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Existence and Uniqueness of Solutions for the First-Order Non-Linear Differential Equations with Three-Point Boundary Conditions

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Abstract. In this paper the existence and uniqueness of the solutions to boundary value problems for the first order non-linear system of the ordinary differential equations with three-point boundary conditions are investigated. For the first time the Green function is constructed and the considered problem is reduced to the equivalent integral equations that allow us to prove the existence and uniqueness theorems in differ from existing works, applying the Banach contraction mapping principle and Schaefer's fixed point theorem. An example is given to illustrate the obtained results.

1. Introduction

The multipoint boundary value problems for ODEs and their systems are intensively investigated in recent years. This is related with their strong relation with a broad range of applications in different fields of physics and mathematics [3, 4]. As examples for application we can note the vibrations of a uniform cross-section string with composed of N parts of different densities, some problems in the theory of elastic stability [16], etc. In mathematical formulations these problems are described by the multipoint boundary value problems.

The study of multi-point boundary-value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [7]. Since then, nonlinear multi-point boundary-value problems have been studied by several authors using the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory, and fixed point theorem in cones.

The existence questions for such problems have been studied by various authors. For example, this problem for the second-order multipoint boundary value problems have been studied in [5, 8, 17, 18] and references therein. We can note also Gupta [6] where the existence of solutions for the generalized multi-point boundary-value problem is studied.

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But it should be noted that this problem is one of the less studied ones in the case of the first-order systems. Multy and Sivasundaram considered this problem using the successive over relaxation iteration and the Banach contraction mapping principle [14]. In that work the authors set the continuous derivative condition that depends on the fundamental matrix of the variational system. R. Ma in [9] gives sufficient conditions for the existence and uniqueness of solutions of the first-order system for the case of Caratheodory function using the Leray-Schauder continuation theorem. Different existence and uniqueness theorems for three-point boundary value problems in resonance problems have been studied in papers [11–13, 15].

Here for the first time the Green function is constructed for the three point boundary value problem and the considered problem is reduced to the equivalent integral equations. In differ from [14] we do not use fundamental matrix of the equation. The advantage of this fact is that we do not impose the existence of the derivative of the right hand side of the equation with respect to the phase coordinates. Then the existence and uniqueness of the solutions is studied using the Banach contraction mapping principle. The existence of the solution is also proved by applying Schaefer's fixed point theorem

2. Problem Statement

We study the existence and uniqueness of solutions of nonlinear differential equations of the type

$$\dot{x} = f(t, x), \quad t \in [0, T], \tag{1}$$

with three-point boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = d, \tag{2}$$

where A, B, C are constant square matrices of order n such that $detN \neq 0,\ N = (A+B+C)$; $f : [0, T]\times R^n \to R^n$ is a given function; t_1 satisfies the condition of $0 < t_1 < T$.

We denote by $C([0, T]; R^n)$ the Banach space of all continuous functions from $[0, T]$ into R^n with the norm

$$\|x\| = \max\{|x(t)| : t \in [0.T]\},$$

where $|\cdot|$ is the norm in the space R^n.

The purpose of this paper is to prove new existence and uniqueness results using Banach contraction principle and Schaefer's fixed point theorem.

3. Preliminaries

We define the solution of problem (1)-(2) as follows:

Definition 3.1. The function $x \in C([0, T]; R^n)$ is said to be a solution of problem (1)-(2) if $\dot{x}(t) = f(t, x(t))$, for each $t \in [0, T]$, and boundary conditions (2) are satisfied.

For the sake of simplicity, we can consider the following problem:

$$\dot{x} = y(t), \quad t \in [0, T], \tag{3}$$

$$Ax(0) + Bx(t_1) + Cx(T) = d. \tag{4}$$

Lemma 3.2. *Let $y \in C([0, T]; R^n)$. The unique solution of the boundary value problem for differential equation (3) with boundary conditions (4) is given by*

$$x(t) = N^{-1}d + \int_0^T G(t.\tau)y(\tau)d\tau, \tag{5}$$

where

$$G(t,\tau) = \begin{cases} G_1(t,\tau), & 0 \le t \le t_1, \\ G_2(t,\tau), & t_1 < t \le T, \end{cases}$$

with

$$G_1(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t, \\ -N^{-1}(B+C), & t < \tau \le t_1, \\ -N^{-1}C, & t_1 < \tau \le T, \end{cases}$$

and

$$G_2(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t_1, \\ N^{-1}(A+B), & t_1 < \tau \le t, \\ -N^{-1}C, & t < \tau \le T. \end{cases}$$

