Filomat 33:6 (2019), 1517–1530
https://doi.org/10.2298/FIL1906517M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by Jakimovski-Leviatan-Stancu-Durrmeyer Type Operators

M. Mursaleen[a], Shagufta Rahman[a], Khursheed J. Ansari[b]

[a]*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India*
[b]*Department of Mathematics, College of Science, King Khalid University, 61413, Abha, Saudi Arabia*

Abstract. In the present paper, we introduce Stancu type modification of Jakimovski-Leviatan-Durrmeyer operators. First, we estimate moments of these operators. Next, we study the problem of simultaneous approximation by these operators. An upper bound for the approximation to r^{th} derivative of a function by these operators is established. Furthermore, we obtain A-statistical approximation properties of these operators with the help of universal korovkin type statistical approximation theorem.

1. Introduction

In 1950, Szász [16] introduced the following operators

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1}$$

where $x \geq 0$ and $f \in C[0,\infty)$. In 1969, Jakimovski and Leviatan [8] gave a generalization of Szász operators by using Appell polynomials. Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the disk $|z| < R, R > 1$ and $g(1) \neq 0$. Appell polynomials $p_k(x)$ are defined by the generating function

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{2}$$

Jakimovski and Leviatan constructed the operators $P_n(f;x)$ by

$$P_n(f;x) = \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{3}$$

2010 *Mathematics Subject Classification*. 41A10; 41A28; 41A36.
Keywords. Durrmeyer operators; Jakimovski-Leviatan operators; Simultaneous approximation; Statistical approximation; Modulus of continuity.
Received: 22 September 2017; Revised: 22 April 2018; Accepted: 12 August 2018
Communicated by Dragan S. Djordjević
Email addresses: mursaleenm@gmail.com (M. Mursaleen), rahmanshagufta14@gmail.com (Shagufta Rahman), ansari.jkhursheed@gmail.com (Khursheed J. Ansari)

For $g(u) = 1$, we obtain the Appell polynomials $p_k(x) = \frac{x^k}{k!}$ and from (3) we meet Szász operators given by (1). In [9] Durrmeyer type modification of positive linear operators (3) is defined and their approximation properties is investigated. For more modifications of Szász and Durrmeyer type operators, one can see [5, 6, 10, 12–14, 18, 19].

In this paper, we study simultaneous and statistical approximation properties of Stancu type modification of Jakimovski-Leviatan-Durrmeyer operators [17] given by

$$L_{n,\alpha,\beta}(f;x) = \frac{e^{-b_n x}}{g(1)} \sum_{k=0}^{\infty} p_k(b_n x) \int_0^{\infty} v_{k,b_n}(t) f\Big(\frac{b_n t + \alpha}{b_n + \beta}\Big) dt, \tag{4}$$

where $0 \le \alpha \le \beta$ are two real parameters and

$$v_{k,b_n}(x) = \frac{1}{B(k+1, b_n)} \frac{x^k}{(1+x)^{b_n+k+1}}, n \in \mathbb{N}, \tag{5}$$

(b_n) is an increasing sequence of positive real numbers, $b_n \to \infty$ as $n \to \infty$. It is assumed that $b_1 + \beta \ge 1$. If we take $b_n = n, \alpha = \beta = 0$ then the operators (4) reduces to the operators given by

$$L_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \int_0^{\infty} v_{k,n}(t) f(t) dt. \tag{6}$$

2. Moments estimation

To obtain the moments of the operators (4) we need the following lemmas:

Lemma 2.1. *By* (2), *we obtain that*

$$\begin{aligned}\sum_{k=0}^{\infty} p_k(nx) =& e^{nx} g(1),\\ \sum_{k=0}^{\infty} k p_k(nx) =& e^{nx}[nxg(1) + g'(1)],\\ \sum_{k=0}^{\infty} k^2 p_k(nx) =& e^{nx}[n^2x^2 g(1) + nx(2g'(1) + g(1)) + g''(1) + g'(1)].\end{aligned}$$

Lemma 2.2. *Let $e_i(x) = x^i$, $i = 0, 1, 2$ and $L_n(.;.)$ be the sequence of operators given by* (6). *Then for each $x \ge 0$ and $b_n > 2$, the following equalities hold:*

1. $L_n(e_0;x) = 1$,
2. $L_n(e_1;x) = \frac{1}{b_n-1}\Big(b_n x + \frac{g'(1)}{g(1)} + 1\Big)$,
3. $L_n(e_2;x) = \frac{1}{(b_n-1)(b_n-2)}\Big\{b_n^2x^2 + b_n x\Big(2\frac{g'(1)}{g(1)} + 4\Big) + \frac{g''(1)}{g(1)} + 4\frac{g'(1)}{g(1)} + 2\Big\}$.

Lemma 2.3. *Let $e_i(x) = x^i$, $i = 0, 1, 2$ and $L_{n,\alpha,\beta}(.;.)$ be the sequence of operators given by* (4). *Then for each $x \ge 0$ and $b_n > 2$, we have the following equalities:*

1. $L_{n,\alpha,\beta}(e_0;x) = 1$,
2. $L_{n,\alpha,\beta}(e_1;x) = \frac{b_n^2 x}{(b_n+\beta)(b_n-1)} + \frac{b_n}{(b_n+\beta)(b_n-1)}\Big(\frac{g'(1)}{g(1)} + 1\Big) + \frac{\alpha}{(b_n+\beta)}$,
3. $L_{n,\alpha,\beta}(e_2;x) = \frac{b_n^4 x^2}{(b_n+\beta)^2(b_n-1)(b_n-2)} + \frac{b_n^2 x}{(b_n+\beta)^2(b_n-1)}\Big\{\frac{b_n}{(b_n-2)}\Big(2\frac{g'(1)}{g(1)} + 4\Big) + 2\alpha\Big\}$
$+ \frac{1}{(b_n+\beta)^2}\Big\{\frac{b_n^2}{(b_n-1)(b_n-2)}\Big(\frac{g''(1)}{g(1)} + 4\frac{g'(1)}{g(1)} + 2\Big) + 2\alpha\frac{b_n}{(b_n-1)}\Big(\frac{g'(1)}{g(1)} + 1\Big) + \alpha^2\Big\}$.

