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A New Type of Paranorm Intuitionistic Fuzzy Zweier I-convergent Double Sequence Spaces

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Abstract. In this article we introduce the paranorm type intuitionistic fuzzy Zweier I-convergent double sequence spaces ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ and ${}_2\mathcal{Z}^I_{0(\mu,\nu)}(p)$ for $p = (p_{ij})$ a double sequence of positive real numbers and study the fuzzy topology on these spaces.

1. Introduction and Preliminaries

After the pioneering work of Zadeh [37], a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields: population dynamics [2], chaos control [8], computer programming [9], nonlinear dynamical system [11], etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down. The concept of intuitionistic fuzzy normed space [31] and of intuitionistic fuzzy 2-normed space [26] are the latest developments in fuzzy topology. Recently Khan and Yasmeen([18, 19]) studied the intuitionistic fuzzy Zweier I-convergent sequence spaces defined by a modulus function and an Orlicz function.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical problems/matrices(double sequences) through the concept of density [6, 7]. The notion of I-convergence, which is a generalization of statistical convergence, was introduced by Kostyrko et al. [21] by using the idea of I of subsets of the set of natural numbers $\mathbb{N}$ and further studied in [27]. Recently, the notion of statistical convergence of double sequences has been defined and studied by Mursaleen and Edely [25]; and for fuzzy numbers by Savaş and Mursaleen [32]. The notion of ideal convergence of double sequences in the topology induced by fuzzy 2-norm has been studied by Kočinac and Rashid [30], in 2-fuzzy 2-norm spaces by Rashid and Kočinac [20]. Quite recently, Das et al. [5] studied the notion of I and I^*-convergence of double sequences in $\mathbb{R}$.

We recall some notations and basic definitions used in this paper.

Definition 1.1. Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is said to be I-convergent to a number L if for every $\epsilon > 0$ the set $\{k \in \mathbb{N} :| x_k - L |\geq \epsilon\} \in I$.

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Definition 1.2. Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is said to be I-Cauchy if for each $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that the set $\{k \in \mathbb{N} :| x_k - x_N |\geq \epsilon\} \in I$.

Recall that a continuous t-norm is a binary operation $*$ on [0, 1] satisfying: (i) $*$ is commutative and associative, (ii) $*$ is continuous, (iii) $a * 1 = 1$, for each $a \in [0, 1]$, (iv) $a * b \leq c * d$ whenever $a \leq c$. $b \leq d$, $a, b, c, d \in [0, 1]$. A binary operation $\diamond$ on [0,1] is called a continuous t-conorm if it satisfies: (1) $\diamond$ is commutative and associative, (2) $\diamond$ is continuous, (3) $a \diamond 0 = a$ for each $a \in [0, 1]$, (4) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

Definition 1.3. The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space(for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $X\times(0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:
(a) $\mu(x, t) + \nu(x, t) \leq 1$,
(b) $\mu(x, t) > 0$,
(c) $\mu(x, t) = 1$ if and only if $x = 0$,
(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(e) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
(f) $\mu(x, .) : (0, \infty) \to [0, 1]$ is continuous,
(g) $\lim\limits_{t\to\infty} \mu(x, t) = 1$ and $\lim\limits_{t\to 0} \mu(x, t) = 0$,
(h) $\nu(x, t) < 1$,
(i) $\nu(x, t) = 0$ if and only if $x = 0$,
(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(k) $\nu(x, t) \diamond \nu(y, s) \geq \nu(x + y, t + s)$,
(l) $\nu(x, .) : (0, \infty) \to [0, 1]$ is continuous,
(m) $\lim\limits_{t\to\infty} \nu(x, t) = 0$ and $\lim\limits_{t\to 0} \nu(x, t) = 1$.
In this case (μ, ν) is called an intuitionistic fuzzy norm.

Definition 1.4. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$ there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \epsilon$ and $\nu(x_k - L, t) < \epsilon$ for all $k \geq k_0$. In this case we write $(\mu, \nu) - \lim x = L$.

