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Investigation of Spectrum and Scattering Function of Impulsive Matrix Difference Operators

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Abstract. In this paper, we consider a second–order impulsive matrix difference operators. Using the asymptotic and analytical properties of the Jost function, we investigate eigenvalues, spectral singularities, resolvent operator, spectrum and scattering function of this problem. Finally, we study spectrum and scattering function of an unperturbated impulsive matrix difference equation.

This paper is dedicated to the 80th birthday of *Professor A. M. Samoilenko*

1. Introduction

In this study, we handle an impulsive discrete matrix Sturm–Liouville boundary value problem on the set of non-negative integers. For clarity, we shortly review some facts from the existing literature about these problems with scalar coefficients without such discontinuities called *impulse*.

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} \tag{1}$$

is a well known second–order difference equation which is the discrete analogue of the Sturm–Liouville equation given by

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \le x < \infty. \tag{2}$$

Here, q is a real (or complex) function, $\{a_n\}$ and $\{b_n\}$ are real (or complex) sequences satisfying certain conditions, λ is a spectral parameter.

Investigation of spectral analysis of boundary value problems (BVP) was first started with the continuous case [24]. Numerous works are devoted to spectral and scattering problems of (2) [8, 11, 12, 16, 19, 21, 25, 29]. Over the years, as a result of developing technology on engineering, physics, control theory, economy and

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other areas, difference equations have taken a prominent attention owing to the necessity of modeling linear and nonlinear problems to discrete equations. For the studies on the spectral and scattering theory of difference equations, we refer to papers [1, 2, 4, 9, 10, 15, 17]. In particular, in the case where $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are real sequences and $a_n > 0$, inverse problems of scattering theory for (1) was intensively investigated in [15] under the condition

$$\sum_{n\in\mathbb{N}} n\left(|1-a_n| + |b_n|\right) < \infty.$$

Besides, in the case, where n is allowed to take all integer numbers and $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}}$ are complex sequences, [2] and [1] are concerned the spectral properties of BVP associated with (1) under the following condition

$$\sum_{n\in\mathbb{Z}} |n|\left(|1-a_n| + |b_n|\right) < \infty.$$

Observe that, the operator corresponds (1) is selfadjoint in [15], while it is nonselfadjoint in [2] and [1]. In the nonselfadjoint case, spectral singularities appear in the continuous spectrum which correspond to the resonance states having a real energy. This physical interpretation of spectral singularities is very essential for quantum mechanics. Consequently, as mentioned above, the spectral theory of difference equations with scalar coefficients is well developed and most of the results have already been obtained. Even though there are some studies about the spectral theory of difference equations with matrix coefficients [3, 6], there is still lack of literature on spectral analysis of such problems. For instance, the spectral and scattering theory of matrix impulsive difference equations have not been treated elsewhere yet.

Impulsive equations have attracted great attention of many researchers as they appear naturally in several real world problems. They are a basic tool to study dynamics that are related to sudden changes in their states. One can encounter some discontinuities or abrupt changes at certain moments during the process of mathematical modeling and simulations including population dynamics, infectious diseases, physiological and pharmaceutical kinetics, chemical kinetics, navigational control of ships, mathematical economy and general control problems. These impulsive actions may cause some important results for mathematical theory. Samoilenko and Peretsyuk have great contribution to this area [26–28]. Rudiments of the general theory of impulsive differential equations can be found in [5, 18, 28] but for recent works on this topic in spectral theory, we can refer to [7, 22, 23, 30, 31]. In the present paper, we propose to discuss some scattering and spectral problems of a matrix difference operator under certain impulsive conditions. Outline of the paper is as follows:

- We first determine the Jost solution, Jost function of the impulsive BVP, then obtain the asymptotic of the Jost function.

- Later, we define the scattering function and obtain the classical results of scattering function for our problem.

- Next, we find the resolvent operator of the difference operator generated by related impulsive BVP, by the poles of the resolvent operator's kernel, we introduce the sets of eigenvalues and spectral singularities of the operator.

- Using the asymptotic equations and uniqueness theorems of analytic functions, we get some results about spectral singularities and eigenvalues.

- At last, we provide an example to illustrate the validity of methods and theory we proposed.

