400}) =& (0,1,0) \\ E_3(\frac{\pi^2}{400}) =& (0,0,1). \end{aligned} \tag{11}$$

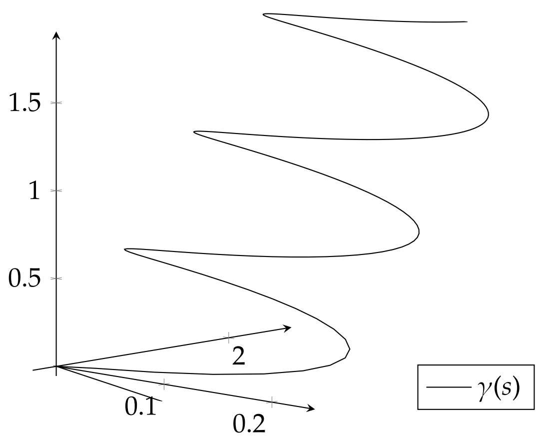
Then the solutions of the system of $\alpha-$*differential equations* (10) *are*

$$\begin{aligned} E_1(s) =& (\frac{9}{25}\sin 10\sqrt{s} + \frac{16}{25}, -\frac{3}{5}\cos 10\sqrt{s}, -\frac{12}{25}\sin 10\sqrt{s} + \frac{12}{25}) \\ E_2(s) =& (\frac{3}{5}\cos 10\sqrt{s}, \sin 10\sqrt{s}, -\frac{4}{5}\cos 10\sqrt{s}) \\ E_3(s) =& (-\frac{12}{25}\sin 10\sqrt{s} + \frac{12}{25}, \frac{4}{5}\cos 10\sqrt{s}, \frac{16}{25}\sin 10\sqrt{s} + \frac{9}{25}). \end{aligned}$$

Finally, the conformable curve corresponding to fractional system (10) *with initial conditions* (11) *is the curve*

$$\gamma(s) = \frac{1}{125}\Big(160\sqrt{s} - 9\cos 10\sqrt{s} - 8\pi, -15\sin 10\sqrt{s} + 15, 120\sqrt{s} + 12\cos 10\sqrt{s} - 6\pi\Big).$$

Graph of γ



6. Conclusions

In our study, the classical differential geometry of curves is investigated with respect to conformable fractional derivative and conformable fractional integral. In this sense, a conformable curve is defined and $\alpha-$Frenet formulas are given. If $\alpha = 1$, the theory of conformable curves coincides with the classical theory of curves. A conformable curve is a natural generalization of a classical curve. Studying this problem, we use conformable derivatives because it provides useful properties such as the product rule and chain rule compared to other fractional derivatives including Caputo and Riemann-Liouville. Thus the fundamental theorem of the local theory of the conformable curves is given. And, for $\alpha-$differentiable functions $\kappa > 0$ and τ, a system of fractional order differential equations is formed, then a conformable curve is obtained by the solution of this system.

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