Definition 1.5. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$ there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_l, t) > 1 - \epsilon$ and $\nu(x_k - x_l, t) < \epsilon$ for all $k, l \geq k_0$.

Definition 1.6. Let K be the subset of the set $\mathbb{N}$ of natural numbers. Then the asymptotic density of K, denoted by $\delta(K)$, is defined as $\delta(K) = \lim\limits_n \frac{1}{n}|\{k \leq n : k \in K\}|$, where the vertical bars denotes the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be statistically convergent to a number ℓ if for each $\epsilon > 0$ the set $K(\epsilon) = \{k \leq n :| x_k - \ell |> \epsilon\}$ has asymptotic density zero, i.e. $\lim\limits_n \frac{1}{n}|\{k \leq n :| x_k - \ell |> \epsilon\}| = 0$. In this case we write $st - \lim x = \ell$.

Definition 1.7. A number sequence $x = (x_k)$ is said to be statistically Cauchy sequence if for every $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that $\lim\limits_n \frac{1}{n}|\{j \leq n :| x_j - x_N |\geq \epsilon\}| = 0$.

The concepts of statistical convergence and statistical Cauchy for double sequences in intuitionistic fuzzy normed spaces have been studied by Mursaleen and Mohiuddine [14].

Definition 1.8. Let $I \subset 2^{\mathbb{N}}$ be a non trivial ideal and $(X, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $x = (x_k)$ of elements of X is said to be I-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$ the set

$\{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \epsilon$ or $\nu(x_k - L, t) \geq \epsilon\} \in I$.

In this case L is called the I-limit of the sequence (x_k) with respect to the intuitionistic fuzzy norm (μ, ν) and we write $I_{(\mu,\nu)} - \lim x_k = L$.

2. I_2-Convergence in an IFNS

Definition 2.1. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{ij})$ is said to be statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$
$\delta(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - L, t) \leq 1 - \epsilon$ or $\nu(x_{ij} - L, t) \geq \epsilon\}) = 0.$
or equivalently
$\lim_{mn} \frac{1}{mn} |\{i \leq m, j \leq n, : \mu(x_{ij} - L, t) \leq 1 - \epsilon$ or $\nu(x_{ij} - L, t) \geq \epsilon\}| = 0.$
In this case we write $st^2_{(\mu,\nu)} - \lim x = L$.

Definition 2.2. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a double sequence $x = (x_{ij})$ is said to be statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$ there exist $N = N(\epsilon)$ and $M = M(\epsilon)$ such that for all $i, p \geq N$ and $j, q \geq M$,
$\delta(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - x_{pq}, t) \leq 1 - \epsilon$ or $\nu(x_{ij} - x_{pq}, t) \geq \epsilon\}) = 0.$

Definition 2.3. Let I_2 be a non trivial ideal of $\mathbb{N} \times \mathbb{N}$ and $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. A double sequence $x = (x_{ij})$ of elements of X is said to be I_2 convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for each $\epsilon > 0$ and $t > 0$
$\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - L, t) \leq 1 - \epsilon$ or $\nu(x_{ij} - L, t) \geq \epsilon\} \in I_2.$
In this case we write $I_2^{(\mu,\nu)} - \lim x = L$.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay et al.[1], Başar [4], Talo and Başar [34], Kadak and Başar [12]), Malkowsky [24], Ng and Lee [28], and Wang [35]. Şeng*önül* [33] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transformation of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1 - p)x_{i-1},$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & \text{if } (i = k), \\ 1 - p, & (i - 1 = k); (i, k \in \mathbb{N}) \\ 0, & \text{otherwise}. \end{cases}$$

Analogous to Başar and Altay [3], Şeng*önü*l [33] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c\};$$
$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c_0\}.$$