2. Statement of the Problem

Let us introduce a matrix difference operator $\mathcal{T}$ in the Hilbert space $\ell_2(\mathbb{N}, \mathbb{C}^\mu)$ such that

$$\ell_2(\mathbb{N}, \mathbb{C}^\mu) := \left\{ Y = \{Y_n\}_{n\in\mathbb{N}}, Y_n \in \mathbb{C}^\mu, \|Y\|^2 = \sum_{n\in\mathbb{N}} \|Y_n\|^2 < \infty \right\},$$

where $\mathbb{C}^\mu$ is a $\mu-$ dimensional ($\mu < \infty$) Euclidian space, $\|.\|$ denotes the matrix norm in $\mathbb{C}^\mu$. Consider that the operator $\mathcal{T}$ is created by the following difference expression

$$Y_{n-1} + B_n Y_n + Y_{n+1} = \lambda Y_n, \quad n \in \mathbb{N}\backslash\{m_0 - 1, m_0, m_0 + 1\} \tag{3}$$

with the boundary condition

$$Y_0 = 0 \tag{4}$$

and the impulsive conditions

$$\begin{cases} Y_{m_0+1} = KY_{m_0-1} \\ Y_{m_0+2} = MY_{m_0-2}, \end{cases} \tag{5}$$

where $\lambda = 2\cos z$ is a spectral parameter, $\{B_n\}_{n\in\mathbb{N}} := B$ is a selfadjoint matrix acting in $\mathbb{C}^\mu$ satisfying

$$\sum_{n\in\mathbb{N}} n\,\|B_n\| < \infty \tag{6}$$

and m_0 is an arbitrary natural number. Throughout the paper, we will assume that K and M are selfadjoint diagonal matrices in $\mathbb{C}^\mu$ such that all eigenvalues of K and M are different and nonzero. Now, we give some preliminaries to help us for further results.

At first, we should remind that if $Y_n(z)$ is the solution of (3), then $Y_n^T(z)$ will be a solution for (3) since B is selfadjoint, where "T" denotes the transpose operator. Thus, in the case that $Y_n(z)$ and $Z_n(z)$ are any solutions of (3), then we have the following Wronskian

$$W[Y, Z^T](n) := Z_{n-1}^T Y_n - Z_n^T Y_{n-1} \tag{7}$$

independently of n. Next, we define the two semi–strips as follows:

$$D := \left\{ z \in \mathbb{C} : z = x + iy, y > 0, -\frac{\pi}{2} \le x \le \frac{3\pi}{2} \right\}, \quad D_* := D \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right].$$

Then, we shall denote by $P_n(z)$ and $Q_n(z)$ the fundamental solutions of (3) for $z \in D_*$ and $n = 0, 1, 2, ..., m_0 - 1$. They are entire functions of z satisfying the initial conditions

$$\begin{aligned} P_0(z) &= 0, \quad P_1(z) = I \\ Q_0(z) &= I, \quad Q_1(z) = 0. \end{aligned}$$

Equation (3) has another solution $E(z) := \{E_n(z)\}$ represented by

$$E_n(z) = e^{inz}\left[I + \sum_{m=1}^{\infty} K_{nm} e^{imz} \right], \quad n \in \{m_0 + 1, m_0 + 2, ...\}$$

for $z \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C},\ \operatorname{Im} z \geq 0\}$, where K_{nm} is expressed in terms of $\{B_n\}$. We remark that, $E(z)$ is called Jost solution of (3) satisfying following asymptotic equalities for $z \in \overline{\mathbb{C}}_+$

$$E_n(z) = e^{inz}\left[I + o(1)\right], \quad n \to \infty \tag{8}$$
$$E_n(z) = e^{inz}\left[I + o(1)\right], \quad \operatorname{Im} z \to \infty.$$

Additionally, it is analytic in $\mathbb{C}_+ := \{z \in \mathbb{C},\ \operatorname{Im} z > 0\}$, continuous in $\overline{\mathbb{C}}_+$ and 2π periodic, i.e., $E_n(z) = E_n(z+2\pi)$. It is a fact that $E_n(z)$ is a bounded solution of (3), but there exists an unbounded solution of (3) denoted by $\widehat{E}_n(z)$ satisfying

$$\widehat{E}_n(z) = e^{-inz}\left[I + o(1)\right], \quad z \in \overline{\mathbb{C}}_+, \quad n \to \infty. \tag{9}$$

