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Exact Constants in Direct and Inverse Approximation Theorems for Functions of Several Variables in the Spaces $\mathcal{S}^p$

Fahreddin G. Abdullayev[a], Pelin Özkartepe[b], Viktor V. Savchuk[c], Andrii L. Shidlich[c]

[a] *Kyrgyz-Turkish Manas University, Bishkek, Kyrgyz Republic; Mersin University, Mersin, Turkey*
[b] *Gaziantep University, Gaziantep, Turkey*
[c] *Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine*

Abstract. In the paper, exact constants in direct and inverse approximation theorems for functions of several variables are found in the spaces $\mathcal{S}^p$. The equivalence between moduli of smoothness and some K-functionals is also shown in the spaces $\mathcal{S}^p$.

1. Introduction

Let $C(\mathbb{T})$, $\mathbb{T} := [0, 2\pi]$, denote the space of 2π-periodic continuous functions with the usual max-norm $\|f\| = \max_{x\in\mathbb{T}} |f(x)|$. Let also $E_n(f) = \inf\limits_{t_n} \|f - t_n\|$ be the best approximation of function $f \in C(\mathbb{T})$ by trigonometric polynomials of degree n, $n \in \mathbb{N}$.

The classical theorem of Jackson (1912) says that *i) if for the function $f \in C(\mathbb{T})$, there exists the derivative $f^{(r)} \in C(\mathbb{T})$, then the following inequality is true: $E_n(f) \le K_r n^{-r}\omega(f^{(r)}, n^{-1})$, $n = 1, 2, \ldots$, where $\omega(f, t) := \sup_{|h|\le t} \|f(\cdot + h) - f(\cdot)\|$ is the modulus of continuity of f.* This assertion is a *direct approximation theorem*, which shows that smoothness of the function f yields a quick decrease to zero of its error of approximation by trigonometric polynomials.

On the other hand, the following *inverse theorem* of Bernstein (1912) is well-known: *ii) if for some $0 < \alpha < 1$, we have $E_n(f) \le K_r n^{-r-\alpha}$, $n = 1, 2, \ldots$, then there exists the derivative $f^{(r)} \in C(\mathbb{T})$ and $\omega(f^{(r)}, t) = \mathcal{O}(t^{\alpha})$, $t \to 0+$.* In ideal cases, the two theorems match each other. For example, it follows from i) and ii) that $E_n(f) = \mathcal{O}(n^{-\alpha})$, $0 < \alpha < 1$, is equivalent to the condition $\omega(f, t) = \mathcal{O}(t^{\alpha})$, $t \to 0+$. (i.e. to the inclusion $f \in \mathrm{Lip}\,\alpha$). In such cases, it is said that for the classes $\mathrm{Lip}\,\alpha$, constructive characteristics is obtained in terms of the best approximations of functions.

Direct and inverse theorems are the central theorems of the approximation theory. They were studied by numerous authors. Here, we mention only the monographs [3, 7, 8, 16, 23], where one can find the fundamental results obtained in this field.

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Email addresses: fahreddinabdullayev@gmail.com (Fahreddin G. Abdullayev), pelinozkartepe@gmail.com (Pelin Özkartepe), vicsavchuk@gmail.com (Viktor V. Savchuk), shidlich@gmail.com (Andrii L. Shidlich)

Stepanets and Serdyuk [17] proved direct and inverse approximation theorems in the spaces $\mathcal{S}^p(\mathbb{T})$ of 2π-periodic Lebesgue summable on $\mathbb{T}$ functions f with the finite norm

$$\|f\|_{\mathcal{S}^p(\mathbb{T})} = \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}}\|_{l_p(\mathbb{Z})} = \Big(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^p\Big)^{1/p},$$

where $\widehat{f}(k)$ are the Fourier coefficients of f. In [17], the obtained direct and inverse theorems not only match each other, but in these assertions, the exact constants were found too. Also this topic was actively investigated in [14, 15, 18, 24–26], [16, Ch. 11], [23, Ch. 3]. These results are also interesting because of the fact that by the definition of the norm, the assertions for the spaces $\mathcal{S}^2(\mathbb{T})$ yield the corresponding results for the Lebesgue spaces $L_2(\mathbb{T})$.

In the present paper, the proven results for the spaces $\mathcal{S}^p(\mathbb{T})$ are extended to the multidimensional case. In particular, the direct and inverse approximation theorems for functions of several variables are proved in the spaces $\mathcal{S}^p(\mathbb{T}^d)$ and some applications of the obtained results are given. The equivalence between moduli of smoothness and some K-functionals is also shown in the spaces $\mathcal{S}^p$.

2. Preliminaries

Let d be a fixed natural number, $\mathbb{R}^d$, $\mathbb{Z}^d$, $\mathbb{Z}^d_+$ and $\mathbb{Z}^d_-$ be the sets of all ordered collections $\mathbf{k} := (k_1, \ldots, k_d)$ of d real, integer, nonegative integer and negative integer numbers correspondingly, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Let also $\mathbb{T}^d := [0, 2\pi]^d$ denote d-dimensional torus.

Further, let $L_p := L_p(\mathbb{T}^d)$, $1 \le p < \infty$, be the space of all Lebesgue-measurable on $\mathbb{R}^d$ 2π-periodic in each variable functions f with norm

$$\|f\|_{L_p} = \|f\|_{L_p(\mathbb{T}^d)} := \Big((2\pi)^{-d}\int_{\mathbb{T}^d} |f(\mathbf{x})|^p d\mathbf{x}\Big)^{\frac{1}{p}}.$$

For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we set $(\mathbf{x}, \mathbf{y}) := x_1y_1 + \ldots + x_dy_d$ and denote the Fourier coefficients of $f \in L_1$ by

$$\widehat{f}(\mathbf{k}) = [f]^\frown(\mathbf{k}) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{x}) \mathrm{e}^{-i(\mathbf{k},\mathbf{x})} d\mathbf{x}, \quad \mathbf{k} \in \mathbb{Z}^d.$$

The space $\mathcal{S}^p := \mathcal{S}^p(\mathbb{T}^d)$, $1 \le p < \infty$, [16, Ch. 11] is the space of all functions $f \in L_1$ such that

$$\|f\|_{\mathcal{S}^p} := \|\{\widehat{f}(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}\|_{l_p(\mathbb{Z}^d)} = \Big(\sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{f}(\mathbf{k})|^p\Big)^{1/p} < \infty. \tag{1}$$

Functions $f \in L_1$ and $g \in L_1$ are equivalent in the space $\mathcal{S}^p$, when $\|f - g\|_{\mathcal{S}^p} = 0$.

For an arbitrary $f \in L_1$ and any $\nu \in \mathbb{N}_0$, we set

$$H_\nu(f)(\mathbf{x}) := \sum_{|\mathbf{k}|_1 = \nu} \widehat{f}(\mathbf{k}) \mathrm{e}^{i(\mathbf{k},\mathbf{x})}, \ |\mathbf{k}|_1 := \sum_{j=1}^{d} |k_j|. \tag{2}$$

Then the Fourier series of f can be represented as

$$S[f](\mathbf{x}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} \widehat{f}(\mathbf{k}) \mathrm{e}^{i(\mathbf{x},\mathbf{k})} = \sum_{\nu=0}^{\infty} H_\nu(f)(\mathbf{x}). \tag{3}$$

Consider the set $\mathscr{T}^\triangle_n$, $n \in \mathbb{N}$ of all polynomials of the form $\tau_n(\mathbf{x}) := \sum_{|\mathbf{k}|_1 \le n} a_{\mathbf{k}} \mathrm{e}^{i(\mathbf{k},\mathbf{x})}$, where $a_{\mathbf{k}}$ are arbitrary complex numbers. The quantity

$$E^\triangle_n(f)_{\mathcal{S}^p} = \inf_{\tau_{n-1}\in\mathscr{T}^\triangle_{n-1}} \|f - \tau_{n-1}\|_{\mathcal{S}^p} \tag{4}$$

is called best approximation of $f \in \mathcal{S}^p$ by triangular polynomials of order $n - 1$.

In the spaces $\mathcal{S}^p$, the approximative properties of linear summation methods of multiple Fourier series on triangular regions were studied in [12, 13]. In particular, in [13], the authors proved direct and inverse theorems of approximation of function of several variables by the Taylor-Abel-Poisson operators in $\mathcal{S}^p$.

3. Differences and Modulus of Smoothness of Fractional Order

Similarly to [4], define the difference of $f \in L_1$ of a fractional order $\alpha > 0$ with respect to the increment $h \in \mathbb{R}$ by

$$\Delta_h^\alpha f(\mathbf{x}) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(\mathbf{x} - jh), \tag{5}$$

where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$ for $j \geq 1$, $\binom{\alpha}{0} = 1$ and $(\mathbf{x} - jh) := (x_1 - jh, x_2 - jh, \ldots, x_d - jh)$.

For convenience, let us assemble some basic properties of the fractional difference.