Recently Khan et al.[13] introduced the following classes of sequences
$\mathcal{Z}^I = \{(x_k) \in \omega : \exists\ L \in \mathbb{C}$ such that for a given $\epsilon > 0, \{k \in \mathbb{N} :| x'_k - L |\geq \epsilon\} \in I\};$
$\mathcal{Z}^I_0 = \{(x_k) \in \omega :$ for a given $\epsilon > 0; \{k \in \mathbb{N} :| x'_k |\geq \epsilon\} \in I\},$
where $(x'_k) = (Z^p x)$.
Khan and Khan [15] introduced the following classes of sequences:

$${}_2\mathcal{Z}^I = \{(x_{ij}) \in \Omega : \exists\ L \in \mathbb{C} \text{ such that for a given } \epsilon > 0, \{(i, j) \in \mathbb{N} \times \mathbb{N} :| x''_{ij} - L |\geq \epsilon\} \in I_2\};$$

$${}_2\mathcal{Z}^I_0 = \{(x_{ij}) \in \Omega : \text{ for a given } \epsilon > 0; \{(i, j) \in \mathbb{N} \times \mathbb{N} :| x''_{ij} |\geq \epsilon\} \in I_2\},$$

where $(x''_{ij}) = ({}_2Z^p x)$ and Ω is space of all double sequences.
Throughout the article, for the sake of convenience, we will denote by

$${}_2Z^p(x_{ij}) = x'', {}_2Z^p(y_{ij}) = y'', {}_2Z^p(z_{ij}) = z'', \text{ for } x, y, z \in \Omega.$$

The concept of paranorm is related to the linear metric spaces. It is a generalization of that of absolute value.

Definition 2.4. ([10, 23, 36]) Let X be linear space. A function $p : X \to \mathbb{R}$ is called a paranorm if
$(p_1) p(0) \geq 0$,
$(p_2) p(x) \geq 0, \forall x \in X$,
$(p_3) p(-x) = p(x), \forall x \in X$,
$(p_4) p(x + y) \leq p(x) + p(y), \forall x, y \in X$ (triangle inequality),
(p_5) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(x_n\lambda_n - x\lambda) \to 0$ as $n \to \infty$, (continuity of multiplication of vectors).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total. It is well known that the metric of any linear metric space is given by some total paranorm[22].

Recently Khan and Yasmeen[16] introduced the following sequence spaces:

$$\mathcal{Z}^I_{(\mu,\nu)}(p) = \{(x_k) \in \omega : \{k \in \mathbb{N} : [\mu(x'_k - L, t)]^{p_k} \leq 1 - \epsilon \text{ or } [\nu(x'_k - L, t)]^{p_k} \geq \epsilon\} \in I\},$$

$$\mathcal{Z}^I_{0(\mu,\nu)}(p) = \{(x_k) \in \omega : \{k \in \mathbb{N} : [\mu(x'_k, t)]^{p_k} \leq 1 - \epsilon \text{ or } [\nu(x'_k, t)]^{p_k} \geq \epsilon\} \in I\}.$$

In this article we introduce the paranorm type intuitionistic Zweier I-convergent double sequence spaces as follows:

$${}_2\mathcal{Z}^I_{(\mu,\nu)}(p) = \{(x_{ij}) \in \Omega : \{(i, j) \in \mathbb{N} \times \mathbb{N} : [\mu(x''_{ij} - L, t)]^{p_{ij}} \leq 1 - \epsilon \text{ or } [\nu(x''_{ij} - L, t)]^{p_{ij}} \geq \epsilon\} \in I_2\};$$

$${}_2\mathcal{Z}^I_{0(\mu,\nu)}(p) = \{(x_{ij}) \in \Omega : \{(i, j) \in \mathbb{N} \times \mathbb{N} : [\mu(x''_{ij}, t)]^{p_{ij}} \leq 1 - \epsilon \text{ or } [\nu(x''_{ij}, t)]^{p_{ij}} \geq \epsilon\} \in I_2\}.$$

and we define an open ball with center x'' and radius r with respect to t by

$${}_2B_{x''}(r, t)(p) = \{y \in X : (i, j) \in \mathbb{N} \times \mathbb{N}; \left[\mu(x''_{ij} - y'', t)\right]^{p_{ij}} > 1 - r \text{ or } \left[\nu(x''_{ij} - y'', t)\right]^{p_{ij}} < r\}.$$