3. Jost Solution and Scattering Matrix

In this section, we present some new definitions and results which are slightly different from the continuous case. Using the previous functions P, Q and E, let us consider any solution of (3)-(5) and write as

$$J_n(z) = \begin{cases} P_n(z)C_1(z) + Q_n(z)C_2(z), & n \in \{0, 1, ...m_0 - 1\} \\ E_n(z), & n \in \{m_0 + 1, m_0 + 2, ...\}, \end{cases} \tag{10}$$

for $z \in D_*$, where C_1 and C_2 are z- dependent coefficients. The impulsive conditions (5) imply

$$K^{-1}E_{m_0+1}(z) = P_{m_0-1}(z)C_1(z) + Q_{m_0-1}(z)C_2(z) \tag{11}$$

and

$$M^{-1}E_{m_0+2}(z) = P_{m_0-2}(z)C_1(z) + Q_{m_0-2}(z)C_2(z). \tag{12}$$

Using (7), we obtain that $W[P(z), P^T(z)] = 0$ and $W[Q(z), Q^T(z)] = -I$ for all $z \in \overline{\mathbb{C}}_+$, therefore from (11) and (12), we get the coefficients $C_1(z)$ and $C_2(z)$ uniquely

$$C_1(z) = K^{-1}M^{-1}\left\{MQ^T_{m_0-2}(z)E_{m_0+1}(z) - KQ^T_{m_0-1}(z)E_{m_0+2}(z)\right\},$$
$$C_2(z) = K^{-1}M^{-1}\left\{KP^T_{m_0-1}(z)E_{m_0+2}(z) - MP^T_{m_0-2}(z)E_{m_0+1}(z)\right\}$$

for $z \in D_*$. Substituting the last equations in (10), we see that $J_n(z)$ is the Jost solution of the impulsive BVP (3)-(5). Therefore,

$$J_0(z) = C_2(z) := J(z)$$

is called the Jost function of (3)-(5). Note that, the function J is analytic in $\mathbb{C}_+$ and continuous up to the real axis.

Next, we consider another solution of (3)-(5) by

$$F_n(z) = \begin{cases} P_n(z), & n \in \{0, 1, ...m_0 - 1\} \\ E_n(z)C_3(z) + E_n(-z)C_4(z), & n \in \{m_0 + 1, m_0 + 2, ...\}, \end{cases} \tag{13}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. By (5), it is easy to obtain

$$E_{m_0+1}(z)C_3(z) + E_{m_0+1}(-z)C_4(z) = KP_{m_0-1}(z) \tag{14}$$

and

$$E_{m_0+2}(z)C_3(z) + E_{m_0+2}(-z)C_4(z) = MP_{m_0-2}(z). \tag{15}$$

Similarly, due to the fact that $W[E(z), E^T(z)] = 0$ and $W[E(-z), E^T(z)] = -2i\sin z$ for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$, the coefficients $C_3(z)$ and $C_4(z)$ can be directly obtained from (14) and (15) as

$$C_3(z) = -\frac{1}{2i\sin z}\left\{KE^T_{m_0+2}(-z)P_{m_0-1}(z) - ME^T_{m_0+1}(-z)P_{m_0-2}(z)\right\},$$
$$C_4(z) = \frac{1}{2i\sin z}\left\{KE^T_{m_0+2}(z)P_{m_0-1}(z) - ME^T_{m_0+1}(-z)P_{m_0-2}(z)\right\}$$

for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$.

Corollary 3.1. *The coefficients C_2, C_3 and C_4 have the following relationship for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$*

$$C_4^T(z) = C_3^T(-z) = \frac{1}{2i\sin z}KMC_2(z). \tag{16}$$

As a consequence of Corollary 3.1 and (7), we can immediately get

$$W[J, F^T](n) = \begin{cases} -C_2(z), & n \in \{0, 1, ...m_0 - 1\} \\ KMC_2(z), & n \in \{m_0 + 1, m_0 + 2, ...\} \end{cases}$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$.

Theorem 3.2. *For all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$, $\det J(z) \neq 0$.*

Proof. Assume that, there exists a $z_0 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$ such that

$$\det J(z_0) = \det C_2(z_0) = 0.$$

By (16), we find

$$\det C_4^T(z_0) = \det C_3^T(-z_0) = \frac{1}{4\sin^2 z_0}\det K \det M \det C_2(z_0)$$

and it verifies that $\det C_4(z_0) = \det C_3(z_0) = 0$. Since then, there exists a non-zero vector u such that $C_3(z_0)u = 0$ and $C_4(z_0)u = 0$. From (13), the solution F is equal to zero identically, that is, F is a trivial solution of (3)-(5). This gives a contradiction with our assumption, i.e., $\det C_2(z) = \det J(z) \neq 0$ for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$. It completes the proof. □

By Theorem 3.2, the inverse of the function J exists, so we can give the following definition.