Lemma 3.1. *Assume that $f \in \mathcal{S}^p$, $\alpha, \beta > 0$, $h \in \mathbb{R}$, $\mathbf{k} \in \mathbb{Z}^d$, $\mathbf{x} \in \mathbb{R}^d$. Then*

(i) $\|\Delta_h^\alpha f\|_{\mathcal{S}^p} \leq K(\alpha)\|f\|_{\mathcal{S}^p}$, *where* $K(\alpha) := \sum_{j=0}^\infty \left|\binom{\alpha}{j}\right| \leq 2^{\{\alpha\}}$, $\{\alpha\} = \inf\{j \in \mathbb{N} : j \geq \alpha\}$;

(ii) $[\Delta_h^\alpha f]^\frown(\mathbf{k}) = (1 - \mathrm{e}^{-i(\mathbf{k},h)})^\alpha \widehat{f}(\mathbf{k})$, *where* $(\mathbf{k}, h) := (\mathbf{k}, h \cdot \mathbf{1}) = h(k_1 + \ldots + k_d)$;

(iii) $(\Delta_h^\alpha(\Delta_h^\beta f))(\mathbf{x}) = \Delta_h^{\alpha+\beta} f(\mathbf{x})$ *(a.e.)*;

(iv) $\|\Delta_h^{\alpha+\beta} f\|_{\mathcal{S}^p} \leq 2^{\{\beta\}}\|\Delta_h^\alpha f\|_{\mathcal{S}^p}$;

(v) $\lim_{h\to 0} \|\Delta_h^\alpha f\|_{\mathcal{S}^p} = 0$.

The proof of Lemma 3.1 and other auxiliary statements of the paper will be given in Section 8.

Based on (5), the modulus of smoothness of $f \in \mathcal{S}^p$ of the index $\alpha > 0$ is defined by

$$\omega_\alpha^\Delta(f, t)_{\mathcal{S}^p} := \sup_{|h| \leq t} \|\Delta_h^\alpha f\|_{\mathcal{S}^p}$$

The functions $\omega_\alpha^\Delta(f, t)_{\mathcal{S}^p}$ possess all the usual properties of ordinary modulus of smoothness. Before their formulation let us give the definition of ψ-derivative of a function (see, for example, [16, Ch. 11]).

Let $\psi = \{\psi(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$ be a multiple sequence of complex numbers, $\psi(\mathbf{k}) \neq 0$, $\mathbf{k} \in \mathbb{Z}^d$. If for the function $f \in L_1$, the series $\sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \widehat{f}(\mathbf{k})\mathrm{e}^{i(\mathbf{k},\mathbf{x})}/\psi(\mathbf{k})$ is the Fourier series of a certain function $g \in L_1$, then g is called ψ-derivative of the function f and is denoted as $g := f^\psi$. It is clear that the Fourier coefficients of functions f and f^ψ are related by equality

$$\widehat{f}(\mathbf{k}) = \psi(\mathbf{k})\widehat{f^\psi}(\mathbf{k}), \ \ k \in \mathbb{Z}^d \setminus \{\mathbf{0}\}. \tag{6}$$

In case $\psi(\mathbf{k}) = \nu^{-r}$, $|\mathbf{k}|_1 = \nu$, $\nu = 0, 1, \ldots$, $r \geq 0$, we use the notation $f^\psi =: f^{(r)}$.

Lemma 3.2. *Assume that $f, g \in \mathcal{S}^p$, $\alpha \geq \beta > 0$, $t, t_1, t_2 \geq 0$. Then*

(i) $\omega_\alpha^\Delta(f, t)_{\mathcal{S}^p}$ *is a non-negative continuously increasing function of t on $(0, \infty)$ with* $\lim_{t\to 0+} \omega_\alpha^\Delta(f, t)_{\mathcal{S}^p} = 0$;

(ii) $\omega_\alpha^\Delta(f, t)_{\mathcal{S}^p} \leq 2^{\{\alpha-\beta\}}\omega_\beta^\Delta(f, t)_{\mathcal{S}^p}$;

(iii) $\omega_\alpha^\Delta(f + g, t)_{\mathcal{S}^p} \leq \omega_\alpha^\Delta(f, t)_{\mathcal{S}^p} + \omega_\alpha^\Delta(g, t)_{\mathcal{S}^p}$;

(iv) $\omega_1^\Delta(f, t_1 + t_2)_{\mathcal{S}^p} \leq \omega_1^\Delta(f, t_1)_{\mathcal{S}^p} + \omega_1^\Delta(f, t_2)_{\mathcal{S}^p}$;

(v) $\omega_\alpha(f, t)_{\mathcal{S}^p} \leq 2^{\{\alpha\}}\|f\|_{\mathcal{S}^p}$;

(vi) *if there exists a derivative $f^{(\beta)} \in \mathcal{S}^p$, then* $\omega_\alpha^\Delta(f, t)_{\mathcal{S}^p} \leq t^\beta \omega_{\alpha-\beta}^\Delta(f^{(\beta)}, t)_{\mathcal{S}^p}$.

4. Direct Approximation Theorems

Proposition 4.1. *Let $\psi = \{\psi(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$ be a multiple sequence of complex numbers such that $\psi(\mathbf{k}) \neq 0$ and $\lim_{|\mathbf{k}|_1\to\infty} |\psi(\mathbf{k})| = 0$. If for the function $f \in \mathcal{S}^p$ there exists a derivative $f^\psi \in \mathcal{S}^p$, then for all $n \in \mathbb{N}$,*

$$E_n^\Delta(f)_{\mathcal{S}^p} \leq \varepsilon_n E_n^\Delta(f^\psi)_{\mathcal{S}^p}, \quad \textit{where} \quad \varepsilon_n = \max_{|\mathbf{k}|_1 \geq n} |\psi(\mathbf{k})|.$$

Proof. Let $f \in \mathcal{S}^p$. By virtue of the definition of the norm in $\mathcal{S}^p$, for any polynomial $\tau_{n-1} \in \mathscr{T}^{\triangle}_{n-1}$,

$$\|f - \tau_{n-1}\|^p_{\mathcal{S}^p} = \sum_{\nu=0}^{n-1} \sum_{|\mathbf{k}|_1=\nu} |\widehat{f}(\mathbf{k}) - a_{\mathbf{k}}|^p + \sum_{\nu=n}^{\infty} \|H_\nu(f)\|^p_{\mathcal{S}^p} \geq \sum_{\nu=n}^{\infty} \|H_\nu(f)\|^p_{\mathcal{S}^p}.$$

Therefore, the infimum on the right-hand side of (4) is realized by the triangular partial Fourier sum $S^{\triangle}_n(f)(\mathbf{x}) = \sum_{\nu=0}^{n} H_\nu(f)(\mathbf{x})$ of order n, i.e.,

$$E^{\triangle}_n(f)^p_{\mathcal{S}^p} = \|f - S^{\triangle}_{n-1}(f)\|^p_{\mathcal{S}^p} = \sum_{\nu=n}^{\infty} \|H_\nu(f)\|^p_{\mathcal{S}^p} = \sum_{|\mathbf{k}|_1 \geq n} |\widehat{f}(\mathbf{k})|^p. \tag{7}$$

If for the function $f \in \mathcal{S}^p$ there exists a derivative $f^\psi \in \mathcal{S}^p$, then according to (6), we have

$$E^{\triangle}_n(f)^p_{\mathcal{S}^p} = \sum_{|\mathbf{k}|_1 \geq n} |\widehat{f}(\mathbf{k})|^p = \sum_{|\mathbf{k}|_1 \geq n} |\psi(\mathbf{k})\widehat{f^\psi}(\mathbf{k})|^p \leq \varepsilon_n E_n(f^\psi)_{\mathcal{S}^p}.$$

In this case, if $\varepsilon_n = \max\limits_{|\mathbf{k}|_1 \geq n} |\psi(\mathbf{k})| = |\psi(\mathbf{k}_n)|$, where $\mathbf{k}_n \in \mathbb{Z}^d$ such that $|\mathbf{k}_n|_1 \geq n$, then for an arbitrary polynomial of the form $\tilde{\tau}_n(\mathbf{x}) := c\, \mathrm{e}^{i(\mathbf{k}_n, \mathbf{x})}$, $c \neq 0$, obviously, the equality holds

$$E_n(\tilde{\tau}_n)_{\mathcal{S}^p} = \varepsilon_n E_n(\tilde{\tau}^\psi_n)_{\mathcal{S}^p}.$$

□

Now we give direct approximation theorems in the spaces $\mathcal{S}^p(\mathbb{T}^d)$ in terms of best approximations and moduli of smoothness of function. We establish Jackson inequalities of the form

$$E^{\triangle}_n(f)_{\mathcal{S}^p} \leq K(\tau)\omega^{\triangle}_\alpha\Big(f, \frac{\tau}{n}\Big)_{\mathcal{S}^p}, \quad \tau > 0,$$

and consider the problem of the least constant in these inequalities for fixed values of the parameters n, α, τ and p. In particular, we study the quantity

$$K_{n,\alpha,p}(\tau) = \sup\left\{ \frac{E^{\triangle}_n(f)_{\mathcal{S}^p}}{\omega^{\triangle}_\alpha(f, \frac{\tau}{n})_{\mathcal{S}^p}} \ : f \in \mathcal{S}^p_{\pm}, \ \ f \not\equiv \mathrm{const} \right\}, \tag{8}$$

where $\mathcal{S}^p_{\pm} := \Big\{ f \in \mathcal{S}^p : \widehat{f}(\mathbf{k}) = 0 \ \ \forall\, \mathbf{k} \in \mathbb{Z}^d \setminus \mathbb{Z}^d_{\pm} \Big\}$, $\mathbb{Z}^d_{\pm} := \mathbb{Z}^d_+ \cup \mathbb{Z}^d_-$.