Proof. Using (4) and Lemma 2.2, the proof is established. □

3. Preliminary results

Let $f, f^{(1)}, f^{(2)}, \cdots, f^{(r)}$ be integrable, continuous and bounded functions on $[0,\infty)$. Now, we have the following lemmas:

Lemma 3.1. *Let*

$$T_{n,m} = \frac{e^{-b_nx}}{g(1)} \sum_{k=0}^{\infty} p_k(b_nx) \int_0^{\infty} v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta} - x\Big)^m dt, \tag{7}$$

where $m = 0,1,2,\cdots$; $r = 0,1,2,\cdots$ *and* v_{k,b_n} *are given by* (5)*, then*

$$T_{n,0} = 1, \tag{8}$$

$$T_{n,1} = \frac{b_n\Big(\frac{g'(1)}{g(1)} + (r+1)(x+1)\Big) + (\alpha-\beta x)(b_n-r-1)}{(b_n+\beta)(b_n-r-1)}, \; b_n > r+1 \tag{9}$$

and in general

$$\begin{aligned}\frac{b_n+\beta}{b_n}(b_n-m-r-1)T_{n,m+1} &= xT'_{n,m} + \Big\{\Big((m+r+1)+(\alpha-\beta x)\Big) + (2m+r+1)\Big(\frac{b_n+\beta}{b_n}\Big)\Big(x - \frac{\alpha}{b_n+\beta}\Big)\Big\}T_{n,m} \\ &\quad + m\Big(x-\frac{\alpha}{b_n+\beta}\Big)\Big(b_n(1+x)-(\alpha-\beta x)\Big)T_{n,m-1},\end{aligned}$$

for $b_n > m+r+1$.

Proof. Differentiating $T_{n,m}$, we get

$$\begin{aligned} T'_{n,m} &= \frac{1}{g(1)}\sum_{k=0}^{\infty}(e^{-b_nx}p_k(b_nx))' \int_0^{\infty} v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^m dt \\ &\quad + \frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty} p_k(b_nx)\int_0^{\infty} v_{k+r,b_n-r}(t)(-m)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^{m-1} dt \\ &= \frac{1}{g(1)}\sum_{k=0}^{\infty}(e^{-b_nx}p_k(b_nx))' \int_0^{\infty} v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^m dt - mT_{n,m-1}. \end{aligned}$$

Now,

$$xT'_{n,m} = \frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}(k-b_nx)p_k(b_nx)\int_0^{\infty} v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^m dt - mT_{n,m-1}.$$

Writing

$$k - b_nx = \Big((k+r)-(b_n-r+1)t\Big) + (b_n+\beta)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big) + (1-r)t + (\beta x - \alpha - r)$$

and

$$t = \frac{b_n+\beta}{b_n}\Big\{\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big) - \Big(\frac{\alpha}{b_n+\beta}-x\Big)\Big\}.$$

Since

$$t(1+t)v'_{k,b_n}(t) = \{k-(b_n+1)t\}v_{k,b_n}(t),$$

which implies that

$$t(1+t)v'_{k+r,b_n-r}(t) = \{(k+r)-(b_n-r+1)t\}v_{k+r,b_n-r}(t).$$

Therefore, we have

$$\begin{aligned}
xT'_{n,m} &= \frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}t(1+t)v'_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^m dt\\
&+(b_n+\beta)\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^{m+1} dt\\
&+(1-r)\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^{m} dt\\
&+(\beta x-\alpha-r)\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^{m} dt\\
&-mT_{n,m-1}.
\end{aligned}$$

Now, substitution for t and integration by parts give

$$\begin{aligned}
&\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}t(1+t)v'_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^m dt\\
=\ &(m+1)\Big\{2\Big(\frac{b_n+\beta}{b_n}\Big)\Big(\frac{\alpha}{b_n+\beta}-x\Big)-1\Big\}T_{n,m}-(m+2)\Big(\frac{b_n+\beta}{b_n}\Big)T_{n,m+1}\\
&-m\Big(\frac{\alpha}{b_n+\beta}-x\Big)\Big\{\Big(\frac{b_n+\beta}{b_n}\Big)\Big(\frac{\alpha}{b_n+\beta}-x\Big)-1\Big\}T_{n,m-1},
\end{aligned}$$

therefore,

$$\begin{aligned}
xT'_{n,m} &= \Big[(m+1)\Big\{2\Big(\frac{b_n+\beta}{b_n}\Big)\Big(\frac{\alpha}{b_n+\beta}-x\Big)-1\Big\}-(1-r)\Big(\frac{b_n+\beta}{b_n}\Big)\Big(\frac{\alpha}{b_n+\beta}-x\Big)+(\beta x-\alpha-r)\Big]T_{n,m}\\
&+\Big\{-(m+2)\Big(\frac{b_n+\beta}{b_n}\Big)+(1-r)\Big(\frac{b_n+\beta}{b_n}\Big)+(b_n+\beta)\Big\}T_{n,m+1}\\
&-\Big[m\Big(\frac{\alpha}{b_n+\beta}-x\Big)\Big\{\Big(\frac{b_n+\beta}{b_n}\Big)\Big(\frac{\alpha}{b_n+\beta}-x\Big)-1\Big\}\Big]T_{n,m-1}.
\end{aligned}$$

Rearranging the terms, we get

$$\begin{aligned}
\frac{b_n+\beta}{b_n}(b_n-m-r-1)T_{n,m+1} &= xT'_{n,m}+\Big\{\Big((m+r+1)+(\alpha-\beta x)\Big)+(2m+r+1)\Big(\frac{b_n+\beta}{b_n}\Big)\Big(x-\frac{\alpha}{b_n+\beta}\Big)\Big\}T_{n,m}\\
&+m\Big(x-\frac{\alpha}{b_n+\beta}\Big)\Big(b_n(1+x)-(\alpha-\beta x)\Big)T_{n,m-1}.
\end{aligned}$$

Put $m=0$ in (7), we have

$$\begin{aligned}
T_{n,0} &= \frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)dt\\
&= \frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\\
&= 1,
\end{aligned}$$

and again putting $m = 1$ in (7), we get

$$\begin{aligned} T_{n,1} &= \frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)dt \\ &= \frac{b_n}{b_n+\beta}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)tdt+\frac{\alpha}{b_n+\beta}-x \\ &= \frac{b_n}{b_n+\beta}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\Big(\frac{k+r+1}{b_n-r-1}\Big)+\frac{\alpha}{b_n+\beta}-x \\ &= \frac{b_n}{(b_n+\beta)(b_n-r-1)}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}kp_k(b_nx)+\frac{b_n(1+r)}{(b_n+\beta)(b_n-r-1)}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)+\frac{\alpha}{b_n+\beta}-x \\ &= \frac{b_n}{(b_n+\beta)(b_n-r-1)}\frac{1}{g(1)}\Big(b_nxg(1)+g'(1)\Big)+\frac{b_n(1+r)}{(b_n+\beta)(b_n-r-1)}+\frac{\alpha}{b_n+\beta}-x \\ &= \frac{b_n\Big(\frac{g\prime(1)}{g(1)}+(r+1)(x+1)\Big)+(\alpha-\beta x)(b_n-r-1)}{(b_n+\beta)(b_n-r-1)}. \end{aligned}$$