3. Main Results

Theorem 3.1. *${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ and ${}_2\mathcal{Z}^I_{0(\mu,\nu)}(p)$ are linear spaces.*

Proof. We prove the result for ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$. Similarly the result can be proved for ${}_2\mathcal{Z}^I_{0(\mu,\nu)}(p)$. Let $(x''_{ij}), (y''_{ij}) \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ and let α, β be scalars. Then for a given $\epsilon > 0$, we have
$A_1 = \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \left[\mu\left(x''_{ij} - L_1, \frac{t}{2|\alpha|}\right)\right]^{p_{ij}} \leq 1 - \epsilon \text{ or } \left[\nu\left(x''_{ij} - L_1, \frac{t}{2|\alpha|}\right)\right]^{p_{ij}} \geq \epsilon\right\} \in I_2;$
$A_2 = \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \left[\mu\left(y''_{ij} - L_1, \frac{t}{2|\beta|}\right)\right]^{p_{ij}} \leq 1 - \epsilon \text{ or } \left[\nu\left(y''_{ij} - L_1, \frac{t}{2|\beta|}\right)\right]^{p_{ij}} \geq \epsilon\right\} \in I_2.$
Thus
$A_1^c = \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \left[\mu\left(x''_{ij} - L_1, \frac{t}{2|\alpha|}\right)\right]^{p_{ij}} > 1 - \epsilon \text{ or } \left[\nu\left(x''_{ij} - L_1, \frac{t}{2|\alpha|}\right)\right]^{p_{ij}} < \epsilon\right\} \in \mathcal{F}(I_2);$
$A_2^c = \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \left[\mu\left(y''_{ij} - L_1, \frac{t}{2|\beta|}\right)\right]^{p_{ij}} > 1 - \epsilon \text{ or } \left[\nu\left(y''_{ij} - L_1, \frac{t}{2|\beta|}\right)\right]^{p_{ij}} < \epsilon\right\} \in \mathcal{F}(I_2).$
Define the set $A_3 = A_1 \cup A_2$, so that $A_3 \in I_2$. It follows that A_3^c is a non-empty set in $\mathcal{F}(I_2)$.

We shall show that for each $(x''_{ij}), (y''_{ij}) \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$,

$$A_3^c \subset \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \left[\mu\left((\alpha x''_{ij} + \beta y''_{ij}) - (\alpha L_1 + \beta L_2), t\right)\right]^{p_{ij}} > 1 - \epsilon \text{ or } \left[\nu\left((\alpha x''_{ij} + \beta y''_{ij}) - (\alpha L_1 + \beta L_2), t\right)\right]^{p_{ij}} < \epsilon\right\}.$$

Let $(m, n) \in A_3^I$. In this case
$\left[\mu\left(x''_{mn} - L_1, \frac{t}{2|\alpha|}\right)\right]^{p_{ij}} > 1 - \epsilon$ or $\left[\nu\left(x''_{mn} - L_1, \frac{t}{2|\alpha|}\right)\right]^{p_{ij}} < \epsilon$ and $\left[\mu\left(y''_{mn} - L_2, \frac{t}{2|\beta|}\right)\right]^{p_{ij}} > 1 - \epsilon$ or $\left[\nu\left(y''_{mn} - L_2, \frac{t}{2|\beta|}\right)\right]^{p_{ij}} < \epsilon$.
We have
$\left[\mu\left((\alpha x''_{mn} + \beta y''_{mn}) - (\alpha L_1 + \beta L_2), t\right)\right]^{p_{ij}}$