Definition 3.3. The matrix function

$$\mathcal{S}(z) = J^{-1}(z)J(-z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$$

is called the *scattering matrix* of the impulsive BVP (3)-(5).

Theorem 3.4. *The matrix function $\mathcal{S}(z)$ satisfies*

$$\mathcal{S}(-z) = \mathcal{S}^{-1}(z) = \mathcal{S}^{*}(z)$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$ *and it is an uniter matrix, where "$*$" denotes the adjoint operator.*

Proof. From the definition of the scattering matrix, we get

$$\mathcal{S}(-z) = J^{-1}(-z)J(z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$$

and it concludes

$$\mathcal{S}(z)\mathcal{S}(-z) = \mathcal{S}(-z)\mathcal{S}(z) = I, \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\},$$

which yields

$$\mathcal{S}(-z) = \mathcal{S}^{-1}(z), \quad z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}.$$

Next, in order to prove $\mathcal{S}^{*}(z) = \mathcal{S}(-z)$, we consider the solutions $F_n(z)$, $J_n(z)$ and $J_n(-z)$, when z belongs to $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}$. Hence, we write

$$\begin{aligned} F_n(z) &= J_n(z)\alpha + J_n(-z)\beta, \\ F_{n+1}(z) &= J_{n+1}(z)\alpha + J_{n+1}(-z)\beta, \end{aligned}$$

where α, β are matrices not depending on n. Premultiplying the first of these by $J^{*}_{n+1}(z)$ and the second by $J^{*}_{n}(z)$ and then subtracting the results, we have

$$\alpha = W^{-1}[J(z), J^{*}(z)] \left\{ J^{*}_{n}(z)F_{n+1}(z) - J^{*}_{n+1}(z)F_n(z) \right\}.$$

In exactly the same way, we find

$$\beta = W^{-1}[J(-z), J^{*}(-z)] \left\{ J^{*}_{n}(-z)F_{n+1}(z) - J^{*}_{n+1}(-z)F_n(z) \right\}.$$

Because of the characteristic features of impulsive equations, we obtain that $W^{-1}[J(z), J^{*}(z)] = -W^{-1}[J(-z), J^{*}(-z)]$. Hence, letting $n = 0$ in the expressions for α and β, we arrive at

$$\alpha = W^{-1}[J(z), J^{*}(z)]J^{*}(z), \quad \beta = -W^{-1}[J(z), J^{*}(z)]J^{*}(-z).$$

As a consequence,

$$F_n(z) = W^{-1}[J(z), J^{*}(z)] \{ J_n(z)J^{*}_{n}(z) - J_n(-z)J^{*}_{n}(-z) \}$$

and upon setting $n = 0$ in this equation, we get

$$J(z)J^{*}(z) = J(-z)J^{*}(-z). \tag{17}$$

By (17), we get

$$J^{*}(z) = J^{-1}(z)J(-z)J^{*}(-z)$$

and so

$$J^{*}(z)[J^{*}(-z)]^{-1} = J^{-1}(z)J(-z).$$

The right hand side of the last equation gives $\mathcal{S}(z)$ and it proves that

$$\mathcal{S}^{*}(z) = J^{-1}(-z)J(z) = \mathcal{S}(-z).$$

Finally, it is obvious that

$$\mathcal{S}\mathcal{S}^{*} = \mathcal{S}^{*}\mathcal{S} = I, \quad \|\mathcal{S}\| = I,$$

that is, $\mathcal{S}$ is uniter. $\square$

4. Eigenvalues, Spectral Singularities and Continuous Spectrum of $\mathcal{T}$

Let us consider a solution of (3)-(5) by

$$G_n(z) = \begin{cases} P_n(z), & n \in \{0, 1, ...m_0 - 1\} \\ E_n(z)C_5(z) + \widehat{E}_n(z)C_6(z), & n \in \{m_0 + 1, m_0 + 2, ...\}, \end{cases}$$

for $z \in D_*$, where $\widehat{E}_n(z)$ denotes the unbounded solution of (3) satisfying (9). Similar to previous ones, it is possible to solve $C_5(z)$ and $C_6(z)$ uniquely. Using (5), we get