□

Remark 3.2. *Using* (8) *and* (9) *in* (7) *for* $m = 1$, *we have*

$$\begin{aligned} T_{n,2} &= \frac{b_n}{(b_n+\beta)^2(b_n-r-1)(b_n-r-2)}\Bigg[\Big\{(b_n+\beta)\Big((r+1)(r+4)+(r+3)\frac{g'(1)}{g(1)} \\ &+\frac{\alpha}{b_n}(r+3)(b_n-r-1)+(b_n-2\alpha)(b_n-r-1)\Big)-\beta b_n\Big(r+2+\frac{g'(1)}{g(1)}+\frac{\alpha}{b_n}(b_n-r-1)\Big) \\ &+\Big((\alpha+r+2)-\frac{\alpha}{b_n}(r+3)\Big)\Big(b_n(1+r)-\beta(b_n-r-1)\Big)\Big\}x \\ &+\Big\{\frac{1}{b_n}\Big((b_n+\beta)^2(r^2+4r+3)-2\beta b_n(b_n+\beta)(r+2)+\beta^2b_n^2\Big)+(b_n+\beta)^2\Big\}x^2 \\ &+\Big(\frac{g'(1)}{g(1)}+(1+r)\Big)\Big(b_n(\alpha+r+2)-\alpha(r+3)\Big)+\frac{\alpha(b_n-r-1)}{b_n}\Big(b_n(\alpha+r+2)-\alpha(r+3)-\alpha b_n(b_n-\alpha)\Big)\Bigg]. \end{aligned}$$

Lemma 3.3. *For* $r = 0, 1, 2, \ldots,$

$$L^{(r)}_{n,\alpha,\beta}(f;x) = \frac{(b_n)^r(b_n-r-1)!}{(b_n-1)!}\Big(\frac{b_n}{b_n+\beta}\Big)^r\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)f^{(r)}\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)dt. \quad (10)$$

Proof. Differentiating (4) r-times, we have

$$L^{(r)}_{n,\alpha,\beta}(f;x) = \frac{1}{g(1)}\sum_{k=0}^{\infty}(e^{-b_nx}p_k(b_nx))^{(r)}\int_0^{\infty}v_{k,b_n}(t)f\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)dt.$$

By Leibnitz theorem, we get

$$\begin{aligned} L^{(r)}_{n,\alpha,\beta}(f;x) &= \frac{1}{g(1)}\sum_{i=0}^{r}\sum_{k=i}^{\infty}\binom{r}{i}(-1)^{r-i}(b_n)^re^{-b_nx}p_{k-i}(b_nx)\int_0^{\infty}v_{k,b_n}(t)f\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)dt \\ &= \frac{1}{g(1)}\sum_{k=0}^{\infty}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}(b_n)^re^{-b_nx}p_k(b_nx)\int_0^{\infty}v_{k+i,b_n}(t)f\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)dt \\ &= \frac{(b_n)^re^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}(-1)^rp_k(b_nx)\int_0^{\infty}\Big(\sum_{i=0}^{r}\binom{r}{i}(-1)^iv_{k+i,b_n}(t)\Big)f\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)dt. \end{aligned}$$

Again by Leibnitz theorem, we have

$$v^{(r)}_{k+r,b_n-r}(t)=\frac{(b_n-1)!}{(b_n-r-1)!}\sum_{i=0}^{r}\binom{r}{i}(-1)^i v_{k+i,b_n}(t).$$

Hence,

$$L^{(r)}_{n,\alpha,\beta}(f;x)=\frac{(b_n)^r(b_n-r-1)!}{(b_n-1)!}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}(-1)^r v^{(r)}_{k+r,b_n-r}(t)f\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)dt.$$

Further, integrating by parts r-times, we have

$$L^{(r)}_{n,\alpha,\beta}(f;x)=\frac{(b_n)^r(b_n-r-1)!}{(b_n-1)!}\Big(\frac{b_n}{b_n+\beta}\Big)^r\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)f^{(r)}\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)dt.$$

□

4. Main results

Theorem 4.1. *If f is integrable in $[0,\infty)$, admits its $(r+1)^{th}$ and $(r+2)^{th}$ derivatives, which are bounded on $[0,\infty)$, and $f^{(r)}(x)=O(x^{\xi})$ (ξ is a positive integer≥2) as $x\to\infty$, then*

$$\lim_{n\to\infty}(b_n+\beta)\Big(L^{(r)}_{n,\alpha,\beta}(f;x)-f^{(r)}(x)\Big)=\Big(\frac{g'(1)}{g(1)}+(r+1)(x+1)+(\alpha-\beta x)\Big)f^{(r+1)}(x)+\frac{x^2}{2}f^{(r+2)}(x).$$

Proof. By Taylor's formula, we can write

$$f^{(r)}\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)-f^{(r)}(x)=\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)f^{(r+1)}(x)+\frac{1}{2}\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2f^{(r+2)}(x)+\frac{1}{2}\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2\eta\Big(\frac{b_nt+\alpha}{b_n+\beta},x\Big),$$

where

$$\eta\Big(\frac{b_nt+\alpha}{b_n+\beta},x\Big)=\frac{f^{(r)}\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)-f^{(r)}(x)-\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)f^{(r+1)}(x)-\frac{1}{2}\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2f^{(r+2)}(x)}{\frac{1}{2}\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2},$$

if $\frac{b_nt+\alpha}{b_n+\beta}\neq x$, and $\eta(x,x)=0$. Also, for an arbitrary $\varepsilon>0$, $A>0$ $\exists$ a $\delta>0$ such that $\Big|\eta\Big(\frac{b_nt+\alpha}{b_n+\beta},x\Big)\Big|\leq\varepsilon$ for $\Big|\frac{b_nt+\alpha}{b_n+\beta}-x\Big|<\delta$, $x\leq A$.