Proof. If $x = x(\cdot)$ is a solution of differential equation (3), then for $t \in (0,T)$

$$x(t) = x_0 + \int_0^t y(\tau)d\tau, \tag{6}$$

where x_0 is an arbitrary constant vector. In order the function in equality (6) satisfy condition (4) we determine x_0 as follows

$$x_0 = N^{-1}d - N^{-1}B\int_0^{t_1} y(t)dt - N^{-1}C\int_0^T y(t)dt. \tag{7}$$

Now taking into account the value x_0 determined from equality (7) in (6), we obtain

$$x(t) = N^{-1}d - N^{-1}B\int_0^{t_1} y(t)dt - N^{-1}C\int_0^T y(t)dt + \int_0^t y(\tau)d\tau. \tag{8}$$

Now suppose that $t \in [0,t_1]$. Then we can rewrite equality (8) as follows:

$$x(t) = N^{-1}d - N^{-1}B(\int_0^t y(\tau)d\tau + \int_t^{t_1} y(\tau)d\tau) - N^{-1}C(\int_0^t y(\tau)d\tau + \int_t^{t_1} y(\tau)d\tau)$$

$$-N^{-1}C\int_{t_1}^T y(t)dt + \int_0^t y(\tau)d\tau.$$

Here grouping the like terms, and then simplifying we get

$$x(t) = N^{-1}d + (E - N^{-1}B - N^{-1}C)\int_0^t y(\tau)d\tau$$

$$-(N^{-1}B + N^{-1}C)\int_t^{t_1} y(\tau)d\tau - N^{-1}C\int_{t_1}^T y(t)dt$$

$$= N^{-1}d + N^{-1}A\int_0^t y(\tau)d\tau - N^{-1}(B+C)\int_t^{t_1} y(\tau)d\tau - N^{-1}C\int_{t_1}^T y(t)dt. \tag{9}$$

Here E is an identity matrix. Let us introduce the new function as follows:

$$G_1(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t, \\ -N^{-1}(B+C), & t < \tau \le t_1, \\ -N^{-1}C, & t_1 < \tau \le T. \end{cases}$$

Using this equality, relation (9) may be written as an integral equation

$$x(t) = N^{-1}d + \int_0^T G_1(t,\tau)y(\tau)d\tau.$$

For the case $t \in (t_1, T]$ we can write equality (8) as follows

$$x(t) = N^{-1}d - N^{-1}B\int_0^{t_1} y(t)dt - N^{-1}C\int_0^{t_1} y(t)dt$$

$$-N^{-1}C(\int_{t_1}^t y(\tau)d\tau + \int_t^T y(\tau)d\tau) + \int_0^{t_1} y(t)dt + \int_{t_1}^t y(\tau)d\tau$$

$$= N^{-1}d + (E - N^{-1}B - N^{-1}C)\int_0^{t_1} y(t)dt + (E - N^{-1}C)\int_{t_1}^t y(\tau)d\tau$$

$$-N^{-1}C\int_t^T y(\tau)d\tau = N^{-1}d + N^{-1}A\int_0^{t_1} y(t)dt$$

$$+N^{-1}(A+B)\int_{t_1}^t y(\tau)d\tau - N^{-1}C\int_t^T y(\tau)d\tau.$$

Here we introduce the new function

$$G_2(t,\tau) = \begin{cases} N^{-1}A, & 0 \le \tau \le t_1, \\ N^{-1}(A+B), & t_1 < \tau \le t, \\ -N^{-1}C, & t < \tau \le T. \end{cases}$$