$$E_{m_0+1}(z)C_5(z) + \widehat{E}_{m_0+1}(z)C_6(z) = KP_{m_0-1}(z)$$

and

$$E_{m_0+2}(z)C_5(z) + \widehat{E}_{m_0+2}(z)C_6(z) = MP_{m_0-2}(z).$$

Since

$$W[E(z), E^T(z)] = 0, \quad W[\widehat{E}(z), E^T(z)] = -2i \sin z$$

and

$$W[\widehat{E}(z), \widehat{E}^T(z)] = 0, \quad W[E(z), \widehat{E}^T(z)] = 2i \sin z,$$

we clearly obtain

$$C_5(z) = -\frac{1}{2i \sin z} \left\{K\widehat{E}^T_{m_0+2}(z)P_{m_0-1}(z) - M\widehat{E}^T_{m_0+1}(z)P_{m_0-2}(z)\right\},$$
$$C_6(z) = \frac{1}{2i \sin z} \left\{KE^T_{m_0+2}(z)P_{m_0-1}(z) - ME^T_{m_0+1}(z)P_{m_0-2}(z)\right\}$$

for $z \in D_*$. Note that,

$$C_6^T(z) = \frac{1}{2i \sin z} KMC_2(z), \quad z \in D_*. \tag{18}$$

In view of (18), we arrive at the following

$$H(z) := W[J(z), G^T(z)] = \begin{cases} -C_2(z), & n \in \{0, 1, ...m_0 - 1\} \\ KMC_2(z), & n \in \{m_0 + 1, m_0 + 2, ...\}, \end{cases} \tag{19}$$

for $z \in D_*$. Hence, it is clear that the resolvent operator of $\mathcal{T}$ is defined by the following

$$(\mathcal{R}_\lambda(\mathcal{T})\varphi)_n := \sum_{k=0}^{\infty} \mathcal{G}_{n,k}(z)\varphi(k), \quad \varphi := \{\varphi_k\} \in \ell_2(\mathbb{N}, \mathbb{C}^\mu),$$

where

$$\mathcal{G}_{n,k}(z) = \begin{cases} J_n(z)H^{-1}(z)G_k^T(z), & k < n \\ G_n(z)[H^{-1}(z)]^T J_k^T(z), & k \geq n \end{cases}$$

for $z \in D_*$ and $k, n \neq m_0$.

Therefore, due to (19), we will show the sets of eigenvalues and spectral singularities of the operator $\mathcal{T}$ by σ_d and σ_{ss}, respectively as

$$\sigma_d(\mathcal{T}) = \{\lambda = 2\cos z : z \in D_*, \det J(z) = 0\} \tag{20}$$

and

$$\sigma_{ss}(\mathcal{T}) = \left\{\lambda = 2\cos z : z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \backslash \{0, \pi\}, \det J(z) = 0\right\}. \tag{21}$$

So, we can deduce the following by Theorem 3.2 :

Corollary 4.1. *The operator $\mathcal{T}$ has no spectral singularity.*

Theorem 4.2. *Under the condition* (6)*, the Jost function J satisfies the following asymptotic equation*

$$J(z) = (KM)^{-1}(K-M)\,[I + o(1)]\,e^{4iz}, \quad z \in D_*, \quad |z| \to \infty. \tag{22}$$

Proof. Since the polynomial function P is of $(n-1)$. degree according to λ, we can immediately obtain that

$$P_n^T(z)e^{i(n-1)z} = [I + o(1)]\,, \quad z \in D_*, \quad |z| \to \infty. \tag{23}$$

We know that $J(z) = C_2(z)$, so we write

$$J(z) = K^{-1}M^{-1}\left\{KP_{m_0-1}^T(z)E_{m_0+2}(z) - MP_{m_0-2}^T(z)E_{m_0+1}(z)\right\}. \tag{24}$$

It is clear from that

$$KP_{m_0-1}^T(z)E_{m_0+2}(z) = KP_{m_0-1}^T(z)e^{i(m_0-2)z}e^{-i(m_0-2)z}E_{m_0+2}e^{i(m_0+2)z}e^{-i(m_0+2)z} \tag{25}$$

and

$$MP_{m_0-2}^T(z)E_{m_0+1}(z) = MP_{m_0-2}^T(z)e^{i(m_0-3)z}e^{-i(m_0-3)z}E_{m_0+1}e^{i(m_0+1)z}e^{-i(m_0+1)z}. \tag{26}$$

If we use (8), (23), (25) and (26) in (24), we obtain

$$J(z)e^{-4iz} = (KM)^{-1}(K-M)\,[I + o(1)]$$

for $|z| \to \infty$. This gives the asymptotic equation (22). □

Theorem 4.3. *If the condition* (6) *satisfies, then*

i) the set of eigenvalues of $\mathcal{T}$ is bounded and countable,

ii) each eigenvalue of the operator $\mathcal{T}$ is of finite multiplicity,

iii) the limit points of eigenvalues can lie only in $[-2, 2]$.