Since

$$\frac{(b_n-1)!}{(b_n)^r(b_n-r-1)!}\Big(\frac{b_n+\beta}{b_n}\Big)^rL^{(r)}_{n,\alpha,\beta}(f;x)-f^{(r)}(x)=\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\Big(f^{(r)}\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)-f^{(r)}(x)\Big),$$

using Taylor's formula, we have

$$\begin{aligned}\frac{(b_n-1)!}{(b_n)^r(b_n-r-1)!}\Big(\frac{b_n+\beta}{b_n}\Big)^rL^{(r)}_{n,\alpha,\beta}(f;x)-f^{(r)}(x) &= L_{n,\alpha,\beta}\Big(\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big);x\Big)f^{(r+1)}(x)\\ &+\frac{1}{2}L_{n,\alpha,\beta}\Big(\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2;x\Big)f^{(r+2)}(x)\\ &+\frac{1}{2}L_{n,\alpha,\beta}\Big(\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2\eta\Big(\frac{b_nt+\alpha}{b_n+\beta},x\Big);x\Big)\\ &=T_{n,1}f^{(r+1)}(x)+\frac{1}{2}T_{n,2}f^{(r+2)}(x)+E_{n,r},\end{aligned}$$

where

$$E_{n,r} = \frac{1}{2}L_{n,\alpha,\beta}\Big(\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2\eta\Big(\frac{b_nt+\alpha}{b_n+\beta},x\Big);x\Big).$$

We shall show that $(b_n+\beta)E_{n,r}\to 0$ as $n\to\infty$. Let

$$R_{n,r,1} = \frac{b_n+\beta}{2}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_{|\frac{b_nt+\alpha}{b_n+\beta}-x|<\delta}v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2\eta\Big(\frac{b_nt+\alpha}{b_n+\beta},x\Big)dt$$

and

$$R_{n,r,2} = \frac{b_n+\beta}{2}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_{|\frac{b_nt+\alpha}{b_n+\beta}-x|\geq\delta}v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2\eta\Big(\frac{b_nt+\alpha}{b_n+\beta},x\Big)dt,$$

then $(b_n+\beta)E_{n,r} = R_{n,r,1}+R_{n,r,2}$. We have

$$\begin{aligned}|R_{n,r,1}| &\leq \varepsilon\frac{b_n+\beta}{2}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_{|\frac{b_nt+\alpha}{b_n+\beta}|<\delta}v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2dt\\ &\leq \varepsilon\frac{x^2}{2}\end{aligned}$$

as $n\to\infty$. It is assumed that $f^{(r)}(t) = O\Big[\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)^{\xi}\Big]$ for some $\xi\geq 2$ as $t\to\infty$, $f^{(r+1)}$ and $f^{(r+2)}$ are bounded on $[0,\infty)$, we have

$$\begin{aligned}R_{n,r,2} &= O\Big(\frac{b_n+\beta}{2}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_{|\frac{b_nt+\alpha}{b_n+\beta}-x|\geq\delta}v_{k+r,b_n-r}(t)\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)^{\xi}dt\Big)\\ &= O\Big(\frac{b_n+\beta}{2}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_{|\frac{b_nt+\alpha}{b_n+\beta}|\geq\delta}v_{k+r,b_n-r}(t)\Big(\sum_{i=0}^{\xi}\binom{\xi}{i}\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^{i}x^{\xi-i}\Big)dt\Big)\\ &= O\Big(\frac{b_n+\beta}{2}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\frac{\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^3}{\delta^3}\Big(\sum_{i=0}^{\xi}\binom{\xi}{i}\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^{i}x^{\xi-i}\Big)dt\Big)\\ &= O\Big(\frac{b_n+\beta}{2\delta^3}\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\Big(\sum_{i=0}^{\xi}\binom{\xi}{i}\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^{i+3}x^{\xi-i}\Big)dt\Big).\end{aligned}$$

Hence

$$\begin{aligned}R_{n,r,2} &= O\Big(\frac{b_n+\beta}{2\delta^3}\sum_{i=0}^{\gamma}\binom{\gamma}{i}x^{\gamma-i}T_{n,i+3}\Big)\\ &= O\Big(\frac{1}{b_n+\beta}\Big).\end{aligned}$$

Therefore, $(b_n+\beta)E_{n,r} = R_{n,r,1}+R_{n,r,2}\to 0$ and

$$(b_n+\beta)\Big[L^{(r)}_{n,\alpha,\beta}(f;x)-f^{(r)}(x)\Big]\to\Big(\frac{g\prime(1)}{g(1)}+(r+1)(x+1)+(\alpha-\beta x)\Big)f^{(r+1)}(x)+\frac{x^2}{2}f^{(r+2)}(x)$$

as $n\to\infty$. □

Theorem 4.2. *Let $f \in C^{r+1}[0,a]$ and let $\omega(f^{r+1};.)$ be the modulus of continuity of f^{r+1}. Then for r=1,2,. . . ,*

$$\begin{aligned}\left\|\frac{(b_n-1)!}{(b_n)^r(b_n-r-1)!}\left(\frac{b_n+\beta}{b_n}\right)^r L^{(r)}_{n,\alpha,\beta}(f;x)-f^{(r)}(x)\right\| \leq\ & \frac{1}{(b_n+\beta)(b_n-r-1)}\Big\{b_n\Big(\frac{g\prime(1)}{g(1)}+(r+1)(a+1)\Big)\\ &+(\alpha+\beta a)(b_n-r-1)\Big\}\|f^{r+1}\|+\Big\{\sqrt{\lambda}+\frac{\lambda}{2}\Big\}\\ &\times\omega\Big(f^{r+1};\frac{b_n}{(b_n+\beta)^2(b_n-r-1)(b_n-r-2)}\Big),\end{aligned}$$

where

$$\begin{aligned}\lambda =&\Big\{(b_n+\beta)\Big((r+1)(r+4)+(r+3)\frac{g\prime(1)}{g(1)}+\frac{\alpha}{b_n}(r+3)(b_n-r-1)+(b_n-\alpha)(b_n-r-1)\Big)\\ &+\beta b_n\Big(r+2+\frac{g\prime(1)}{g(1)}+\frac{\alpha}{b_n}(b_n-r-1)\Big)+\Big((\alpha+r+2)+\frac{\alpha}{b_n}(r+3)\Big)\Big(b_n(1+r)+\beta(b_n-r-1)\Big)\Big\}a\\ &+\Big\{\frac{1}{b_n}\Big((b_n+\beta)^2(r^2+4r+3)+2\beta b_n(b_n+\beta)(r+2)+\beta^2b_n^2\Big)+(b_n+\beta)^2\Big\}a^2\\ &+\Big(\frac{g\prime(1)}{g(1)}+(1+r)\Big)\Big(b_n(\alpha+r+2)+\alpha(r+3)\Big)+\frac{\alpha(b_n-r-1)}{b_n}\Big(b_n(\alpha+r+2)+\alpha(r+3)+\alpha b_n(b_n-\alpha)\Big),\end{aligned}$$

$\alpha \leq r-1$, $b_n > r+2$ and $\|.\|$ is the supremum norm over $[0,a]$, $a > 0$.