$\geq \left[\mu\left(\alpha x''_{mn} - \alpha L_1, \frac{t}{2}\right)\right]^{p_{ij}} * \left[\mu\left(\beta y''_{mn} - \beta L_2, \frac{t}{2}\right)\right]^{p_{ij}}$
$= \left[\mu\left(x''_{mn} - L_1, \frac{t}{2|\alpha|}\right)\right]^{p_{ij}} * \left[\mu\left(y''_{mn} - L_2, \frac{t}{2|\beta|}\right)\right]^{p_{ij}}$
$> (1-\epsilon) * (1-\epsilon) = (1-\epsilon)$
and
$\left[\nu\left((\alpha x''_{mn} + \beta y''_{mn}) - (\alpha L_1 + \beta L_2), t\right)\right]^{p_{ij}}$
$\leq \left[\nu\left(\alpha x''_{mn} - \alpha L_1, \frac{t}{2}\right)\right]^{p_{ij}} \diamond \left[\nu\left(\beta y''_{mn} - \beta L_2, \frac{t}{2}\right)\right]^{p_{ij}}$
$= \left[\nu\left(x''_{mn} - L_1, \frac{t}{2|\alpha|}\right)\right]^{p_{ij}} \diamond \left[\nu\left(y''_{mn} - L_2, \frac{t}{2|\beta|}\right)\right]^{p_{ij}}$
$> \epsilon \diamond \epsilon = \epsilon.$
This implies that

$$A_3^c \subset \Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : \left[\mu\left((\alpha x''_{ij} + \beta y''_{ij}) - (\alpha L_1 + \beta L_2), t\right)\right]^{p_{ij}} > 1-\epsilon \text{ or}$$
$$\left[\nu\left((\alpha x''_{ij} + \beta y''_{ij}) - (\alpha L_1 + \beta L_2), t\right)\right]^{p_{ij}} < \epsilon\Big\}.$$

Hence ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ is a linear space. □

Theorem 3.2. *Every open ball ${}_2B_{x''}(r,t)(p)$ is an open set in ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$.*

Proof. Let ${}_2B_{x''}(r,t)(p)$ be an open ball with center x'' and radius r with respect to t. That is

$${}_2B_{x''}(r,t)(p) = \{y \in X : (i,j) \in \mathbb{N} \times \mathbb{N}; \left[\mu(x^{//}_{ij} - y'', t)\right]^{p_{ij}} > 1-r \text{ or } \left[\nu(x''_{ij} - y'', t)\right]^{p_{ij}} < r\}.$$

Let $y'' \in {}_2B_{x''}(r,t)(p)$. Then $\left[\mu(x'' - y'', t)\right]^{p_{ij}} > 1-r$ and $\left[\nu(x'' - y'', t)\right]^{p_{ij}} < r$.
Since $\left[\mu(x''-y'', t)\right]^{p_{ij}} > 1-r$, there exists $t_0 \in (0,1)$ such that $\left[\mu(x''-y'', t_0)\right]^{p_{ij}} > 1-r$ and $\left[\nu(x''-y'', t_0)\right]^{p_{ij}} < r$.
Putting $r_0 = \left[\mu(x''-y'', t_0)\right]^{p_{ij}}$, we have $r_0 > 1-r$, and there exists $s \in (0,1)$ such that $r_0 > 1-s > 1-r$.
For $r_0 > 1-s$, we have $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1-s$ and $(1-r_0) \diamond (1-r_2) \leq s$.
Putting $r_3 = \max\{r_1, r_2\}$, consider the ball ${}_2B_{y^{//}}(1-r_3, t-t_0)(p)$. We prove that