Let $M(\tau)$, $\tau > 0$, be a set of bounded nondecreasing functions μ that differ from a constant on $[0, \tau]$.

Theorem 4.2. *Assume that $f \in \mathcal{S}^p_{\pm}$. Then for any $\tau > 0$, $n \in \mathbb{N}$ and $\alpha > 0$ the following inequality holds:*

$$E^{\triangle}_n(f)_{\mathcal{S}^p} \leq C_{n,\alpha,p}(\tau)\omega^{\triangle}_\alpha\Big(f, \frac{\tau}{n}\Big)_{\mathcal{S}^p}, \tag{9}$$

where

$$C_{n,\alpha,p}(\tau) := \left(\inf_{\mu \in M(\tau)} \frac{\mu(\tau) - \mu(0)}{2^{\frac{\alpha p}{2}} I_n(\tau, \mu)} \right)^{1/p}, \tag{10}$$

and

$$I_n(\tau, \mu) = I_{n,\alpha,p}(\tau, \mu) = \inf_{\nu \in \mathbb{N}: \nu \geq n} \int_0^\tau \Big(1 - \cos\frac{\nu}{n} t\Big)^{\frac{\alpha p}{2}} d\mu(t). \tag{11}$$

Futhermore, there exists a function $\mu_* \in M(\tau)$ that realizes the greatest lower bound in (10). *Inequality* (9) *is unimprovable on the set of all functions $f \in \mathcal{S}^p_{\pm}$, $f \not\equiv$ const, in the sense that for any $\alpha > 0$ and $n \in \mathbb{N}$ the following equality is true:*

$$C_{n,\alpha,p}(\tau) = K_{n,\alpha,p}(\tau). \tag{12}$$

Proof. In the spaces $L_2(\mathbb{T}^1)$, for $\alpha = 1$ this theorem was proved by Babenko in [1]. For the spaces $\mathcal{S}^p(\mathbb{T})$ of 2π-periodic functions of one variable this result was obtained in [17]. In the proof, we mainly use the notation and arguments presented in [1, 5, 6, 17].

By virtue of Lemma 3.1 ii) for any $f \in \mathcal{S}^p$ and $h \in \mathbb{R}$, we see that

$$\|\Delta_h^\alpha f\|_{\mathcal{S}^p}^p = \sum_{\nu=0}^{\infty}\sum_{|\mathbf{k}|_1=\nu} |\widehat{f}(\mathbf{k})|^p |1-\mathrm{e}^{-i(\mathbf{k},h)}|^{\alpha p}$$

$$= 2^{\alpha p}\sum_{\nu=0}^{\infty}\sum_{|\mathbf{k}|_1=\nu} |\widehat{f}(\mathbf{k})|^p \Big|\sin\frac{(\mathbf{k},h)}{2}\Big|^{\alpha p} = 2^{\frac{\alpha p}{2}}\sum_{\nu=0}^{\infty}\sum_{|\mathbf{k}|_1=\nu} |\widehat{f}(\mathbf{k})|^p (1-\cos(\mathbf{k},h))^{\frac{\alpha p}{2}}. \tag{13}$$

If $f \in \mathcal{S}^p_{\pm}$, then $\widehat{f}(\mathbf{k}) = 0$ for $\mathbf{k} \in \mathbb{Z}^d \setminus \mathbb{Z}^d_{\pm}$ and $|(\mathbf{k},h)| = h|\mathbf{k}|_1$ for $\mathbf{k} \in \mathbb{Z}^d_{\pm}$. Therefore, by virtue of (2), (13) and (11), we have

$$\|\Delta_h^\alpha f\|_{\mathcal{S}^p}^p = 2^{\frac{\alpha p}{2}}\sum_{\nu=0}^{\infty}\|H_\nu(f)\|_{\mathcal{S}^p}^p (1-\cos\nu h)^{\frac{\alpha p}{2}} \geq 2^{\frac{\alpha p}{2}}\sum_{\nu=n}^{\infty}\|H_\nu(f)\|_{\mathcal{S}^p}^p (1-\cos\nu h)^{\frac{\alpha p}{2}}$$

$$= \frac{2^{\frac{\alpha p}{2}} I_n(\tau,\mu)}{\mu(\tau)-\mu(0)} E_n^{\triangle}(f)^p_{\mathcal{S}^p} + 2^{\frac{\alpha p}{2}}\sum_{\nu=n}^{\infty}\|H_\nu(f)\|_{\mathcal{S}^p}^p\Big((1-\cos\nu h)^{\frac{\alpha p}{2}} - \frac{I_n(\tau,\mu)}{\mu(\tau)-\mu(0)}\Big), \tag{14}$$

where the quantity $I_n(\tau,\mu)$ in defined by (11). Therefore, for arbitrary $t \in [0,\tau]$,

$$E_n^{\triangle}(f)^p_{\mathcal{S}^p} \leq \frac{\mu(\tau)-\mu(0)}{2^{\frac{\alpha p}{2}} I_n(\tau,\mu)}\Bigg(\|\Delta^\alpha_{\frac{t}{n}} f\|^p_{\mathcal{S}^p} - 2^{\frac{\alpha p}{2}}\sum_{\nu=n}^{\infty}\|H_\nu(f)\|_{\mathcal{S}^p}^p\Big(\Big(1-\cos\frac{\nu t}{n}\Big)^{\frac{\alpha p}{2}} - \frac{I_n(\tau,\mu)}{\mu(\tau)-\mu(0)}\Big)\Bigg). \tag{15}$$

Since the both sides of inequality (15) are nonnegative and the series on its right-hand side is majorized on the entire real axis by the absolutely convergent series $2^{\alpha p}\sum_{\nu=n}^{\infty}\|H_\nu(f)\|^p_{\mathcal{S}^p}$, then integrating this inequality with respect to $d\mu(t)$ from 0 to τ, we get

$$E_n^{\triangle}(f)^p_{\mathcal{S}^p}(\mu(\tau)-\mu(0)) \leq \frac{\mu(\tau)-\mu(0)}{2^{\frac{\alpha p}{2}} I_n(\tau,\mu)}\Bigg(\int_0^\tau \|\Delta^\alpha_{\frac{t}{n}} f\|^p_{\mathcal{S}^p} d\mu(t)$$

$$-2^{\frac{\alpha p}{2}}\sum_{\nu=n}^{\infty}\|H_\nu(f)\|_{\mathcal{S}^p}^p\Big(\int_0^\tau \Big(1-\cos\frac{\nu t}{n}\Big)^{\frac{\alpha p}{2}} d\mu(t) - I_n(\tau,\mu)\Big)\Bigg). \tag{16}$$

By virtue of the definition of $I_n(\tau,\mu)$, we see that the second term on the right-hand side of (16) is nonnegative. Therefore, for any function $\mu \in M(\tau)$, we have

$$E_n^{\triangle}(f)^p_{\mathcal{S}^p} \leq \frac{1}{2^{\frac{\alpha p}{2}} I_n(\tau,\mu)}\int_0^\tau \|\Delta^\alpha_{\frac{t}{n}} f\|^p_{\mathcal{S}^p} d\mu(t) \leq \frac{1}{2^{\frac{\alpha p}{2}} I_n(\tau,\mu)}\int_0^\tau \omega_\alpha^{\triangle}\Big(f,\frac{t}{n}\Big)^p d\mu(t), \tag{17}$$

whence we immediately obtain inequality (9) and the estimate

$$K^p_{n,\alpha,p}(\tau) \leq \inf_{\mu\in M(\tau)} \frac{\mu(\tau)-\mu(0)}{2^{\frac{\alpha p}{2}} I_n(\tau,\mu)} = C^p_{n,\alpha,p}(\tau). \tag{18}$$

Let us show that relation (18) is the equality. In view of (14), for any $f \in \mathcal{S}^p_{\pm}$,

$$\omega^p_\alpha(f,t)_{\mathcal{S}^p} = 2^{\frac{\alpha p}{2}}\sup_{0<h\leq t}\sum_{\nu=0}^{\infty}\|H_\nu(f)\|_{\mathcal{S}^p}^p(1-\cos\nu h)^{\frac{\alpha p}{2}}. \tag{19}$$

Therefore, using (7) and (19), we get

$$K^{p}_{n,\alpha,p}(\tau) = \sup_{\rho_\nu \geq 0} \frac{\sum_{\nu=n}^{\infty} \rho_\nu}{2^{\frac{\alpha p}{2}} \sup\limits_{0\leq u\leq \tau} \sum_{\nu=n}^{\infty} \rho_\nu (1-\cos\frac{\nu}{n}u)^{\frac{\alpha p}{2}}}, \tag{20}$$

where the outer supremum is taken over all sequences of nonnegative real numbers ρ_ν, $\nu = 1, 2, \ldots$, such that $\sum_{\nu=1}^{\infty} \rho_\nu < \infty$.