Proof. The boundedness of eigenvalues of $\mathcal{T}$ directly obtained by asymptotic equation (22). Moreover, by using the definition of $J(z)$, $P_n(z)$, and $E_n(z)$, we get the following representation of the Jost function J

$$\begin{aligned} J(z) &= K^{-1}M^{-1}\left\{KP_{m_0-1}^T(z)E_{m_0+2}(z) - MP_{m_0-2}^T(z)E_{m_0+1}(z)\right\} \\ &= Ke^{2im_0(z)}[I + \mathcal{A}(z)], \end{aligned} \tag{27}$$

where $\mathcal{A}(z) = \mathcal{A}(z + 2\pi)$ and $\mathcal{A}(z)$ is a finite dimensional matrix-valued analytic function in $\mathbb{C}_+$ with respect to z. Using (27) and Theorem 5.1 in [14], the rest of the proof of Theorem can be found easily. □

Theorem 4.4. *Assume* (6)*. Then $\sigma_c(\mathcal{T}) = [-2, 2]$, where $\sigma_c(\mathcal{T})$ denotes the continuous spectrum of $\mathcal{T}$.*

Proof. Introduce the difference operators $\mathcal{T}_0$ and $\mathcal{T}_1$ generated by the following difference expressions in $\ell_2(\mathbb{N}, \mathbb{C}^\mu)$ together with (4) and (5)

$$(\mathcal{T}_0 y)_n = Y_{n-1} + Y_{n+1}, \quad n \in \mathbb{N} \backslash \{m_0 - 1, m_0 + 1\}$$
$$(\mathcal{T}_1 y)_n = B_n Y_n, \quad n \in \mathbb{N} \backslash \{m_0\},$$

respectively. It is clear to see the compactness of $\mathcal{T}_1$ [20]. We also can write $\mathcal{T} = \mathcal{T}_0^1 + \mathcal{T}_0^2 + \mathcal{T}_1$, where $\mathcal{T}_0^1$ is a selfadjoint operator with $\sigma_c(\mathcal{T}_0^1) = [-2, 2]$ and $\mathcal{T}_0^2$ is a finite dimensional operator in $\ell_2(\mathbb{N}, \mathbb{C}^\mu)$. Hence, by using Weyl theorem [13] of a compact perturbation, we get the continuous spectrum of the operator $\mathcal{T}$. □

5. An Example

In this part, we give a special example as an application to draw attention to the validity of our results.

Example 5.1. In the problem (3)-(5), suppose that B is a zero matrix in $\mathbb{C}^\mu$, $m_0 = 3$, $K := [a_{ij}]_{nxn}$ and $M := [b_{ij}]_{nxn}$. Then let us investigate the difference operator $\mathcal{L}$ corresponding to the following impulsive BVP

$$\begin{cases} Y_{n-1} + Y_{n+1} = 2\cos z Y_n, \quad n \in \mathbb{N} \backslash \{2, 3, 4\} \\ Y_0 = 0 \\ Y_4 = K Y_2 \\ Y_5 = M Y_1. \end{cases} . \tag{28}$$

Then, the solution $E_n(z)$ turns into e^{inz} and the fundamental solutions $P_n(z)$ and $Q_n(z)$ of (28) have the following values for $n = 0, 1, 2$.

$$P_0(z) = 0, \quad P_1(z) = I, \quad P_2(z) = \lambda I$$
$$Q_0(z) = I, \quad Q_1(z) = 0, \quad Q_1(z) = -I.$$

Thus, the Jost function of (28) is given as

$$J_0(z) = C_2(z) := J(z) = (KM)^{-1} \left\{ K P_2^T(z) E_5(z) - M P_1^T(z) E_4(z) \right\}. \tag{29}$$