$${}_2B_{y^{//}}(1-r_3, t-t_0)(p) \subset {}_2B_{x^{//}}(r,t)(p).$$

Let $z'' \in {}_2B^c_{y''}(1-r_3, t-t_0)(p)$.
$\left[\mu(y''-z'', t-t_0)\right]^{p_{ij}} > r_3$ and $\left[\nu(y''-z'', t-t_0)\right]^{p_{ij}} < r_3$.
Therefore,
$\left[\mu(x''-z'', t)\right]^{p_{ij}} \geq \left[\mu(x''-y'', t_0)\right]^{p_{ij}} * \left[\mu(y''-z'', t-t_0)\right]^{p_{ij}} \geq (r_0 * r_3) \geq (r_0 * r_1) \geq (1-s) > (1-r).$

and
$\left[\nu(x''-z'', t)\right]^{p_{ij}} \leq \left[\nu(x''-y'', t_0)\right]^{p_{ij}} \diamond \left[\nu(y''-z'', t-t_0)\right]^{p_{ij}} \leq (1-r_0) \diamond (1-r_3) \leq (1-r_0) > (1-r_2) < s < r.$

Thus $z'' \in {}_2B^c_{x''}(r,t)(p)$ and hence ${}_2B^c_{y''}(1-r_3, t-t_0)(p) \subset {}_2B^c_{x''}(r,t)(p)$. □

Remark 3.3. ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ is an IFNS.

Define
${}_2\tau^I_{(\mu,\nu)}(p) = \{A \subset {}_2\mathcal{Z}^I_{(\mu,\nu)}(p) :$ for each $x \in A$ there exists $t > 0$ and $r \in (0,1)$ s. t. ${}_2B^c_{x''}(r,t)(p) \subset A\}$.

Then ${}_2\tau^I_{(\mu,\nu)}(p)$ is a topology on ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$.

Theorem 3.4. *The topology ${}_2\tau^I_{(\mu,\nu)}(p)$ on ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ is first countable.*

Proof. $\left\{{}_2B_{x''}\left(\frac{1}{n},\frac{1}{n}\right)(p) : n = 1, 2, 3, ...\right\}$ is a local base at x''. Hence the topology ${}_2\tau^I_{(\mu,\nu)}(p)$ on ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ is first countable. □

Theorem 3.5. *${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ and ${}_2\mathcal{Z}^I_{0(\mu,\nu)}(p)$ are Hausdorff spaces.*

Proof. We prove the result for ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$. Similarly the result can be proved for ${}_2\mathcal{Z}^I_{0(\mu,\nu)}(p)$. Let $x'', y'' \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ such that $x'' \neq y''$. Then $0 < \left[\mu(x'' - y'', t)\right]^{p_{ij}} < 1$ and $0 < \left[\nu(x'' - y'', t)\right]^{p_{ij}} < 1$. Put $r_1 = \left[\mu(x'' - y'', t)\right]^{p_{ij}}$, $r_2 = \left[\nu(x'' - y'', t)\right]^{p_{ij}}$ and $r = \max\{r_1, 1 - r_2\}$.

For each $r_0 \in (r, 1)$, there exists r_3 and r_4 such that $r_3 * r_3 \geq r_0$ and $(1 - r_4) \diamond (1 - r_4) \leq (1 - r_0)$.

Putting $r_5 = \max\{r_3, r_4\}$ consider the open balls ${}_2B_{x''}(1 - r_5, \frac{t}{2})(p)$ and ${}_2B_{y''}(1 - r_5, \frac{t}{2})(p)$.

Then clearly ${}_2B_{x''}(1 - r_5, \frac{t}{2})(p) \cap {}_2B_{y''}(1 - r_5, \frac{t}{2})(p) = \emptyset$.