Consider the set

$$W_{n,\alpha,p} = \left\{ \omega(t) = \sum_{\nu=n}^{\infty} \rho_\nu \Big(1-\cos\frac{\nu t}{n}\Big)^{\frac{\alpha p}{2}} \;:\; \rho_\nu \geq 0,\; \sum_{\nu=n}^{\infty} \rho_\nu = 1 \right\} \tag{21}$$

and the quantity $\dot{\mathcal{J}}_n(\tau) := \dot{\mathcal{J}}_{n,\alpha,p}(\tau) = \inf\limits_{\omega\in W_{n,\alpha,p}} \|\omega\|_{C_{[0,\tau]}}$. By virtue of (20), we have

$$K^{-p}_{n,\alpha,p}(\tau) = 2^{\frac{\alpha p}{2}} \dot{\mathcal{J}}_n(\tau). \tag{22}$$

For what follows, we need a duality relation in the space $C_{[a,b]}$, (see, e.g., [9, p. 30–31]).

Proposition 4.3. ([9, p. 30–31]) *If F is a convex set in the space $C_{[a,b]}$, then for any $x \in C_{[a,b]}$,*

$$\inf_{u\in F} \|x-u\|_{C_{[a,b]}} = \sup_{\overset{b}{\underset{a}{V}}(g)\leq 1} \Big(\int_a^b x(t)dg(t) - \sup_{u\in F} \int_a^b u(t)dg(t) \Big). \tag{23}$$

For $x \in C_{[a,b]} \setminus \bar{F}$, where $\bar{F}$ is the closure of a set F, there exists a function g_* with variation equal to 1 on $[a,b]$ that realizes the least upper bound in (23).

It is easy to show that the set $W_{n,\alpha,p}$ is a convex subset of the space $C_{[0,\tau]}$. Therefore, setting $a = 0$, $b = \tau$, $x(t) \equiv 0$, $u(t) = w(t) \in W_{n,\alpha,p}$, $F = W_{n,\alpha,p}$, from relation (23) we get

$$\dot{\mathcal{J}}_n(\tau) = \inf_{w\in W_{n,\alpha,p}} \|0-w\|_{C_{[0,\tau]}} = \sup_{\overset{\tau}{\underset{0}{V}}(g)\leq 1} \Big(0 - \sup_{w\in W_{n,\alpha,p}} \int_0^\tau w(t)dg(t)\Big) = \sup_{\overset{\tau}{\underset{0}{V}}(g)\leq 1} \inf_{w\in W_{n,\alpha,p}} \int_0^\tau w(t)dg(t). \tag{24}$$

Furthermore, according to Proposition 4.3, there exists a function $g_*(t)$, that realizes the least upper bound in (24) and such that $\overset{\tau}{\underset{0}{V}}(g_*) = 1$. Since every function $w \in W_{n,\alpha,p}$ is nonnegative, it suffices to take the supremum on the right-hand side of (24) over the set of nondecreasing functions $\mu(t)$ for which $\mu(\tau) - \mu(0) \leq 1$. For such functions, by virtue of (11) and (21), the following equality is true:

$$\inf_{w\in W_{n,\alpha,p}} \int_0^\tau w(t)d\mu(t) = I_n(\tau,\mu). \tag{25}$$

Hence, there exists a function $\mu_* \in M(\tau)$ such that $\mu_*(\tau) - \mu_*(0) = 1$ and

$$I_n(\tau,\mu_*) = \sup_{\mu\in M(\tau):\overset{\tau}{\underset{0}{V}}(\mu)\leq 1} I_n(\tau,\mu) = \dot{\mathcal{J}}_n(\tau). \tag{26}$$

Relations (18), (22) and (26) yield

$$K^{p}_{n,\alpha,p}(\tau) \geq K^{p}_{n,\alpha,p}(\tau) = \frac{1}{2^{\frac{\alpha p}{2}} \dot{\mathcal{J}}_n(\tau)} = \frac{1}{2^{\frac{\alpha p}{2}} I_n(\tau,\mu_*)} = \frac{\mu_*(\tau)-\mu_*(0)}{2^{\frac{\alpha p}{2}} I_n(\tau,\mu_*)} = C^{p}_{n,\alpha,p}(\tau).$$

□

It should be noted that for $f \in \mathcal{S}^p$, condition $\widehat{f}(\mathbf{k}) = 0$, $\mathbf{k} \in \mathbb{Z}^d \setminus \mathbb{Z}^d_\pm$ in Theorem 4.2 generally speaking cannot be omitted. For example, consider the function $f(\mathbf{x}) = \mathrm{e}^{i(\mathbf{k}^*,\mathbf{x})}$, where $k^* = (l, -l, 0, \ldots)$, $l \in \mathbb{N}$. Then for all $n < 2l$, we have $E_n(f^*)_{\mathcal{S}^p} = 1$ and $\omega(f^*, t) \equiv 0$.

Corollary 4.4. *Assume that $f \in \mathcal{S}^p_\pm$. Then for any $n \in \mathbb{N}$ and $\alpha > 0$, the following inequality holds:*

$$E_n^\triangle(f)^p_{\mathcal{S}^p} \leq \frac{1}{2^{\frac{\alpha p}{2}} I_n(\frac{\alpha p}{2})} \int_0^\pi \omega_\alpha^\triangle\Big(f, \frac{t}{n}\Big)^p_{\mathcal{S}^p} \sin t dt, \tag{27}$$

where

$$I_n(\lambda) := \inf_{\nu \in \mathbb{N}: \nu \geq n} \int_0^\pi \Big(1 - \cos \frac{\nu}{n} t\Big)^\lambda \sin t dt, \quad \lambda > 0, \quad n \in \mathbb{N}. \tag{28}$$

If, in addition $\frac{\alpha p}{2} \in \mathbb{N}$, then

$$I_n\Big(\frac{\alpha p}{2}\Big) = \frac{2^{\frac{\alpha p}{2}+1}}{\frac{\alpha p}{2} + 1}, \tag{29}$$

and inequality (27) cannot be improved for any $n \in \mathbb{N}$, that is for any $n \in \mathbb{N}$, there exists a function $f^* \in \mathcal{S}^p_\pm$ such that

$$E_n^\triangle(f^*)^p_{\mathcal{S}^p} = \frac{\frac{\alpha p}{2} + 1}{2^{\alpha p+1}} \int_0^\pi \omega_\alpha^\triangle\Big(f^*, \frac{t}{n}\Big)^p_{\mathcal{S}^p} \sin t dt. \tag{30}$$

Proof. Setting $\tau = \pi$ and $\mu(t) = 1 - \cos t$, $t \in [0, \pi]$, we see that inequality (17) yields (27). For $\frac{\alpha p}{2} \in \mathbb{N}$, equality (29) was proved in [17]. To prove the unimprovability of inequality (27) for $\frac{\alpha p}{2} \in \mathbb{N}$, consider the function $f^*(\mathbf{x}) = \gamma + \beta \mathrm{e}^{-(\mathbf{k}^*,\mathbf{x})} + \delta \mathrm{e}^{(\mathbf{k}^*,\mathbf{x})}$, where γ, β, δ are arbitrary complex numbers and the vector $\mathbf{k}^* = (k_1^*, k_2^*, \ldots, k_d^*) \in \mathbb{Z}^d_+$ is such that $|\mathbf{k}^*|_1 = n$. Let us show that relation (30) holds.

Since the function $\|\Delta^\alpha_{t/n} f^*\|^p_{\mathcal{S}^p} = 2^{\frac{\alpha p}{2}}(|\beta|^p + |\delta|^p)(1 - \cos t)^{\frac{\alpha p}{2}}$ doesn't decrease with respect to t on $[0, \pi]$, then $\omega_\alpha^\triangle(f^*, \frac{t}{n})_{\mathcal{S}^p} = \|\Delta^\alpha_{t/n} f^*\|_{\mathcal{S}^p}$. Furthermore,

$$\frac{2^{\alpha p+1}}{\frac{\alpha p}{2} + 1} E_n^\triangle(f^*)^p_{\mathcal{S}^p} - \int_0^\pi \omega_\alpha^\triangle\Big(f^*, \frac{t}{n}\Big)^p_{\mathcal{S}^p} \sin t dt = (|\beta|^p + |\delta|^p)\Big(\frac{2^{\alpha p+1}}{\frac{\alpha p}{2} + 1} - 2^{\frac{\alpha p}{2}} \int_0^\pi (1 - \cos t)^{\frac{\alpha p}{2}} \sin t dt\Big) = 0.$$

□

In the following assertion we give the upper bounds for the constants $K_{n,\alpha,p}(\tau)$ with $\tau = \pi$ which are independent of n and unimprovable in several important cases.