Hence for the case $t \in (t_1, T]$ we can write equality (8) in the following form

$$x(t) = N^{-1}d + \int_0^T G_2(t,\tau)y(\tau)d\tau.$$

So, we conclude that the solution of boundary-value problem (3)-(4) is in the form

$$x(t) = N^{-1}d + \int_0^T G(t,\tau)y(\tau)d\tau.$$

So, we showed the validity of formula (5). The proof is completed. □

Lemma 3.3. *Assume that $f \in C([0,T] \times R^n; R^n)$. Then the function $x(t)$ is a solution of boundary-value problem (1)-(2) if and only if $x(t)$ is a solution of the integral equation*

$$x(t) = N^{-1}d + \int_0^T G(t,\tau)f(\tau, x(\tau))d\tau.$$

Proof. Let $x(t)$ be a solution of boundary-value problem (1)-(2). Similarly as in Lemma 3.2 one can prove that it is also a solution of integral equation (8). It is clear that, the solution of integral equation (8) satisfies boundary-value problem (1)-(2). Lemma 3.3 is proved. □

4. Main results

Let us define the operator $P : C([0, T]; R^n) \to P([0, T]; R^n)$ as

$$Px(t) = N^{-1}d + \int_0^T G(t, \tau) f(\tau, x(\tau)) d\tau.$$

It is known that problem (1), (2) is equivalent to the fixed point problem $x = Px$. So, problem (1), (2) has a solution if and only if the operator P has a fixed point.

In Lemma 3.2, we used the most basic fixed point theorem called the contraction mapping principle that uses the assumption:
H1) There exists a continuous function $M(t) \geq 0$ such that

$$|f(t, x) - f(t, y)| \leq M(t)|x - y|$$

for each $t \in [0, T]$ and all $x, y \in R^n$.

Theorem 4.1. *Assume that the above assumption holds, and*

$$L = TSM < 1, \tag{10}$$

where

$$M = \max_{[0,T]} M(t),$$

$$S = \max_{[0,T]\times[0,T]} \|G(t, \tau)\|.$$

Then boundary-value problem (1),(2) has a unique solution on $[0, T]$.

Proof. Denoting $\max_{[0,T]} |f(t, 0)| = M_f$ and choosing $r \geq \frac{\|N^{-1}d\|+M_f TS}{1-L}$, we show that $PB_r \subset B_r$, where

$$B_r = \{x \in C([0, T]; R^n) : \|x\| \leq r\}.$$

For $x \in B_r$ we have

$$\|Px(t)\| \leq \|N^{-1}d\| + \int_0^T |G(t, \tau)|| f(\tau, x(\tau)) - f(\tau, 0)| + |f(\tau, 0)|) d\tau$$

$$\leq \|N^{-1}d\| + S \int_0^T (M|x| + M_f) dt \leq \|N^{-1}d\| + SMrT + M_f TS \leq \frac{\|N^{-1}d\| + M_f TS}{1 - L} \leq r.$$

Now for any $x, y \in B_r$ it is valid

$$|Px - Py| \leq \int_0^T |G(t, \tau)(f(\tau, x(\tau)) - f(\tau, y(\tau))| d\tau$$

$$\leq \int_0^T |G(t, \tau)||f(x(\tau), \tau) - f(\tau, y(\tau))| d\tau$$

$$\leq S \int_0^T M(t)|x(t) - y(t)| dt \leq MST \max_{[0,T]} |x(t) - y(t)| \leq MST\|x - y\|$$

or

$$\|Px - Py\| \leq L\|x - y\|.$$

It is clear that P is contraction by condition (10). So, boundary-value problem (3), (4) has a unique solution. □

Our second result is based on Schafer's fixed point theorem that uses the following assumptions:
(H2) The function $f : [0, T] \times R^n \to R^n$ is continuous;
(H3) There exists a constant $N_1 > 0$ such that $|f(t, x)| \leq N_1$ for each $t \in [0, T]$ and all $x \in R^n$.

Theorem 4.2. *Assume (H2),(H3) hold. Then boundary-value problem (3), (4) has at least one solution on* $[0, T]$.