Proof. By Taylor's formula, we have

$$f^{(r)}\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)-f^{(r)}(x)=\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)f^{(r+1)}(x)+\int_x^{\frac{b_nt+\alpha}{b_n+\beta}}\Big(f^{(r+1)}(y)-f^{(r+1)}(x)\Big)dy. \tag{11}$$

Now,

$$\frac{(b_n-1)!}{(b_n)^r(b_n-r-1)!}\Big(\frac{b_n+\beta}{b_n}\Big)^r L^{(r)}_{n,\alpha,\beta}(f;x)-f^{(r)}(x)=\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\Big(f^{(r)}\Big(\frac{b_nt+\alpha}{b_n+\beta}\Big)-f^{(r)}(x)\Big).$$

Using (11), we get

$$\begin{aligned}\left|\frac{(b_n-1)!}{(b_n)^r(b_n-r-1)!}\Big(\frac{b_n+\beta}{b_n}\Big)^r L^{(r)}_{n,\alpha,\beta}(f;x)-f^{(r)}(x)\right| \leq\ & |T_{n,1}||f^{(r+1)}(x)|+\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\\ &\times\left|\int_x^{\frac{b_nt+\alpha}{b_n+\beta}}\omega(f^{(r+1)};|y-x|)dy\right|dt\\ \leq\ & |T_{n,1}||f^{(r+1)}(x)|+\omega(f^{(r+1)};\delta)\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\\ &\times\left|\int_x^{\frac{b_nt+\alpha}{b_n+\beta}}\Big(1+\frac{|y-x|}{\delta}\Big)dy\right|dt.\end{aligned}$$

The Cauchy-Schwartz inequality implies

$$\begin{aligned}\left|\frac{(b_n-1)!}{(b_n)^r(b_n-r-1)!}\Big(\frac{b_n+\beta}{b_n}\Big)^r L^{(r)}_{n,\alpha,\beta}(f;x)-f^{(r)}(x)\right| \leq\ & |T_{n,1}||f^{(r+1)}(x)|+\omega(f^{(r+1)};\delta)\frac{e^{-b_nx}}{g(1)}\sum_{k=0}^{\infty}p_k(b_nx)\int_0^{\infty}v_{k+r,b_n-r}(t)\\ &\times\Big[\Big|\frac{b_nt+\alpha}{b_n+\beta}-x\Big|+\frac{1}{2\delta}\Big(\frac{b_nt+\alpha}{b_n+\beta}-x\Big)^2\Big]dt\\ \leq\ & |T_{n,1}||f^{(r+1)}(x)|+\omega(f^{(r+1)};\delta)\Big\{\sqrt{T_{n,2}}+\frac{T_{n,2}}{2\delta}\Big\}.\end{aligned}$$

Choosing $\delta = \frac{b_n}{(b_n+\beta)^2(b_n-r-1)(b_n-r-2)}$, using remark (3.2), (9) and taking the supremum norm on [0,a], $a > 0$, we get the result. □

5. Statistical approximation

There is another notion of convergence known as the statistical convergence which was introduced by Fast [2] and Steinhaus [15]. In approximation theory, the concept of statistical convergence was used in the year 2002 by Gadjiev and Orhan [4]. They proved the Bohman-Korovkin type approximation theorem for statistical convergence.
In this section we obtain Korovkin type theorem for A-statistical convergence and weighted A-statistical convergence of the operators defined in (4). Recently, the statistical approximation properties have also been investigated for several operators (see [1, 11]).

Let $A := (a_{kn})$, be an infinite summability matrix. For a given sequence $x := (x_n)$, the A-transform of x is denoted by $Ax := ((Ax)_k)$, is given by $(Ax)_k = \sum\limits_{n=1}^{\infty} a_{kn}x_n$ provided the series converges for each n.

A is said to be regular if $\lim\limits_n (Ax)_n = L$ whenever $\lim\limits_n x_n = L$ (see [7]).

If $A = (a_{kn})$ is a non-negative regular summability matrix, then we say that a sequence $x := (x_n)$, is A-statistically convergent to L provided that for every $\varepsilon > 0$

$$\lim_{k\to\infty} \sum_{n:|x_n-L|\geq\epsilon} a_{kn} = 0. \tag{12}$$

In this case we write $st_A - \lim x = L$.

If $A = C_1$, the Cesáro matrix of order one then A-statistical convergence reduces to the statistical convergence (see [2, 3]). Further, If A is the identity matrix, then A-statistical convergence coincide with the ordinary convergence.

Theorem 5.1. *Let $A = (a_{nk})$ be a non negative regular summability matrix and (b_n) an increasing sequence of positive real numbers, $b_n \to \infty$ as $n \to \infty$. Then for every $f \in C[0, v] \subset C[0, \infty)$, we have*

$$st_A - \lim_n \|L_{n,\alpha,\beta}(f; x) - f(x)\| = 0,$$

uniformly with respect to $x \in [0, v]$ with $v > 0$.

Proof. From [[1],p-191,Th.3], it is enough to show that $st_A - \lim\limits_n \|L_{n,\alpha,\beta}(e_i; x) - e_i(x)\| = 0$ where $e_i(x) = x^i$, $i = 0, 1, 2$. Using $L_{n,\alpha,\beta}(e_0; x) = 1$, it is clear that

$$st_A - \lim_n \|L_{n,\alpha,\beta}(e_0; x) - e_0(x)\| = 0.$$

Now by Lemma 2.3 (2), we have

$$\begin{aligned} \|L_{n,\alpha,\beta}(e_1; x) - e_1(x)\| &= \left\| \frac{b_n^2 x}{(b_n+\beta)(b_n-1)} + \frac{b_n}{(b_n+\beta)(b_n-1)}\left(\frac{g'(1)}{g(1)} + 1\right) + \frac{\alpha}{(b_n+\beta)} - x \right\| \\ &\leq \frac{b_n - \beta(1-b_n)x}{(b_n+\beta)(b_n-1)} + \frac{b_n\left(\frac{g'(1)}{g(1)} + 1\right) + \alpha(b_n-1)}{(b_n+\beta)(b_n-1)}. \end{aligned}$$