Corollary 4.5. *For any $\alpha > 0$ and $n \in \mathbb{N}$, the following inequalities are true:*

$$K^p_{n,\alpha,p}(\pi) \leq \frac{1}{2^{\frac{\alpha p}{2}-1} I_n(\frac{\alpha p}{2})} \leq \frac{\frac{\alpha p}{2} + 1}{2^{\alpha p} + 2^{\frac{\alpha p}{2}-1}(\frac{\alpha p}{2} + 1)\sigma(\frac{\alpha p}{2})}, \tag{31}$$

where $I_n(\lambda)$ are defined by (28), and

$$\sigma(\lambda) := -\sum_{m=[\frac{\lambda}{2}]+1}^\infty \binom{\lambda}{2m} \frac{1}{2^{2m-1}} \Big(\frac{1 - (-1)^{[\lambda]}}{2} \binom{2m}{m} - \sum_{j=0}^{m-1} \binom{2m}{j} \frac{2}{2(m-j)^2 - 1}\Big), \quad \lambda > 0, \tag{32}$$

([λ] *is the integer part of the number* λ). *If, in addition* $\frac{\alpha p}{2} \in \mathbb{N}$, *then the value* $\sigma(\frac{\alpha p}{2}) = 0$ *and*

$$K^p_{n,\alpha,p}(\pi) \le \frac{\frac{\alpha p}{2}+1}{2^{\alpha p}}, \quad \frac{\alpha p}{2} \in \mathbb{N}, \quad n \in \mathbb{N}. \tag{33}$$

Proof. The first inequality in (31) and inequality (33) follow from Corollary 4.4. The second inequality in (31) follows from the relation

$$I_n(\frac{\alpha p}{2}) \ge \frac{2^{\frac{\alpha p}{2}+1}}{\frac{\alpha p}{2}+1} + \sigma(\frac{\alpha p}{2}), \ \ n \in \mathbb{N}, \ \ \alpha > 0, \tag{34}$$

which is a consequence of the following inequality [17]:

$$\int_0^\pi (1-\cos \tau t)^\lambda \sin t dt \ge \frac{2^{\lambda+1}}{\lambda+1} + \sigma(\lambda), \ \ \tau \ge 1, \ \ \lambda > 0. \tag{35}$$

□

The statement below establishes the uniform boundedness of the constants $K_{n,\alpha,p}(\pi)$ with respect to all parameters under consideration $\alpha > 0$, $n \in \mathbb{N}$, $1 \le p < \infty$.

Corollary 4.6. *Assume that* $f \in \mathcal{S}^p_{\pm}$ *and* $f \not\equiv$ const. *Then for any* $\alpha > 0$ *and* $n \in \mathbb{N}$,

$$E^{\Delta}_n(f)_{\mathcal{S}^p} < \frac{4}{3\cdot 2^{\alpha/2}} \omega_\alpha\Big(f, \frac{\pi}{n}\Big)_{\mathcal{S}^p}, \ \ \alpha > 0, \ \ n \in \mathbb{N}. \tag{36}$$

Proof. In [17], it was proved that $I_n(\lambda) > 2$ for $\lambda \ge 1$ and $I_n(\lambda) \ge 1 + 2^{\lambda-1}$ for $\lambda \in (0,1)$. Combining these estimates and relation (31) we obtain (36). □

In the spaces $\mathcal{S}^p(\mathbb{T})$ of 2π-periodic functions of one variable, Theorem 4.2 and Corollaries 4.4–4.6 were proved in [17]. In the spaces $L_2(\mathbb{T}^1)$, for $\alpha = 1$ inequality (27) was proved by Chernykh (see, [5, 6]). The inequalities of this type were also investigated in [10, 14, 19, 20, 24–27] and others.

5. Inverse Approximation Theorems for Functions of Several Variables in the Spaces $\mathcal{S}^p$.

Before proving the inverse approximation theorems, let us formulate the known Bernstein inequality in which the norm of the derivative of a trigonometric polynomial is estimated in terms of the norm of this polynomial (see, e.g. [23, Ch. 4]).

Proposition 5.1. *Let* $\psi = \{\psi(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$ *be a multiple sequence of complex numbers,* $\psi(\mathbf{k}) \ne 0$, $\mathbf{k} \in \mathbb{Z}^d$. *Then for any* $\tau_n \in \mathscr{T}_n$, $n \in \mathbb{N}$, *the following inequality holds:*

$$\|\tau^\psi_n\|_{\mathcal{S}^p} \le \frac{1}{\epsilon_n}\|\tau_n\|_{\mathcal{S}^p}, \quad \textit{where } \epsilon_n := \min_{0<|\mathbf{k}|_1\le n} |\psi(\mathbf{k})|.$$

Proof. Let $\tau_n(\mathbf{x}) = \sum_{|\mathbf{k}|_1\le n} a_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}$, $a_{\mathbf{k}} \in \mathbb{C}$. By the definition of the ψ-derivative and equalities (6), we get

$$\|\tau^\psi_n\|^p_{\mathcal{S}^p} = \sum_{0<|\mathbf{k}|_1\le n} |[\tau^\psi_n]^\frown(\mathbf{k})|^p = \sum_{0<|\mathbf{k}|_1\le n} |a_{\mathbf{k}}|^p/|\psi(\mathbf{k})|^p \le \max_{0<|\mathbf{k}|_1\le n} |\psi(\mathbf{k})|^{-p} \sum_{0<|\mathbf{k}|_1\le n} |a_{\mathbf{k}}|^p \le \frac{1}{\epsilon^p_n}\|\tau_n\|^p_{\mathcal{S}^p}. \tag{37}$$

In this case, if $\min_{0<|\mathbf{k}|_1\le n} |\psi(\mathbf{k})| = |\psi(\mathbf{k}_0)|$, then for an arbitrary polynomial of the form $\tilde{\tau}_{\mathbf{k}_0}(x) := c\, e^{i(\mathbf{k}_0,\mathbf{x})}$, $c \ne 0$, we have

$$\|\tilde{\tau}^\psi_{\mathbf{k}_0}\|_{\mathcal{S}^p} = |c_{\mathbf{k}_0}|/|\psi(\mathbf{k}_0)| = \|\tau_{\mathbf{k}_0}\|_{\mathcal{S}^p}/\epsilon_n.$$

□

Corollary 5.2. *Let $\psi(\mathbf{k}) = \nu^{-r}$, $|\mathbf{k}|_1 = \nu$, $\nu = 0, 1, \ldots$, $r \geq 0$. Then for any $\tau_n \in \mathscr{T}_n$, $n \in \mathbb{N}$*

$$\|\tau_n^{\psi}\|_{\mathcal{S}^p} = \|\tau_n^{(r)}\|_{\mathcal{S}^p} \leq n^r \|\tau_n\|_{\mathcal{S}^p}.$$

Now, we prove assertion similar to the theorem of Bernstein ii) formulated above. Further, in the formulations and proofs of assertions, we mainly use the scheme and some of the methods proposed in [17], where the corresponding results were obtained in the case of approximation of 2π-periodic functions of one variable and moduli of smoothness of integer order.

Theorem 5.3. *Assume that $f \in \mathcal{S}^p$. Then for any $n \in \mathbb{N}$ and $\alpha > 0$ the following inequality holds:*

$$\omega_\alpha^{\triangle}\Big(f, \frac{\pi}{n}\Big)_{\mathcal{S}^p} \leq \frac{\pi^\alpha}{n^\alpha} \Big(\sum_{\nu=1}^{n} (\nu^{\alpha p} - (\nu-1)^{\alpha p}) E_\nu^{\triangle}(f)_{\mathcal{S}^p}^p \Big)^{1/p}. \tag{38}$$

Proof. Let $f \in \mathcal{S}^p$. By virtue of (13), for any $n \in \mathbb{N}$ and $h \in \mathbb{R}$, we have

$$\|\Delta_h^\alpha f\|_{\mathcal{S}^p}^p = 2^{\alpha p} \sum_{|\mathbf{k}|_1 \leq n-1} |\widehat{f}(\mathbf{k})|^p \Big| \sin \frac{(\mathbf{k}, h)}{2} \Big|^{\alpha p} + 2^{\alpha p} \sum_{|\mathbf{k}|_1 \geq n} |\widehat{f}(\mathbf{k})|^p \Big| \sin \frac{(\mathbf{k}, h)}{2} \Big|^{\alpha p}. \tag{39}$$

It is clear that the second term on the right-hand side of (39) does not exceed the value

$$2^{\alpha p} \sum_{|\mathbf{k}|_1 \geq n} |\widehat{f}(\mathbf{k})|^p = 2^{\alpha p} \sum_{\nu=n}^{\infty} \|H_\nu(f)\|_{\mathcal{S}^p}^p = 2^{\alpha p} E_n^{\triangle}(f)_{\mathcal{S}^p}^p,$$

and for $h \in [0, \pi/n]$, we have

$$2^{\alpha p} \sum_{|\mathbf{k}|_1 \leq n-1} |\widehat{f}(\mathbf{k})|^p \Big| \sin \frac{(\mathbf{k}, h)}{2} \Big|^{\alpha p} \leq \sum_{\nu=0}^{n-1} \sum_{|\mathbf{k}|_1 = \nu} |\widehat{f}(\mathbf{k})|^p |(\mathbf{k}, h)|^{\alpha p} \leq \Big(\frac{\pi}{n}\Big)^{\alpha p} \sum_{\nu=1}^{n-1} \nu^{\alpha p} \|H_\nu(f)\|_{\mathcal{S}^p}^p.$$

Therefore,

$$\|\Delta_h^\alpha f\|_{\mathcal{S}^p}^p \leq 2^{\alpha p} E_n^{\triangle}(f)_{\mathcal{S}^p}^p + \Big(\frac{\pi}{n}\Big)^{\alpha p} \sum_{\nu=1}^{n-1} \nu^{\alpha p} \|H_\nu(f)\|_{\mathcal{S}^p}^p. \tag{40}$$

Now we use the following assertion from [17].