Equation (29) yields

$$J(z) = (KM)^{-1} e^{4iz} \left\{ K e^{2iz} + K - M \right\}.$$

Since all eigenvalues of K and M are different from zero, $\det J(z) = 0$ if and only if

$$\det \begin{bmatrix} a_{11}e^{2iz} + a_{11} - b_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22}e^{2iz} + a_{22} - b_{22} & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & a_{nn}e^{2iz} + a_{nn} - b_{nn} \end{bmatrix} = 0,$$

it implies that

$$\prod_{j=1}^{n} \left[a_{jj} e^{2iz} + a_{jj} - b_{jj} \right] = 0.$$

Last equation gives $\det J(z) = 0$ whenever

$$a_{jj}e^{2iz} + a_{jj} - b_{jj} = 0 \tag{30}$$

for any j integer in $\{1, 2, ...n\}$. It follows from (30) that

$$e^{2iz} = \frac{b_{jj} - a_{jj}}{a_{jj}} := R_j \neq 0. \tag{31}$$

In accordance with (20)-(21), the operator $\mathcal{L}$ has eigenvalues and spectral singularities if and only if (31) holds.

Case 5.2. Let $a_{jj} > b_{jj}$ for all j integers in $\{1, 2, ...n\}$. Thus, there appear two special cases:

(i) If $a_{jj} > 0$, in this case, since $R_j < 0$, we obtain

$$z = -\frac{i}{2}\ln(-R_j) + \frac{\pi}{2} + k\pi, \quad k = -1, 0, 1, \quad j = 1, 2, ..., n. \tag{32}$$

Hence, the problem doesn't have any spectral singularity due to the fact that $R_j \neq -1$, but it has eigenvalues whenever $\ln(-R_j) < 0$, namely,

$$-1 < R_j < 0, \quad j = 1, 2, .., n.$$

Consequently, the necessary condition for the impulsive BVP (28) to have an eigenvalue is that b_{jj} is positive for all j integers in $\{1, 2, ...n\}$.

(ii) If $a_{jj} < 0$, in this case, since $R_j > 0$, we obtain

$$z = -\frac{i}{2}\ln(R_j) + k\pi, \quad k = 0, 1, \quad j = 1, 2, ..., n. \tag{33}$$

Then, there appears a spectral singularity whenever $R_j = 1$ and it yields $z = 0, \pi$. It is concluded from (21) that the impulsive BVP (28) doesn't have any spectral singularity, too. But it has eigenvalues if and only if $\ln(R_j) > 0$, namely,

$$0 < R_j < 1, \quad j = 1, 2, .., n,$$

i.e.,

$$1 < \frac{b_{jj}}{a_{jj}} < 2, \quad j = 1, 2, .., n.$$

Case 5.3. Let $a_{jj} < b_{jj}$ for all j integers in $\{1, 2, ...n\}$. Similarly, there appear two special cases:

(i) Assume $a_{jj} < 0$. Similar with Case 1(i), we get (32). Thus, the impulsive BVP (28) does not have any spectral singularity owing to the fact that $R_j \neq -1$ but it has eigenvalues if and only if $b_{jj} < 0$.

(ii) Assume $a_{jj} > 0$. In this case, similar with Case 2(i), we find (33). Likewise the other cases, there is no spectral singularity. However, there are eigenvalues if and only if $1 < \dfrac{b_{jj}}{a_{jj}} < 2$.

Finally, in order to find the continuous spectrum of $\mathcal{L}$, we consider the Jacobi matrices

$$(\mathcal{L}_0)_{ij} = [t]_{ij} = \begin{cases} I, & i = 5, j = 6 \\ I, & i > 5, j = i - 1, j = i + 1 \\ 0, & otherwise, \end{cases}$$

$$(\mathcal{L}_1)_{ij} = [t]_{ij} = \begin{cases} I, & t_{ij} = t_{12}, t_{32}, t_{34}, t_{54} \\ 0, & otherwise \end{cases}$$

for $i \in \mathbb{N}\backslash\{2, 4\}$, $j \in \mathbb{N}$. Then, it is obvious that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, $\mathcal{L}_0$ is a self-adjoint matrix, $\mathcal{L}_1$ is a compact operator [20] in $\ell_2(\mathbb{N}, \mathbb{C}^\mu)$ as it is finite dimensional. So, we have $\sigma_c(\mathcal{L}) = \sigma_c(\mathcal{L}_0) = [-2, 2]$.

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