For given $\varepsilon > 0$, we define the following sets

$$S_1 := \Big\{ n : \|L_{n,\alpha,\beta}(e_1;x) - e_1(x)\| \geq \varepsilon \Big\},$$
$$S_2 := \left\{ n : \frac{b_n - \beta(1-b_n)x + b_n\left(\frac{g'(1)}{g(1)} + 1\right) + \alpha(b_n - 1)}{(b_n+\beta)(b_n-1)} \geq \varepsilon \right\},$$

it is obvious that $S_1 \subset S_2$ which implies that $\sum\limits_{n\in S_1} a_{nk} \leq \sum\limits_{n\in S_2} a_{nk} = 0$ and hence

$$st_A - \lim_n \|L_{n,\alpha,\beta}(e_1;x) - e_1(x)\| = 0.$$

Similarly by Lemma 2.3 (3), we have

$$\begin{aligned}
\|L_{n,\alpha,\beta}(e_2;x) - e_2(x)\| &= \Bigg\| \left(\frac{b_n^4}{(b_n+\beta)^2(b_n-1)(b_n-2)} - 1 \right) x^2 + \frac{b_n^2 x}{(b_n+\beta)^2(b_n-1)} \left\{ \frac{b_n}{(b_n-2)} \left(2\frac{g'(1)}{g(1)} + 4 \right) + 2\alpha \right\} \\
&\quad + \frac{1}{(b_n+\beta)^2} \left\{ \frac{b_n^2}{(b_n-1)(b_n-2)} \left(\frac{g''(1)}{g(1)} + 4\frac{g'(1)}{g(1)} + 2 \right) + 2\alpha \frac{b_n}{(b_n-1)} \left(\frac{g'(1)}{g(1)} + 1 \right) + \alpha^2 \right\} \Bigg\| \\
&\leq \left| \left(\frac{b_n^4}{(b_n+\beta)^2(b_n-1)(b_n-2)} - 1 \right) \right| v^2 + \left| \frac{b_n^2}{(b_n+\beta)^2(b_n-1)} \left\{ \frac{b_n}{(b_n-2)} \left(2\frac{g'(1)}{g(1)} + 4 \right) + 2\alpha \right\} \right| v \\
&\quad + \left| \frac{1}{(b_n+\beta)^2} \left\{ \frac{b_n^2}{(b_n-1)(b_n-2)} \left(\frac{g''(1)}{g(1)} + 4\frac{g'(1)}{g(1)} + 2 \right) + 2\alpha \frac{b_n}{(b_n-1)} \left(\frac{g'(1)}{g(1)} + 1 \right) + \alpha^2 \right\} \right| \\
&\leq \mu^2 \Bigg[\left(\frac{b_n^4}{(b_n+\beta)^2(b_n-1)(b_n-2)} - 1 \right) + \frac{b_n^2}{(b_n+\beta)^2(b_n-1)} \left\{ \frac{b_n}{(b_n-2)} \left(2\frac{g'(1)}{g(1)} + 4 \right) + 2\alpha \right\} \\
&\quad + \frac{1}{(b_n+\beta)^2} \left\{ \frac{b_n^2}{(b_n-1)(b_n-2)} \left(\frac{g''(1)}{g(1)} + 4\frac{g'(1)}{g(1)} + 2 \right) + 2\alpha \frac{b_n}{(b_n-1)} \left(\frac{g'(1)}{g(1)} + 1 \right) + \alpha^2 \right\} \Bigg]
\end{aligned}$$

where $\mu^2 = \max\{1, v, v^2\}$.
Choose

$$\begin{aligned}
\alpha_n &= \frac{b_n^4}{(b_n+\beta)^2(b_n-1)(b_n-2)} - 1, \\
\beta_n &= \frac{b_n^2}{(b_n+\beta)^2(b_n-1)} \left\{ \frac{b_n}{(b_n-2)} \left(2\frac{g'(1)}{g(1)} + 4 \right) + 2\alpha \right\}, \\
\gamma_n &= \frac{1}{(b_n+\beta)^2} \left\{ \frac{b_n^2}{(b_n-1)(b_n-2)} \left(\frac{g''(1)}{g(1)} + 4\frac{g'(1)}{g(1)} + 2 \right) \right. \\
&\quad \left. + \frac{2\alpha b_n}{(b_n-1)} \left(\frac{g'(1)}{g(1)} + 1 \right) + \alpha^2 \right\}.
\end{aligned}$$

Hence

$$\|L_{n,\alpha,\beta}(e_2;x) - e_2(x)\| \leq \mu^2(\alpha_n + \beta_n + \gamma_n).$$

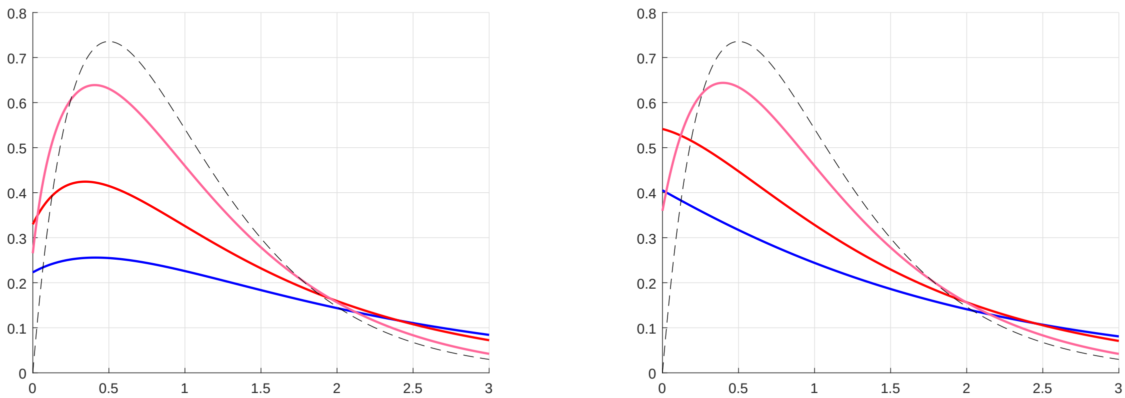
Now for given $\varepsilon > 0$, we define the following four sets

$$S_3 := \left\{ n : \|L_{n,\alpha,\beta}(e_2; x) - e(x)\| \geq \varepsilon \right\},$$
$$S_4 := \left\{ n : \alpha_n \geq \frac{\varepsilon}{3\mu^2} \right\},$$
$$S_5 := \left\{ n : \beta_n \geq \frac{\varepsilon}{3\mu^2} \right\},$$
$$S_6 := \left\{ n : \gamma_n \geq \frac{\varepsilon}{3\mu^2} \right\}.$$

It is obvious that $S_3 \subset S_4 \cup S_5 \cup S_6$, which implies that $\sum\limits_{n \in S_3} a_{nk} \leq \sum\limits_{n \in S_4} a_{nk} + \sum\limits_{n \in S_5} a_{nk} + \sum\limits_{n \in S_6} a_{nk} = 0$ and hence

$$st_A - \lim_n \|L_{n,\alpha,\beta}(e_2; x) - e_2(x)\| = 0.$$

□



(a) (b)

Figure 1: The convergence of $L_{n,\alpha,\beta}(f; x)$ when $\alpha = \beta = 0$, taking $g(u) = e^u$ *(left)* and $g(u) = u$ *(right)*.