Lemma 5.4. ([17]) *Assume that the numerical series $\sum_{\nu=1}^{\infty} c_\nu$ is convergent. Then for any sequence a_ν, $\nu \in \mathbb{N}$, the following equality holds for all natural m and M, $m \leq M$:*

$$\sum_{\nu=m}^{M} a_\nu c_\nu = a_m \sum_{\nu=m}^{\infty} c_\nu + \sum_{\nu=m+1}^{M} (a_\nu - a_{\nu-1}) \sum_{i=\nu}^{\infty} c_i - a_M \sum_{\nu=M+1}^{\infty} c_\nu. \tag{41}$$

Setting $a_\nu = \nu^{\alpha p}$, $c_\nu = \|H_\nu(f)\|_{\mathcal{S}^p}^p$, $m = 1$ and $M = n-1$ in (41), we get

$$\sum_{\nu=1}^{n-1} \nu^{\alpha p} \|H_\nu(f)\|_{\mathcal{S}^p}^p = \sum_{\nu=1}^{\infty} \|H_\nu(f)\|_{\mathcal{S}^p}^p + \sum_{\nu=2}^{n-1} (\nu^{\alpha p} - (\nu-1)^{\alpha p}) \sum_{i=\nu}^{\infty} \|H_{i,r}(f)\|_{\mathcal{S}^p}^p$$

$$-(n-1)^{\alpha p} \sum_{\nu=n}^{\infty} \|H_\nu(f)\|_{\mathcal{S}^p}^p = \sum_{\nu=1}^{n-1} (\nu^{\alpha p} - (\nu-1)^{\alpha p}) E_\nu(f)_{\mathcal{S}^p}^p - (n-1)^{\alpha p} E_n^{\triangle}(f)_{\mathcal{S}^p}^p.$$

Thus,

$$\|\Delta_h^\alpha f\|_{\mathcal{S}^p}^p \le \Big(\frac{\pi}{n}\Big)^{\alpha p}\Big(\sum_{\nu=1}^{n-1}(\nu^{\alpha p}-(\nu-1)^{\alpha p})E_\nu^\triangle(f)_{\mathcal{S}^p}^p-(n-1)^{\alpha p}E_n^\triangle(f)_{\mathcal{S}^p}^p\Big)$$

$$+2^{\alpha p}E_n^\triangle(f)_{\mathcal{S}^p}^p \le \Big(\frac{\pi}{n}\Big)^{\alpha p}\Big(\sum_{\nu=1}^{n}(\nu^{\alpha p}-(\nu-1)^{\alpha p})E_\nu^\triangle(f)_{\mathcal{S}^p}^p, \tag{42}$$

which yields (38). □

Let us note that in (38) the constant π^α is exact in the sense that for any positive number $\varepsilon > 0$, there exists a function $f^* \in \mathcal{S}^p$ such that for all n greater that a certain number n_0, we have

$$\omega_\alpha^\triangle\Big(f^*,\frac{\pi}{n}\Big)_{\mathcal{S}^p} > \frac{\pi^\alpha-\varepsilon}{n^\alpha}\Big(\sum_{\nu=1}^{n}(\nu^{\alpha p}-(\nu-1)^{\alpha p})E_\nu^\triangle(f^*)_{\mathcal{S}^p}^p\Big)^{1/p}. \tag{43}$$

Indeed, according to (13), for any $f\in\mathcal{S}_\pm^p$ and $h\in\mathbb{R}$, we have

$$\|\Delta_h^\alpha f\|_{\mathcal{S}^p}^p = 2^{\alpha p}\sum_{\nu=0}^{\infty}\sum_{|\mathbf{k}|_1=\nu}|\widehat{f}(\mathbf{k})|^p\Big|\sin\frac{(\mathbf{k},h)}{2}\Big|^{\alpha p} = 2^{\alpha p}\sum_{\nu=0}^{\infty}\|H_\nu(f)\|_{\mathcal{S}^p}^p\Big|\sin\frac{\nu h}{2}\Big|^{\alpha p}. \tag{44}$$

This yields that if $f\in\mathcal{S}_\pm^p$, then for any $n\in\mathbb{N}$,

$$\omega_\alpha^\triangle\Big(f,\frac{\pi}{n}\Big)_{\mathcal{S}^p}^p \ge \|\Delta_{\frac{\pi}{n}}^\alpha f\|_{\mathcal{S}^p}^p \ge 2^{\alpha p}\sum_{\nu=0}^{\infty}\|H_\nu(f)\|_{\mathcal{S}^p}^p\Big|\sin\frac{\nu\pi}{2n}\Big|^{\alpha p}. \tag{45}$$

Consider the function $f^*(\mathbf{x}) = \mathrm{e}^{i(\mathbf{k_0},\mathbf{x})}$, where $\mathbf{k_0}$ is an arbitrary vector from the set $\mathbb{Z}_\pm^d$ and $|\mathbf{k}|_1 =: \nu_0$. Then $E_\nu^\triangle(f^*)_{\mathcal{S}^p} = 1$ for $\nu = 1,2,\ldots,\nu_0$, $E_\nu^\triangle(f^*)_{\mathcal{S}^p} = 0$ when $\nu > \nu_0$ and

$$\omega_\alpha^\triangle\Big(f^*,\frac{\pi}{n}\Big)_{\mathcal{S}^p}^p \ge \|\Delta_{\frac{\pi}{n}}^\alpha f^*\|_{\mathcal{S}^p}^p \ge 2^{\alpha p}\Big|\sin\frac{\nu_0\pi}{2n}\Big|^{\alpha p}.$$

Since $\sin t/t$ tends to 1 as $t\to 0$, then for all n, greater that a certain number n_0, the inequality $2^\alpha|\sin\nu_0\pi/(2n)|^\alpha > (\pi^\alpha-\varepsilon)\nu_0^\alpha/n^\alpha$ holds, which yields (43).

Since $\nu^{\alpha p}-(\nu-1)^{\alpha p}\le \alpha p\nu^{\alpha p-1}$, it follows from inequality (38) that

$$\omega_\alpha^\triangle\Big(f,\frac{\pi}{n}\Big)_{\mathcal{S}^p} \le \frac{\pi^\alpha(\alpha p)^{1/p}}{n^\alpha}\Big(\sum_{\nu=1}^{n}\nu^{\alpha p-1}E_\nu^\triangle(f)_{\mathcal{S}^p}^p\Big)^{1/p}. \tag{46}$$

This, in particular, yields the following statement:

Corollary 5.5. *Assume that $f\in\mathcal{S}^p$, $1\le p<\infty$, the sequence of the best approximations $E_n^\triangle(f)_{\mathcal{S}^p}$ of the function f satisfies the relation $E_n^\triangle(f)_{\mathcal{S}^p} = O(n^{-\beta})$ with a certain $\beta>0$. Then, for any $\alpha>0$,*

$$\omega_\alpha^\triangle(f,t)_{\mathcal{S}^p} = \begin{cases} O(t^\beta) & for\ \ \beta<\alpha,\\ O(t^\alpha|\ln t|^{1/p}) & for\ \ \beta=\alpha,\\ O(t^\alpha) & for\ \ \beta>\alpha.\end{cases}$$

In the mentioned above spaces $\mathcal{S}^p$ of 2π-periodic functions of one variable, inequalities (46) were obtained in [18] and [17]. For the spaces $L_p(\mathbb{T}^d)$, inequalities of the type (46) were proved by M. Timan (see [21, 22] and [23, Ch. 2]).

6. Constructive Characteristics of the Classes of Functions Defined by the αth Moduli of Continuity

In the following two sections some applications of the obtained results are considered. In particular, in this section we give the constructive characteristics of the classes $\mathcal{S}^p_{\pm}H^{\omega}_{\alpha}$ of functions for which the αth moduli of smoothness do not exceed some majorant.

Let ω be a function defined on interval $[0,1]$. For a fixed $\alpha > 0$, we set

$$\mathcal{S}^p_{\pm}H^{\omega}_{\alpha} = \Big\{ f \in \mathcal{S}^p_{\pm} : \quad \omega^{\Delta}_{\alpha}(f;\delta)_{\mathcal{S}^p} = O(\omega(\delta)), \quad \delta \to 0+ \Big\}. \tag{47}$$

Further, we consider the functions $\omega(\delta)$, $\delta \in [0,1]$, satisfying the following conditions 1)–4) and $(\mathscr{B}_{\alpha})$:
1) $\omega(\delta)$ is continuous on $[0,1]$; **2)** $\omega(\delta)\uparrow$; **3)** $\omega(\delta) \neq 0$ for any $\delta \in (0,1]$; **4)** $\omega(\delta) \to 0$ as $\delta \to 0$;

$$(\mathscr{B}_{\alpha}),\ \alpha > 0: \quad \sum_{v=1}^{n} v^{\alpha-1}\omega\Big(\frac{1}{v}\Big) = O\Big[n^{\alpha}\omega\Big(\frac{1}{n}\Big)\Big]. \tag{48}$$

The condition $(\mathscr{B}_{\alpha})$ is a well-known condition (see, e.g. [2]).