Proof. To prove the theorem we show the existence of the fixed point for the operator P under the assumptions of the theorem. As is accepted in the existing literature we give the proof of the theorem in following steps.
Step 1. Here we prove the continuity of the operator P. For this purpose let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0, T]; R^n)$. Then for any $t \in T$

$$|P(x_n)(t) - P(x)(t)| \leq |\int_0^T G(t, \tau)(f(\tau, x_n(\tau)) - f(\tau, x(\tau)))d\tau|$$

$$\leq S \int_0^T |f(\tau, x_n(\tau)) - f(\tau, x(\tau))|d\tau \leq S \max_{[0,T]} |f(\tau, x_n(\tau)) - f(\tau, x(\tau))|T.$$

From the continuity of the function f we have

$$\|P(x_n)(t) - P(x)(t)\| \to 0, \quad \text{as} \quad n \to \infty.$$

Step 2. The purpose of this step is to prove that the operator P maps bounded sets in $C([0, T]; R^n)$. To do this it is enough to show that, for any $\eta > 0$ there exists a positive constant l such that for each $\{x \in C([0, T]; R^n) : \|x\| \leq \eta\}$, it is true $\|P(x)\| \leq l$. For each $t \in [0, T]$, by $H(3)$ we have

$$|P(x)(t)| \leq \|N^{-1}d\| + \int_0^T |G(t, \tau)\|f(\tau, x(\tau))|d\tau$$

Hence,

$$|P(x)(t)| \leq \|N^{-1}d\| + SN_1T.$$

Thus,

$$\|P(x)(t)\| \leq \|N^{-1}d\| + SN_1T = l.$$

Step 3. Here we prove that the operator P maps bounded sets into equicontinuous sets from $C([0, T]; R^n)$. Let $\tau_1, \tau_2 \in [0, T], \tau_1 < \tau_2, B_\eta$ be a bounded set in $C([0, T]; R^n)$. As in Step 2 we assume that $x \in B_\eta$.
Case 1:
Let $\tau_1, \tau_2 \in [0, t_1]$. Then

$$|P(x)(\tau_2) - P(x)(\tau_1)| = |N^{-1}A \int_0^{\tau_2} f(\tau, x(\tau))d\tau - N^{-1}(B + C) \int_{\tau_2}^{t_1} f(\tau, x(\tau))d\tau$$

$$-N^{-1}A \int_0^{\tau_1} f(\tau, x(\tau))d\tau + N^{-1}(B + C) \int_{\tau_1}^{t_1} f(\tau, x(\tau))d\tau| \leq \|N^{-1}A\| \int_{\tau_1}^{\tau_2} |f(\tau, x(\tau))|d\tau$$

$$+\|N^{-1}(B + C)\| \int_{\tau_1}^{\tau_2} |f(\tau, x(\tau))|d\tau \leq S \int_{\tau_1}^{\tau_2} |f(\tau, x(\tau))d\tau|.$$

Case 2: For the case $\tau_1 \in [0, t_1]$, and $\tau_2 \in (t_1, T]$ we can write

$$|P(x)(\tau_2) - P(x)(\tau_1)| = |N^{-1}A \int_0^{t_1} f(\tau, x(\tau))d\tau + N^{-1}(A + B) \int_{t_1}^{\tau_2} f(\tau, x(\tau))d\tau$$

$$-N^{-1}C \int_{\tau_2}^{T} f(\tau, x(\tau))d\tau - N^{-1}A \int_0^{\tau_1} f(\tau, x(\tau))d\tau + N^{-1}(B + C) \int_{\tau_1}^{t_1} f(\tau, x(\tau))d\tau$$

$$+N^{-1}C\int\limits_{t_1}^{T} f(\tau,x(\tau))d\tau| \leq \int_{\tau_1}^{\tau_2}|f(\tau,x(\tau))|d\tau.$$

Case 3: Finally if $\tau_1,\tau_2\in[t_1,T]$ then

$$|P(x)(\tau_2)-P(x)(\tau_1)| = |N^{-1}(A+B)\int_{t_1}^{\tau_2} f(\tau,x(\tau))d\tau - N^{-1}C\int_{\tau_2}^{T} f(\tau,x(\tau))d\tau$$

$$-N^{-1}(A+B)\int_{t_1}^{\tau_1} f(\tau,x(\tau))d\tau + N^{-1}C\int_{\tau_1}^{T} f(\tau,x(\tau))d\tau|$$

$$\leq \|N^{-1}(A+B)\|\int_{\tau_1}^{\tau_2}|f(\tau,x(\tau))|d\tau + \|N^{-1}C\|\int_{\tau_1}^{\tau_2}|f(\tau,x(\tau))d\tau|$$

$$\leq \max\{\|N^{-1}C\|,\|N^{-1}(A+B)\|\}\int_{\tau_1}^{\tau_2}|f(\tau,x(\tau))|d\tau \leq S\int_{\tau_1}^{\tau_2}|f(\tau,x(\tau))|d\tau.$$

The right-hand side of the above inequalities for all three cases 1-3 tends to zero by $\tau_1\to\tau_2$. From this due to Arzela-Ascoli theorem and Step 1-3 follows that the mapping $P: C([0,T];R^n)\to C([0,T];R^n)$ is completely continuous.