For $g(u) = e^u$ and $g(u) = u$, the convergence of $L_{n,\alpha,\beta}(f; x)$ defined by (1.5) to $f(x) = 4xe^{-2x}$ is illustrated in Figure 1 and Figure 2, respectively, where $b_n = \sqrt{n}$ and $n = 4(blue), 16(red), 256(pink)$, the function $f(x)$ is plotted with dashed line.

A real function ρ is called a weight function if it is continuous on $\mathbb{R}$ and $\lim\limits_{|x| \to \infty} \rho(x) = \infty$, $\rho(x) \geq 1$ for all $x \in \mathbb{R}$. Let $B_\rho(\mathbb{R})$ be the weighted space of real-valued functions f defined on $\mathbb{R}$ with the property $|f(x)| \leq M_f \rho(x)$ for all $x \in \mathbb{R}$, where M_f is a constant depending on the functions f. Introduce

$$C_\rho(\mathbb{R}) = \left\{ f \in B_\rho(\mathbb{R}) : f \text{ is continuous on } \mathbb{R} \right\}.$$

Clearly, $C_\rho(\mathbb{R})$ is a subspace of $B_\rho(\mathbb{R})$. Note that $B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are Banach spaces with $\|f\|_\rho = \sup\limits_{x \in \mathbb{R}} \frac{|f(x)|}{\rho(x)}$. In case of weight function $\rho(x) = 1 + x^2$, we have $\|f\|_\rho = \sup\limits_{x \in \mathbb{R}} \frac{|f(x)|}{1+x^2}$. In the following result we prove a weighted Korovkin theorem via A-statistical convergence.

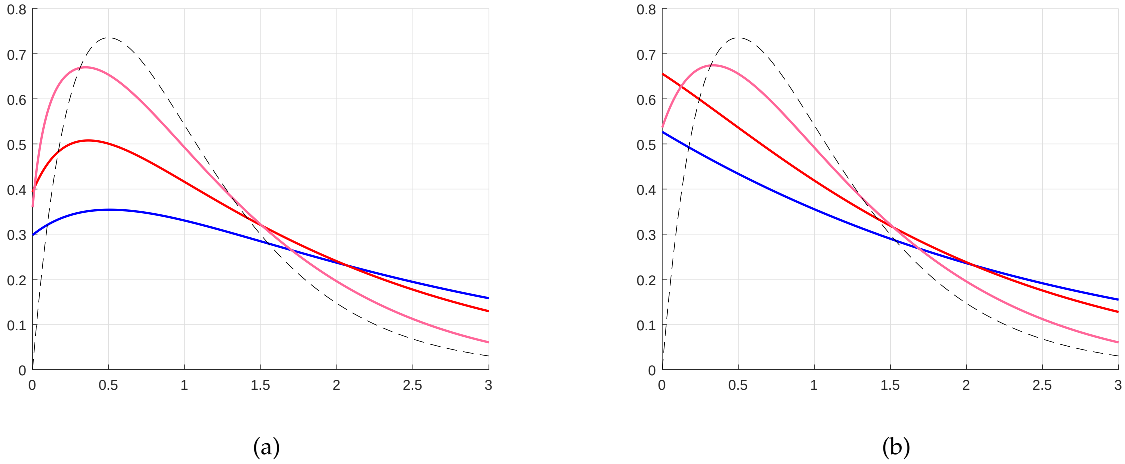


Figure 2: The convergence of $L_{n,\alpha,\beta}(f;x)$ to $f(x)$ when $\alpha = 2, \beta = 3$, taking $g(u) = e^u$ *(left)* and $g(u) = u$ *(right)*.

Theorem 5.2. *Let $A = (a_{nk})$ be a non negative regular summability matrix and (b_n) an increasing sequence of positive real numbers, $b_n \to \infty$ as $n \to \infty$. Then for every $f \in C[0,\infty)$ we have*

$$st_A - \lim_n \|L_{n,\alpha,\beta}(f;x) - f(x)\|_\rho = 0$$

where $\rho(x) = 1 + x^2$.

Proof. Let $e_i(t) = t^i$, where $i = 0, 1, 2$. Using $L_{n,\alpha,\beta}(e_0;x) = 1$, it is clear that

$$st_A - \lim_n \|L_{n,\alpha,\beta}(e_0;x) - e_0(x)\|_\rho = 0.$$

Now by Lemma 2.3 (2), we have

$$\begin{aligned}\lim_n \|L_{n,\alpha,\beta}(e_1;x) - e_1(x)\|_\rho &= \sup_{x\in[0,\infty)} \left| \frac{b_n - \beta(1-b_n)}{(b_n+\beta)(b_n-1)} \frac{x}{1+x^2} + \frac{b_n}{(b_n+\beta)(b_n-1)} \left(1 + \frac{g'(1)}{g(1)} + \frac{\alpha(b_n-1)}{b_n}\right) \frac{1}{1+x^2} \right| \\ &\leq \frac{b_n - \beta(1-b_n)}{(b_n+\beta)(b_n-1)} + \frac{b_n}{(b_n+\beta)(b_n-1)} \left(1 + \frac{g'(1)}{g(1)} + \frac{\alpha(b_n-1)}{b_n}\right).\end{aligned}$$

Let

$$\begin{aligned}U_1 &:= \left\{ n : \|L_{n,\alpha,\beta}(e_1;x) - e_1(x)\|_\rho \geq \varepsilon \right\}, \\ U_2 &:= \left\{ n : \frac{1}{(b_n+\beta)(b_n-1)} \left(b_n - \beta(1-b_n) + b_n\left(\frac{g'(1)}{g(1)} + 1\right) + \alpha(b_n-1) \right) \geq \varepsilon \right\}.\end{aligned}$$