For if there exists $z'' \in {}_2B_{x''}(1 - r_5, \frac{t}{2})(p) \cap {}_2B_{y''}(1 - r_5, \frac{t}{2})(p)$, then

$r_1 = \left[\mu(x'' - y'', t)\right]^{p_{ij}} \geq \left[\mu(x'' - z'', \frac{t}{2})\right]^{p_{ij}} * \left[\mu(z'' - y'', \frac{t}{2})\right]^{p_{ij}} \geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1.$

and

$r_2 = \left[\nu(x^{//} - y^{//}, t)\right]^{p_{ij}} \leq \left[\mu(x^{//} - z^{//}, \frac{t}{2})\right]^{p_{ij}} \diamond \left[\nu(z^{//} - y^{//}, \frac{t}{2})\right]^{p_{ij}} \leq (1 - r_5) \diamond (1 - r_5) \leq (1 - r_4) \diamond (1 - r_4) \leq (1 - r_0) < r_2,$

which is a contradiction.

Hence ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ is Hausdorff. □

Theorem 3.6. *${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ is an IFNS. ${}_2\tau^I_{(\mu,\nu)}(p)$ is a topology on ${}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$. Then a sequence $(x''_{ij}) \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$, $x''_{ij} \to x''$ if and if $\left[\mu(x''_{ij} - x'', t)\right]^{p_{ij}} \to 1$ and $\left[\nu(x''_{ij} - x'', t)\right]^{p_{ij}} \to 0$ as $i \to \infty, j \to \infty$.*

Proof. Fix $t_0 > 0$. Suppose $x''_{ij} \to x''$. Then for $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $x''_{ij} \in {}_2B_{x''}(r, t)(p)$ for all $i \geq n_0, j \geq n_0$.

${}_2B_{x''}(r, t)(p) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : \left[\mu(x''_{ij} - x'', t)\right]^{p_{ij}} \leq 1 - r$ or $\left[\nu(x''_{ij} - x'', t)\right]^{p_{ij}} \geq r\} \in I_2$

such that ${}_2B^c_{x''}(r, t)(p) \in \mathcal{F}(I_2)$.

Then $1 - \left[\mu(x''_{ij} - x'', t)\right]^{p_{ij}} < r$ and $\left[\nu(x''_{ij} - x'', t)\right]^{p_{ij}} < r$. This implies $\left[\mu(x''_{ij} - x'', t)\right]^{p_{ij}} \to 1$ and $\left[\nu(x''_{ij} - x'', t)\right]^{p_{ij}} \to 0$ as $i \to \infty, j \to \infty$.

Conversely, if for each $t > 0$, $\left[\mu(x''_{ij} - x'', t)\right]^{p_{ij}} \to 1$ and $\left[\nu(x''_{ij} - x'', t)\right]^{p_{ij}} \to 0$ as $i \to \infty, j \to \infty$, then for $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - \left[\mu(x''_{ij} - x'', t)\right]^{p_{ij}} < r$ for all $i \geq n_0, j \geq n_0$.

Thus $x''_{ij} \in {}_2B^c_{x_{ij}}(r, t)(p)$ for all $i \geq n_0, j_0 \geq n_0$ and hence $x''_{ij} \to x''$. □

Theorem 3.7. *A double sequence $x = (x''_{ij}) \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$ is said to be I-convergent to L if and only if for every $\epsilon > 0$ and $t > 0$ there exist the numbers $M = M(x, \epsilon, t)$ and $N = N(x, \epsilon, t)$ such that*

$\{(M, N) \in \mathbb{N} \times \mathbb{N} : [\mu(x''_{MN} - L, \frac{t}{2})]^{p_{ij}} > 1 - \epsilon$ or $[\nu(x''_{MN} - L, \frac{t}{2})]^{p_{ij}} < \epsilon\} \in \mathcal{F}(I_2)$.