Step 4. Here we prove the necessary apriori bounds. Indeed we show that the set $\Omega=\{x\in C([0,T];R^n): x=\lambda P(x),$ for some $0<\lambda<1\}$ is bounded.

Suppose that $x=\lambda(Px)$ for some $0<\lambda<1$. Then for each $t\in[0,T]$ one can write

$$x(t)=\lambda N^{-1}d+\lambda\int_0^T G(t,\tau)f(\tau,x(\tau))d\tau.$$

This fact in combination with H(3) shows that for each $t\in[0,T]$,

$$|P(x)(t)|\leq \|N^{-1}d\|+SN_1T.$$

Therefore we conclude that

$$\|x\|\leq \|N^{-1}d\|+SN_1T.$$

for each $t\in[0,T]$.

Thus the set Ω is bounded and P has a fixed point by Schaefer's fixed point theorem, that is a solution of problem (3)-(4). □

Similar problems for two-point boundary value problems are considered in [1, 2, 10].

5. Examples

In this section, we give some examples to illustrate the main results obtained in this paper.

Example 5.1. Let us consider the following system of differential equations with three-point boundary condition

$$\begin{cases}\dot{x}_1 = 0.1\cos x_2, \\ \dot{x}_2 = \frac{|x_1|}{(9+e^t)(1+|x_1|)}, \end{cases} \qquad t\in[0,2], \tag{11}$$

$$x_1(0)+\frac{1}{2}x_2(1)-\frac{1}{2}x_2(2)=1, \qquad x_2(2)=1. \tag{12}$$

We can rewrite the problem (11), (12) in the equivalent form:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}+\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix}+\begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Obviously,

$$N=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here the matrix N is invertible, and $N^{-1}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Condition (H1) holds with $G_{\max} \leq 1.5$ and $M = 0.1$ and condition (10) is satisfied.
Hence,

$$L = G_{\max}MT = 1.5 \cdot 0.1 \cdot 2 = 0.3 < 1.$$

So, by Theorem 3.1 boundary-value problem (11)-(12) has a unique solution on [0,2].

Example 5.2. Let us consider the following boundary-value problem on [0,2]

$$\begin{cases} \dot{x}_1 = \frac{1}{1+x_2^2}, \\ \dot{x}_2 = \frac{1}{1+x_1^2}, \end{cases} \tag{13}$$

with

$$\begin{cases} x_1(0) + \frac{1}{2}x_2(1) = 1, \\ x_1(1) + \frac{1}{4}x_2(2) = 1. \end{cases} \tag{14}$$

Conditions (14) in the equivalent form will be as follows

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}+\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Obviously,

$$N=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}=\begin{pmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{4} \end{pmatrix}.$$

The matrix N here is non-singular and invertible. The function

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}=\begin{pmatrix} \frac{1}{1+x_2^2} \\ \frac{1}{1+x_1^2} \end{pmatrix}$$

is continuous and bounded. By Theorem 3.2 boundary-value problem (13)-(14) has at least one solution on [0,2].

6. Conclusion

In this paper the sufficient conditions are found for the existence and uniqueness of the solutions for the boundary value problems for the first order non-linear system of the ordinary differential equations with three-point boundary conditions. The approach used in the work is general enough and can be applied to the investigation of the similar multi-point problems for the ordinary differential equations as below:

$$\dot{x} = f(t,x), \quad t \in [0,T]$$

$$\sum_{j=0}^{m} L_j x(t_j) = \alpha.$$

Here $0 = t_0 < t_1 < ... < t_{(m-1)} < t_m = T,\ L_j \in R^{n\times n}$ are the given matrices,

$$detN \neq 0, \quad N = \sum_{j=0}^{m} L_j.$$

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