Theorem 6.1. *Assume that $\alpha > 0$ and ω is a function, satisfying conditions* 1)–4) *and* (48). *Then, in order a function $f \in \mathcal{S}^p_{\pm}$ to belong to the class $\mathcal{S}^p_{\pm}H^{\omega}_{\alpha}$, it is necessary and sufficient that*

$$E_n(f)_{\mathcal{S}^p} = O\Big[\omega\Big(\frac{1}{n}\Big)\Big]. \tag{49}$$

Proof. Let $f \in \mathcal{S}^p_{\pm}H^{\omega}_{\alpha}$, by virtue of Corollary 4.6, we have

$$E_n(f)_{\mathcal{S}^p} \le C(\alpha)\omega^{\Delta}_{\alpha}\Big(f;\frac{1}{n}\Big)_{\mathcal{S}^p}, \tag{50}$$

Therefore, relation (47) yields (49). On the other hand, if relation (49) holds, then by virtue of (46), taking into account the condition (48), we obtain

$$\omega^{\Delta}_{\alpha}\Big(f,\frac{1}{n}\Big)_{\mathcal{S}^p} \le \frac{C}{n^{\alpha}}\sum_{\nu=1}^{n}\nu^{\alpha-1}E_{\nu}(f)_{\mathcal{S}^p} \le \frac{C_1}{n^{\alpha}}\sum_{\nu=1}^{n}\nu^{\alpha-1}\omega\Big(\frac{1}{\nu}\Big) = O\Big[\omega\Big(\frac{1}{n}\Big)\Big]. \tag{51}$$

Thus, the function f belongs to the set $\mathcal{S}^p_{\pm}H^{\omega}_{\alpha}$. □

The function $\varphi(t) = t^r$, $r \le \alpha$, satisfies the condition (48). Hence, denoting by $\mathcal{S}^p_{\pm}H^{r}_{\alpha}$ the class $\mathcal{S}^p_{\pm}H^{\omega}_{\alpha}$ for $\omega(t) = t^r$, $0 < r \le \alpha$, we establish the following statement:

Corollary 6.2. *Let $\alpha > 0$, $0 < r \le \alpha$. In order a function $f \in \mathcal{S}^p_{\pm}$ to belong to $\mathcal{S}^p_{\pm}H^{r}_{\alpha}$, it is necessary and sufficient that*

$$E_n(f)_{\mathcal{S}^p} = O(n^{-r}).$$

7. The Equivalence Between αth Moduli of Smoothness and K-Functionals

K-functionals were introduced by Lions and Peetre in 1961, and defined in their usual form by Peetre in the monograph [11]. Unlike the moduli of continuity expressing the smooth properties of functions, K-functionals express some of their approximative properties. In this section, the equivalence between αth moduli of smoothness and certain Peetre K-functionals is proved in the spaces $\mathcal{S}^p$. This connection is important for studying the properties of the modulus of smoothness and the K-functional, and also for their further application to the problems of approximation theory.

In the space $\mathcal{S}^p$, the Petree K-functional of a function f (see, e.g. [7, Ch. 6]), which generated by its derivative of order $\alpha > 0$, is the following quantity:

$$K_{\alpha}(t,f)_{\mathcal{S}^p} = \inf\Big\{\|f-h\|_{\mathcal{S}^p} + t^{\alpha}\|h^{(\alpha)}\|_{\mathcal{S}^p} : h^{(\alpha)} \in \mathcal{S}^p\Big\}, \quad t > 0.$$

Theorem 7.1. *For each $f \in \mathcal{S}^p_{\pm}$, $\alpha > 0$, there exist constants $C_1(\alpha)$, $C_2(\alpha) > 0$, such that for $t > 0$,*

$$C_1(\alpha)\omega^{\Delta}_{\alpha}(f,t)_{\mathcal{S}^p} \le K_\alpha(t,f)_{\mathcal{S}^p} \le C_2(\alpha)\omega^{\Delta}_{\alpha}(f,t)_{\mathcal{S}^p}. \tag{52}$$

Proof. Let $f \in \mathcal{S}^p_{\pm}$. Consider an arbitrary function $h \in \mathcal{S}^p$ such that $h^{(\alpha)} \in \mathcal{S}^p$. Then we have by Lemma 3.2 (iii), (v) and (vi)

$$\omega^{\Delta}_{\alpha}(f,t)_{\mathcal{S}^p} \le \omega^{\Delta}_{\alpha}(f-h,t)_{\mathcal{S}^p} + \omega^{\Delta}_{\alpha}(h,t)_{\mathcal{S}^p} \le 2^{\{\alpha\}}\|f-h\|_{\mathcal{S}^p} + t^\alpha\|h^{(\alpha)}\|_{\mathcal{S}^p}.$$

Taking the infimum over all $h \in \mathcal{S}^p$ such that $h^{(\alpha)} \in \mathcal{S}^p$, we get the left-hand side of (52).

To prove the right-hand side of (52), let us formulate the following auxiliary lemma.

Lemma 7.2. *Assume that $\alpha > 0$, $n \in \mathbb{N}$ and $0 < h < 2\pi/n$. Then for any $\tau_n \in \mathcal{T}_n \cap \mathcal{S}^p_{\pm}$*

$$\|\tau_n^{(\alpha)}\|_{\mathcal{S}^p} \le \Big(\frac{n}{2\sin nh/2}\Big)^\alpha \|\Delta^\alpha_h \tau_n\|_{\mathcal{S}^p}. \tag{53}$$

Now let $t \in (0, 2\pi)$ and $n \in \mathbb{N}$ such that $\pi/n < t < 2\pi/n$. Let also $S_n := S_n(f)$ be the Fourier sum of f. Using Lemma 7.2 with $h = \pi/n$ and property (i) of Lemma 3.1, we obtain

$$\|(S^{\Delta}_n)^{(\alpha)}\|_{\mathcal{S}^p} \le 2^{-\alpha}n^\alpha\|\Delta^\alpha_{\pi/n}S^{\Delta}_n\|_{\mathcal{S}^p} \le (\pi/t)^\alpha\Big(\|\Delta^\alpha_{\pi/n}(S^{\Delta}_n - f)\|_{\mathcal{S}^p} + \|\Delta^\alpha_{\pi/n}f\|_{\mathcal{S}^p}\Big)$$

$$\le (\pi/t)^\alpha\Big(2^{\{\alpha\}}\|f - S^{\Delta}_n\|_{\mathcal{S}^p} + \|\Delta^\alpha_{\pi/n}f\|_{\mathcal{S}^p}\Big). \tag{54}$$

By virtue of (7) and Corollary 4.6, we have

$$\|f - S^{\Delta}_n\|_{\mathcal{S}^p} = E_n(f)_{\mathcal{S}^p} \le C(\alpha)\omega^{\Delta}_{\alpha}(f;\delta)_{\mathcal{S}^p}. \tag{55}$$

Combining (54), (55) and the definition of modulus of smoothness, we obtain the relation

$$\|(S^{\Delta}_n)^{(\alpha)}\|_{\mathcal{S}^p} \le C_2(\alpha)t^{-\alpha}\omega^{\Delta}_{\alpha}(f;\delta)_{\mathcal{S}^p},$$

where $C_2(\alpha) := 2\pi^\alpha(2^{\{\alpha\}}C(\alpha) + 1)$, which yields the right-hand side of (52):

$$K_\alpha(t,f)_{\mathcal{S}^p} \le \|f - S^{\Delta}_n\|_{\mathcal{S}^p} + t^\alpha\|(S^{\Delta}_n)^{(\alpha)}\|_{\mathcal{S}^p} \le C_2(\alpha)\omega^{\Delta}_{\alpha}(f,\delta)_{\mathcal{S}^p}.$$

□

8. Proof of the Auxiliary Statements

Proof of Lemma 3.1. Setting $f_{jh}(\mathbf{x}) := f(\mathbf{x} - jh)$, we see that $\widehat{f_{jh}}(\mathbf{k}) = \widehat{f}(\mathbf{k})\mathrm{e}^{-i(\mathbf{k},jh)} = \widehat{f}(\mathbf{k})\mathrm{e}^{-ijh(k_1+\ldots+k_d)}$ and

$$\|\Delta^\alpha_h f\|^p_{\mathcal{S}^p} = \sum_{\mathbf{k}\in\mathbb{Z}^d} |[\Delta^\alpha_h f]\widehat{\ }(\mathbf{k})|^p = \sum_{\mathbf{k}\in\mathbb{Z}^d}\Big|\Big[\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f_{jh}\Big]\widehat{\ }(\mathbf{k})\Big|^p$$

$$= \sum_{\mathbf{k}\in\mathbb{Z}^d}\Big|\widehat{f}(\mathbf{k})\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\mathrm{e}^{-i(\mathbf{k},jh)}\Big|^p \le \Big(\sum_{j=0}^{\infty}\Big|\binom{\alpha}{j}\Big|\Big)^p\sum_{\mathbf{k}\in\mathbb{Z}^d}|\widehat{f}(\mathbf{k})|^p \le 2^{\{\alpha\}p}\|f\|^p_{\mathcal{S}^p}.$$