Then we obtain $U_1 \subset U_2$ which implies that $\sum_{n\in U_1} a_{nk} \leq \sum_{n\in U_2} a_{nk} = 0$ and hence

$$st_A - \lim_n \|L_{n,\alpha,\beta}(e_1;x) - e_1(x)\|_\rho = 0.$$

Similarly by Lemma 2.3 (3), we have

$$\begin{aligned}
\|L_{n,\alpha,\beta}(e_2;x)-e_2(x)\|_\rho &= \sup_{x\in[0,\infty)}\frac{1}{1+x^2}\Bigg|\Big(\frac{b_n^4}{(b_n+\beta)^2(b_n-1)(b_n-2)}-1\Big)x^2+\frac{b_n^2x}{(b_n+\beta)^2(b_n-1)}\\
&\quad\times\Big\{\frac{b_n}{(b_n-2)}\Big(2\frac{g'(1)}{g(1)}+4\Big)+2\alpha\Big\}+\frac{1}{(b_n+\beta)^2}\Big\{\frac{b_n^2}{(b_n-1)(b_n-2)}\Big(\frac{g''(1)}{g(1)}+4\frac{g'(1)}{g(1)}+2\Big)\\
&\quad+2\alpha\frac{b_n}{(b_n-1)}\Big(\frac{g'(1)}{g(1)}+1\Big)+\alpha^2\Big\}\Bigg|\\
&\le \Big(\frac{b_n^4}{(b_n+\beta)^2(b_n-1)(b_n-2)}-1\Big)+\frac{b_n^2}{(b_n+\beta)^2(b_n-1)}\Big\{\frac{b_n}{(b_n-2)}\Big(2\frac{g'(1)}{g(1)}+4\Big)+2\alpha\Big\}\\
&\quad+\frac{1}{(b_n+\beta)^2}\Big\{\frac{b_n^2}{(b_n-1)(b_n-2)}\Big(\frac{g''(1)}{g(1)}+4\frac{g'(1)}{g(1)}+2\Big)+2\alpha\frac{b_n}{(b_n-1)}\Big(\frac{g'(1)}{g(1)}+1\Big)+\alpha^2\Big\}.
\end{aligned}$$

Now, for given $\varepsilon>0$, we define the following sets:

$$\begin{aligned}
U_3 &:=\Big\{n:\|L_{n,\alpha,\beta}(e_2;x)-e(x)\|_\rho\ge\varepsilon\Big\},\\
U_4 &:=\Big\{n:\frac{b_n^4}{(b_n+\beta)^2(b_n-1)(b_n-2)}-1\ge\frac{\varepsilon}{3}\Big\},\\
U_5 &:=\Big\{n:\frac{b_n^2}{(b_n+\beta)^2(b_n-1)}\Big[\frac{b_n}{(b_n-2)}\Big(2\frac{g'(1)}{g(1)}+4\Big)+2\alpha\Big]\ge\frac{\varepsilon}{3}\Big\},\\
U_6 &:=\Big\{n:\frac{1}{(b_n+\beta)^2}\Big[\frac{b_n^2}{(b_n-1)(b_n-2)}\Big(\frac{g''(1)}{g(1)}+4\frac{g'(1)}{g(1)}+2\Big)\\
&\qquad+\frac{2}{\alpha b_n}(b_n-1)\Big(\frac{g'(1)}{g(1)}+1\Big)+\alpha^2\Big]\ge\frac{\varepsilon}{3}\Big\}.
\end{aligned}$$

It is clear that $U_3\subset U_4\cup U_5\cup U_6$, which implies that $\sum_{n\in U_3}a_{nk}\le\sum_{n\in U_4}a_{nk}+\sum_{n\in U_5}a_{nk}+\sum_{n\in U_6}a_{nk}=0$ and hence

$$st_A-\lim_n\|L_{n,\alpha,\beta}(e_2;x)-e_2(x)\|_\rho=0$$

which completes the proof. □

Acknowledgements

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups program under Grant Number R.G.P.1/79/40

References

[1] O. Duman, C. Orhan, Statistical approximation by positive linear operators, Studia Mathmatica 161 (2004), 187–197.
[2] H. Fast, Sur la convergence statistique, Colloquium Mathematicum 2 (1951) 241–244.
[3] A.J. Fridy, On statistical convergence, Analysis (Munich) 5 (1985) 301–313.
[4] A.D. Gadjiev , C. Orhan, Some approximation theorems via statistical convergence, The Rocky Mountain Journal of Mathematics 32 (2002) 129–138.
[5] A.R. Gairola, Deepmala, L.N. Mishra, Rate of approximation by finite Iterates of q-Durrmeyer operators, Proceedings of National Academy of Sciences, India Section A: Physical Sciences 86(2) (2016) 229–234.
[6] R.B. Gandhi, Deepmala, V.N. Mishra, Local and global results for modified Szász-Mirakjan operators, Mathematical Methods in the Applied Sciences 40(7) (2017) 2491–2504.
[7] G.H. Hardy, Divergent Series, Oxford University Press, London, 1949.
[8] A. Jakimovski, D. Leviatan, Generalized Szász operators for the approximation in the infinite interval, Mathematica (Cluj) 34 (1969) 97–103.

[9] A. Karaisa, Approximation by Durrmeyer type Jakimovski Leviatan operators, Mathematical Methods in Applied Sciences 39(9) (2016) 2401–2410.
[10] M. Mursaleen, A. Al-Abied, K.J. Ansari, Rate of convergence of Chlodowsky type Durrmeyer Jakimovski-Leviatan operators, Tbilisi Mathematical Journal 10(2) (2017) 173–184.
[11] M. Mursaleen, F. Khan, A. Khan, Statistical approximation for new positive linear operators of Lagrange type, Applied Mathematics and Computation 232 (2014) 548–558.
[12] M. Mursaleen, K.J. Ansari, On Chlodowsky variant of Szász operators by Brenke type polynomials, Applied Mathematics and Computation 271 (2015) 991–1003.
[13] M. Mursaleen, K.J. Ansari, Approximation of q-Stancu-Beta operators which preserve x^2, Bulletin of the Malaysian Mathematical Sciences Society 40(4) (2017) 1479–1491.
[14] M. Mursaleen, K.J. Ansari, M. Nasiruzzaman, Approximation by q-analogue of Jakimovski-Leviatan operators involving q-Appell polynomials, Iranian Journal of Science and Technology, Transactions A: Science 41(4) (2017) 891–900.
[15] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloquium Mathematicum 2 (1951) 73–74.
[16] O. Szász, Generalization of S. Bernstein polynomials to the infinite interval, Journal of Research of the National Burean of Standards 45 (1950) 239–245.
[17] V.N. Mishra, R.B. Gandhi, Simultaneous approximation by Szász-Mirakjan-Stancu-Durrmeyer type operators, Periodica Mathematica Hungarica, DOI 10.1007/s10998-016-0145-0.
[18] V.N. Mishra, K. Khatri, L.N. Mishra, Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators, Journal of Inequalities and Applications (2013) 2013:586.
[19] V.N. Mishra, P. Sharma, L.N. Mishra, On statistical approximation properties of q-Baskakov-Szász-Stancu operators, Journal of Egyptian Mathematical Society 24(3) 2016 396–401.