Proof. Suppose that $I_2^{(\mu,\nu)} - \lim x = L$ and let $\epsilon > 0$ and $t > 0$. For a given $\epsilon > 0$ choose, $s > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > 1 - s$ and $\epsilon \diamond \epsilon < s$. Then for each $x \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$,

$A_x(\epsilon, t)(p) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : [\mu(x''_{ij} - L, \frac{t}{2})]^{p_{ij}} \leq 1 - \epsilon$ or $[\nu(x''_{ij} - L, \frac{t}{2})]^{p_{ij}} \geq \epsilon\} \in I_2$

which implies that

$A^c_x(\epsilon, t)(p) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : [\mu(x''_{ij} - L, \frac{t}{2})]^{p_{ij}} > 1 - \epsilon$ or $[\nu(x''_{ij} - L, \frac{t}{2})]^{p_{ij}} < \epsilon\} \in \mathcal{F}(I_2)$.

Conversely, let us choose $N \in A^c_x(\epsilon, t)(p)$. Then $\left[\mu(x''_{MN} - L, \frac{t}{2})\right]^{p_{ij}} > 1 - \epsilon$ or $\left[\nu(x''_{MN} - L, \frac{t}{2})\right]^{p_{ij}} < \epsilon$. Now we want to show that there exist the numbers $M = M(x, \epsilon, t)$ and $N = N(x, \epsilon, t)$ such that

$\left\{(i, j) \in \mathbb{N} \times \mathbb{N} : \left[\mu(x''_{ij} - x''_{MN}, t)\right]^{p_{ij}} \leq 1 - s \text{ or } \left[\nu(x''_{ij} - x''_{MN}, t)\right]^{p_{ij}} \geq s\right\} \in I_2.$

For this, define for each $x \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(p)$

$_2B_x(\epsilon, t)(p) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : [\mu(x''_{ij} - x''_{MN}, t)]^{p_{ij}} \leq 1 - s \text{ or } [\nu(x''_{ij} - x''_{MN}, t)]^{p_{ij}} \geq s\} \in I_2$.
Now we show that $_2B_x(\epsilon, t)(p) \subset {_2A_x}(\epsilon, t)(p)$. Suppose that $_2B_x(\epsilon, t)(p) \not\subseteq {_2A_x}(\epsilon, t)(p)$. Then there exists $(m, n) \in {_2B_x}(\epsilon, t)(p) \backslash {_2A_x}(\epsilon, t)(p)$. Therefore we have $[\mu(x''_{ij} - x''_{MN}, t)]^{p_{ij}} < 1 - s$ and $\left[\mu(x''_{ij} - L, \frac{t}{2})\right]^{p_{ij}} > 1 - \epsilon$. In particular $\left[\mu(x''_{MN} - L, \frac{t}{2})\right]^{p_{ij}} > 1 - \epsilon$.

Therefore we have

$$1 - s \geq [\mu(x''_{mn} - x''_{MN}, t)]^{p_{ij}} \geq \left[\mu(x''_{mn} - L, \frac{t}{2})\right]^{p_{ij}} * \left[\mu(x''_{MN} - L, \frac{t}{2})\right]^{p_{ij}} \geq (1 - \epsilon) * (1 - \epsilon) > 1 - s,$$

which is not possible.

On the other hand, $[\nu(x''_{ij} - x''_{MN}, t)]^{p_{ij}} \geq s$ and $\left[\nu(x''_{ij} - L, \frac{t}{2})\right]^{p_{ij}} < \epsilon$. In particular $\left[\nu(x''_{MN} - L, \frac{t}{2})\right]^{p_{ij}} < \epsilon$. Therefore we have

$$s \leq [\nu(x''_{mn} - x''_{MN}, t)]^{p_{ij}} \leq \left[\nu(x''_{mn} - L, \frac{t}{2})\right]^{p_{ij}} \diamond \left[\nu(x''_{MN} - L, \frac{t}{2})\right]^{p_{ij}} \leq \epsilon \diamond \epsilon < s,$$

which is not possible.

Hence

$$_2B_x(\epsilon, t)(p) \subset {_2A_x}(\epsilon, t)(p). {_2A_x}(\epsilon, t)(p) \in I_2 \Rightarrow {_2B_x}(\epsilon, t)(p) \in I_2.$$

This completes the proof. □

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