The property (ii) is simple:

$$[\Delta^\alpha_h f]\widehat{\ }(\mathbf{k}) = \Big[\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f_{jh}\Big]\widehat{\ }(\mathbf{k}) = \widehat{f}(\mathbf{k})\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}\mathrm{e}^{-i(\mathbf{k},jh)} = (1 - \mathrm{e}^{-i(\mathbf{k},h)})^\alpha\widehat{f}(\mathbf{k}). \tag{56}$$

and it yields (iii). Part (iv) follows by (i)–(iii). Concerning (v), let us note that by virtue of (ii),

$$\|\Delta_h^\alpha f\|_{\mathcal{S}^p}^p = \sum_{\mathbf{k}\in\mathbb{Z}^d} \left|[\Delta_h^\alpha f]^\frown(\mathbf{k})\right|^p = \sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{f}(\mathbf{k})|^p |1-\mathrm{e}^{-i(\mathbf{k},h)}|^{\alpha p} = 2^{\alpha p}\sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{f}(\mathbf{k})|^p \left|\sin\frac{(\mathbf{k},h)}{2}\right|^{\alpha p} < \infty.$$

Therefore, for any $\varepsilon > 0$ there exist numbers $n_0 = n_0(\varepsilon) \in \mathbb{N}_0$ and $\delta = \delta(\varepsilon, n_0) > 0$ such that for $|h| < \delta$,

$$2^{-\alpha p}\|\Delta_h^\alpha f\|_{\mathcal{S}^p}^p = \sum_{|\mathbf{k}|_1\le n_0} |\widehat{f}(\mathbf{k})|^p \left|\sin\frac{(\mathbf{k},h)}{2}\right|^{\alpha p} + \sum_{|\mathbf{k}|_1> n_0} |\widehat{f}(\mathbf{k})|^p \left|\sin\frac{(\mathbf{k},h)}{2}\right|^{\alpha p} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

□

Proof of Lemma 3.2. In (i), the convergence to zero for $t \to 0+$ follows by Lemma 3.1 (v); the property (iii), non-negativity and increasing of the function $\omega_\alpha^\Delta(f,t)_{\mathcal{S}^p}$ follow from the definition of modulus of smoothness. The property (ii) is a consequence of Lemma 3.1 (iv). Part (iv) is proved by the following standard arguments:

$$\omega_1^\Delta(f, t_1+t_2)_{\mathcal{S}^p} = \sup_{|h_1|\le t_1, |h_2|\le t_2} \|f(\cdot+h_1+h_2) - f(\cdot)\|_{\mathcal{S}^p} \le \sup_{|h_2|\le t_2} \|f(\cdot+h_1+h_2) - f(\cdot+h_1)\|_{\mathcal{S}^p}$$
$$+ \sup_{|h_1|\le t_1} \|f(\cdot+h_1) - f(\cdot)\|_{\mathcal{S}^p} \le \omega_1^\Delta(f,t_2)_{\mathcal{S}^p} + \omega_1^\Delta(f,t_1)_{\mathcal{S}^p}.$$

In particular, this yields continuity of the function $\omega_1^\Delta(f,t)_{\mathcal{S}^p}$, since for arbitrary $t_1 > t_2 > 0$, $\omega_1^\Delta(f,t_1)_{\mathcal{S}^p} - \omega_1^\Delta(f,t_2)_{\mathcal{S}^p} \le \omega_1^\Delta(t_1-t_2)_{\mathcal{S}^p} \to 0$ as $t_1 - t_2 \to 0$.

Let us prove the continuity of the function $\omega_\alpha^\Delta(f,t)_{\mathcal{S}^p}$ for arbitrary $\alpha > 0$. Let $0 < t_1 < \delta_2$ and $h = h_1 + h_2$, where $0 < h_1 \le t_1$, $0 < h_2 \le t_2 - t_1$. Since $\Delta_h^\alpha f(\mathbf{x}) = \Delta_{h_1}^\alpha f(\mathbf{x}) + \sum_{j=0}^\infty (-1)^j\binom{\alpha}{j}\Delta_{jh_2}^1 f(\mathbf{x}-jh_1)$, $\mathbf{x}\in\mathbb{R}^d$, and

$$\Big\|\sum_{j=0}^\infty (-1)^j\binom{\alpha}{j}\Delta_{jh_2}^1 f_{jh_1}\Big\|_{\mathcal{S}^p}^p = \sum_{\mathbf{k}\in\mathbb{Z}^d}\Big|\Big[\sum_{j=0}^\infty (-1)^j\binom{\alpha}{j}\Delta_{jh_2}^1 f_{jh_1}\Big]^\frown(\mathbf{k})\Big|^p$$

$$\le \sum_{\mathbf{k}\in\mathbb{Z}^d}\Big|\sum_{j=0}^\infty (-1)^j\binom{\alpha}{j} j\Big|^p |[\Delta_{h_2}^1 f]^\frown(\mathbf{k})|^p \le 2^{\{\alpha\}}\alpha\sum_{\mathbf{k}\in\mathbb{Z}^d} |[\Delta_{h_2}^1 f]^\frown(\mathbf{k})|^p \le 2^{\alpha p}\alpha\|\Delta_{h_2}^1 f\|_{\mathcal{S}^p}^p,$$

then $\|\Delta_h^\alpha f\|_{\mathcal{S}^p} \le \|\Delta_{h_1}^\alpha f\|_{\mathcal{S}^p} + 2^{\{\alpha\}}\alpha\|\Delta_{h_2}^1 f\|_{\mathcal{S}^p}$ and $\omega_\alpha^\Delta(f,t_2) \le \omega_\alpha^\Delta(f,t_1) + 2^{\{\alpha\}}\alpha\,\omega_1^\Delta(f,t_2-t_1)$. Hence, we obtain the necessary relation: $\omega_\alpha^\Delta(f,t_2) - \omega_\alpha^\Delta(f,t_1) \le 2^{\{\alpha\}}\alpha\omega_1^\Delta(f,t_2-t_1) \to 0$, $t_2 - t_1 \to 0$.

If for the function $f \in \mathcal{S}^p$ there exists a derivative $f^{(\beta)} \in \mathcal{S}^p$, $0 < \beta \le \alpha$, then by virtue of (56) and (6), for arbitrary numbers $\mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ and $h \in [0, t]$, we have

$$\left|[\Delta_h^\alpha f]^\frown(\mathbf{k})\right| = 2^\beta\left|\sin\frac{(\mathbf{k},h)}{2}\right|^\beta |1-\mathrm{e}^{-i(\mathbf{k},h)}|^{\alpha-\beta}|\widehat{f}(\mathbf{k})| \le t^\beta|\mathbf{k}|^\beta|1-\mathrm{e}^{-i(\mathbf{k},h)}|^{\alpha-\beta}|\widehat{f}(\mathbf{k})| \le t^\beta\left|[\Delta_h^{\alpha-\beta} f^{(\beta)}]^\frown(\mathbf{k})\right|,$$

and therefore property (vi) holds. □

Proof of Lemma 7.2. Let $\tau_n = \sum_{\nu=0}^n \sum_{|\mathbf{k}|_1=\nu} a_{\mathbf{k}}\mathrm{e}^{i(\mathbf{k},\mathbf{x})}$ is an arbitrary polynomial of the set $\mathscr{T}_n \cap \mathcal{S}_\pm^p$ and $0 < h < 2\pi/n$. By virtue of the definition of the set $\mathcal{S}_\pm^p$ and relation (13), we have

$$\|\Delta_h^\alpha \tau_n\|_{\mathcal{S}^p}^p = 2^{\alpha p}\sum_{\nu=0}^n \left|\sin\nu h/2\right|^{\alpha p}\sum_{|\mathbf{k}|_1=\nu}|a_{\mathbf{k}}|^p = \sum_{\nu=0}^n\left|\frac{\sin\nu h/2}{\nu h/2}\cdot\nu h\right|^{\alpha p}\sum_{|\mathbf{k}|_1=\nu}|a_{\mathbf{k}}|^p.$$

Since the function $\sin t/t$ decrease on $[0,\pi]$, then for any $0 < h < 2\pi/n$, we get

$$\|\Delta_h^\alpha \tau_n\|_{\mathcal{S}^p}^p \ge \Big(\frac{\sin nh/2}{nh/2}\Big)^{\alpha p}\sum_{\nu=0}^n |\nu h|^{\alpha p}\sum_{|\mathbf{k}|_1=\nu}|a_{\mathbf{k}}|^p \ge \Big(\frac{2\sin nh/2}{n}\Big)^{\alpha p}\|\tau_n^{(\alpha)}\|_{\mathcal{S}^p}^p.$$

□

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